

LECTURES ON UNIQUE CONTINUATION FOR WAVES

by

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Abstract. – These notes are intended as an introduction to the question of unique continuation for the wave operator, and some of its applications. The general question is the following: is a solution to a wave equation in a domain identically zero if it is known to vanish in a subdomain? We state and prove two of the main results in the field.

We first give a proof of the classical local Hörmander theorem in this context that holds under a pseudoconvexity condition.

We then specialize to the case of wave operators with time-independent coefficients and prove the Tataru theorem: local unique continuation holds across any non-characteristic hypersurface. This local result implies a global unique continuation statement that can be interpreted as a converse to finite propagation speed.

We finally give an application to approximate controllability, and present without proofs some associated quantitative estimates.

Résumé (Cours sur le prolongement unique pour les ondes). – L’objectif de ces notes est de présenter la question du prolongement unique pour l’opérateur des ondes, ainsi que certaines de ses applications. La question générale est la suivante : une solution d’une équation d’onde dans un domaine est-elle nulle partout si l’on sait qu’elle s’annule dans un sous-domaine ? Nous énonçons et démontrons deux des principaux résultats dans ce domaine.

Nous donnons tout d’abord une preuve du théorème local classique de Hörmander dans ce contexte, qui s’applique sous une condition de pseudo-convexité.

Nous nous spécialisons ensuite au cas des opérateurs d’onde à coefficients indépendants du temps et prouvons le théorème de Tataru : le prolongement unique local s’applique à travers toute hypersurface non caractéristique. Ce résultat local implique un énoncé de prolongement unique global qui peut être interprété comme la réciproque de la vitesse finie de propagation. Nous donnons enfin une application au contrôle approché et présentons, sans démonstration, quelques estimées quantitatives associées.

1. Introduction and generalities

These notes provide an introduction to the question of unique continuation for waves. We present the Hörmander theorem and the Tataru theorem in this context. They are two of the main local results in the field. Before entering the heart of the subject, we discuss motivation for studying unique continuation for wave operators. Related references and further readings are presented in Section 4.

1.1. Motivation and applications. – We start with presenting different applications to motivate the more technical parts of these notes. All these applications are further discussed in Section 3.5.

1.1.1. *Penetration of waves into a shadow region.* – In this section, we consider the wave equation outside a convex obstacle in $\mathbb{R}^d$. Namely, let $\mathcal{O} \subset \mathbb{R}^d$ be a bounded smooth open subset, and consider $\mathcal{M} = \mathbb{R}^d \setminus \mathcal{O}$. We consider the Laplace operator Δ and $u(t, x)$ the solution to the wave equation

$$(1) \quad \begin{cases} \partial_t^2 u - \Delta u = 0, & \text{on } \mathbb{R} \times \text{Int}(\mathcal{M}), \\ u = 0, & \text{on } \mathbb{R} \times \partial\mathcal{M}, \\ (u, \partial_t u)|_{t=0} = (u_0, u_1), & \text{on } \mathcal{M}. \end{cases}$$

Now, we consider a compact set $K \subset \mathcal{M}$, and assume that the initial data (u_0, u_1) are supported in K . If the set K is not too large, there is a whole region of $\mathcal{M}$ that does not intersect any ray of geometric optics passing through $\mathcal{M}$ (i.e., a straight line in $\text{Int}(\mathcal{M})$, that reflects according to the Snell-Descartes laws at the boundary $\partial\mathcal{M}$) passing through K . Taking an open set ω in this *shadow region*, the question under consideration is the following:

Can one recover (u_0, u_1) from the observation of u on the set $(-T, T) \times \omega$?

And if so, what is the required time T ? By the linearity of (1), this can be reformulated as the following unique continuation question:

$$(2) \quad (u \text{ solution to (1), } \text{supp}(u_0, u_1) \subset K, \text{ and } u|_{(-T, T) \times \omega} = 0) \implies (u_0, u_1) = 0?$$

(and hence $u \equiv 0$ on account to the well-posedness of the Cauchy problem, see, e.g., [88] or [22]). We shall see that Property (2) is false if T is too small, but holds true if T is large enough. The limit time will be expressed as a natural geometric quantity.

1.1.2. *Approximate controllability for the wave equation.* – In this section, we consider a wave equation in a compact d -dimensional manifold $\mathcal{M}$ (or the closure of a bounded open set $\mathcal{M} \subset \mathbb{R}^d$), controlled from a subdomain, namely, given $\chi_\omega \in C^\infty(\mathcal{M})$, the

equation

$$(3) \quad \begin{cases} \partial_t^2 u - \Delta_g u = \chi_\omega f, & \text{on } (0, T) \times \text{Int}(\mathcal{M}), \\ u = 0, & \text{on } (0, T) \times \partial \mathcal{M}, \\ (u, \partial_t u)|_{t=0} = (u_0, u_1), & \text{on } \mathcal{M}. \end{cases}$$

Here, f plays the role of a forcing term, acting only on $\omega := \{\chi_\omega \neq 0\}$. Controllability problems study the possibility of driving the solution u to (3) from the initial state (u_0, u_1) to a final target state (v_0, v_1) at time T , using only the action of f on ω . This property depends a priori on the data/target, and is very complicated. More tractable questions, arising from applications in engineering are the following.

Definition 1. – We say that (3) is *exactly controllable* from (χ_ω, T) if for all data $(u_0, u_1) \in L^2(\mathcal{M}) \times H^{-1}(\mathcal{M})$ and all target state $(v_0, v_1) \in L^2(\mathcal{M}) \times H^{-1}(\mathcal{M})$, there is a function $f \in L^2((0, T); H^{-1}(\mathcal{M}))$ such that the solution to (3) satisfies $(u, \partial_t u)|_{t=T} = (v_0, v_1)$.

We say that (3) is *approximately controllable* from (χ_ω, T) if for all data $(u_0, u_1) \in L^2(\mathcal{M}) \times H^{-1}(\mathcal{M})$ all target state $(v_0, v_1) \in L^2(\mathcal{M}) \times H^{-1}(\mathcal{M})$, and any precision $\varepsilon > 0$, there is a function $f = f_\varepsilon \in L^2((0, T); H^{-1}(\mathcal{M}))$ such that the solution to (3) satisfies $\left\| (u, \partial_t u)|_{t=T} - (v_0, v_1) \right\|_{L^2(\mathcal{M}) \times H^{-1}(\mathcal{M})} \leq \varepsilon$.

Notice that multiplication by χ_ω maps $H_0^1(\mathcal{M})$ into itself continuously and thus $H^{-1}(\mathcal{M})$ into itself continuously as well; the Cauchy problem is thus well-defined in these spaces.

Due to finite speed of propagation for waves, if $\bar{\omega} \neq \mathcal{M}$, a minimal time will be required for controllability to hold. Here, we will mostly be interested in the (weaker) approximate controllability question. By the linearity of the equation, it is enough to consider zero initial conditions $(u_0, u_1) = (0, 0)$. Introducing the “final value” linear map

$$F_T : L^2((0, T); H^{-1}(\mathcal{M})) \rightarrow L^2(\mathcal{M}) \times H^{-1}(\mathcal{M})$$

$$f \mapsto (u, \partial_t u)|_{t=T},$$

where u denotes the solution of (3) associated to $(u_0, u_1) = (0, 0)$, approximate controllability is equivalent to having $\text{range}(F_T)$ dense in $L^2(\mathcal{M}) \times H^{-1}(\mathcal{M})$. This can be reformulated as $\ker({}^t F_T) = \{0\}$, where ${}^t F_T$ is an appropriate transpose of F_T . Multiplying Equation (3) by w solution to

$$(4) \quad \begin{cases} \partial_t^2 w - \Delta_g w = 0, & \text{on } (0, T) \times \text{Int}(\mathcal{M}), \\ w = 0, & \text{on } (0, T) \times \partial \mathcal{M}, \\ (w, \partial_t w)|_{t=T} = (w_0, w_1), & \text{on } \mathcal{M}, \end{cases}$$

with $(w_0, w_1) \in H_0^1(\mathcal{M}) \times L^2(\mathcal{M})$, integrating on $(0, T) \times \mathcal{M}$, and integrating by parts in time and space, we obtain

$$\langle \partial_t u(T), w_0 \rangle_{H^{-1}, H_0^1} - (u(T), w_1)_{L^2} = \langle f, \chi_\omega w \rangle_{L^2(0, T; H^{-1}), L^2(0, T; H_0^1)}.$$

Note that this identity is valid both with real inner products and duality products and real-valued solutions to (4), or with complex ones (since the wave Equation (4) has real coefficients). As a consequence, with an adequate choice of duality, one can identify ${}^t F_T$ with the map

$$\begin{aligned} {}^t F_T : H_0^1(\mathcal{M}) \times L^2(\mathcal{M}) &\rightarrow L^2((0, T); H_0^1(\mathcal{M})) \\ (w_0, w_1) &\mapsto \chi_\omega w, \end{aligned}$$

where w is the unique solution to (4). Having $\ker({}^t F_T) = \{0\}$ is the unique-continuation property

$$(5) \quad (w \text{ solution to (4), } (w_0, w_1) \in H_0^1(\mathcal{M}) \times L^2(\mathcal{M}), \text{ and } w|_{(0, T) \times \omega} = 0) \implies (w_0, w_1) = 0,$$

which now appears as a characterization of the approximate controllability of (3).

1.1.3. Inverse problems and the boundary control method. – In this section, we consider a wave equation in the compact manifold $\mathcal{M}$ with $\partial\mathcal{M} \neq \emptyset$, with a time independent potential $q \in C^\infty(\mathcal{M})$ and a forcing term $f \in C_c^\infty((0, \infty) \times \partial\mathcal{M})$ at the boundary and vanishing initial data:

$$(6) \quad \begin{cases} \partial_t^2 u - \Delta_g u + qu = 0, & \text{on } (0, \infty) \times \text{Int}(\mathcal{M}), \\ u = f, & \text{on } (0, \infty) \times \partial\mathcal{M}, \\ (u, \partial_t u)|_{t=0} = (0, 0), & \text{on } \mathcal{M}. \end{cases}$$

Existence and uniqueness of a solution $u \in C^\infty((0, \infty) \times \mathcal{M})$ follows for instance from lifting the (smooth) boundary data to $\mathcal{M}$ and using well-posedness of (3). We define the dynamical Dirichlet-to-Neumann map

$$\Lambda_q : C_c^\infty((0, \infty) \times \partial\mathcal{M}) \ni f \mapsto \partial_n u|_{\partial\mathcal{M}} \in C^\infty((0, \infty) \times \partial\mathcal{M}),$$

where u is the solution to (6) and ∂_n denotes the unit outward normal derivative to $\partial\mathcal{M}$. A general question in inverse problem is whether the knowledge of the Dirichlet-to-Neumann map Λ_q determines the potential q uniquely? That is to say, probing the wave in the domain $\mathcal{M}$ by means of a boundary source f and knowing the response for all possible inputs f , is it possible to determine the potential q . In mathematical terms, do we have $\Lambda_{q_1} = \Lambda_{q_2} \implies q_1 = q_2$? We refer to [71] for a presentation of the boundary control method of [12] to solve this question. This methods relies in a key fashion on a unique continuation result presented in Section 3.5.1 below.

1.2. Generalities about unique continuation

1.2.1. *The unique continuation problem.* – All above described problems amount to a *unique continuation* property for the wave equation. The general problem of *unique continuation* can be set into the following form: given a differential operator $P = \sum_{|\alpha| \leq m} a_\alpha(x) \partial_x^\alpha$ on a connected open set $\Omega \subset \mathbb{R}^n$, and given a small subset U of Ω , do we have (for u regular enough):

$$(7) \quad \begin{cases} Pu = 0 & \text{in } \Omega \\ u = 0 & \text{in } U \end{cases} \implies u = 0 \text{ on } \Omega?$$

A more tractable problem than (7) is the so called *local unique continuation across an hypersurface* problem: given an oriented local hypersurface $S = \{\Psi = 0\}$ at a point x_0 (that is $\Psi(x_0) = 0$ and $d\Psi(x_0) \neq 0$), do we have the following implication:

There is a neighborhood Ω of x_0 such that

$$(8) \quad \begin{cases} Pu = 0 & \text{in } \Omega \\ u = 0 & \text{in } \Omega \cap S^+ \end{cases} \implies u = 0 \text{ in a neighborhood of } x_0.$$

Here $S^+ = \{\Psi > 0\}$ is one side of S . It turns out that proving (8) for a suitable class of hypersurfaces S (with regards to the operator P) is in general a key step in the proof of properties of the type (7).

Let us discuss briefly the local unique continuation property (8) in the simple case where P is a real non-degenerate vector field. More precisely, consider $P = \sum_{k=0}^n a_k(x) \partial_{x_k}$ near 0, with a_k smooth and real-valued, and assume that it is nondegenerate at 0, that is $a(0) = (a_1(0), \dots, a_n(0)) \neq 0$.

Example 2 (Non-characteristic hypersurface). – Take $S = \{\Psi = 0\}$ where $\Psi(0) = 0$ and $d\Psi(0) \neq 0$. Then, a sufficient condition for having local unique continuation (8) is that $\langle d\Psi(0), a(0) \rangle \neq 0$, i.e., the vector field P is transverse to S at 0. This condition is a “non-characteristicity assumption,” see Definition 7 below. Note that the condition $\langle d\Psi(0), a(0) \rangle \neq 0$ is not necessary for unique continuation to hold, see the discussion in Example 3 below.

Example 3 (Constant vector field and curved hypersurface). – Here (as opposed to Example 2), we show that the orientation of the hypersurface can play a role. Consider for simplicity the operator $P = \frac{\partial}{\partial x_1}$ in $\mathbb{R}^2$ in a neighborhood of 0, and the hypersurface $S = \{x = (x_1, x_2) \in \mathbb{R}^2, \Psi(x) = 0\}$, where $\Psi(x_1, x_2) = x_2 - x_1^2$, see Figure 1. Notice first that P is tangent to S at 0 since $\langle d\Psi(0), P \rangle = 0$ (that is to say, S is characteristic for P at the point 0). We claim that unique continuation holds from $S^- = \{\Psi < 0\}$ (outside the parabola) to $S^+ = \{\Psi > 0\}$ (inside the parabola), but not from S^+ to S^- .

Indeed solutions u to $Pu = 0$ read $u(x_1, x_2) = u^0(x_2)$ for all $x_1 \in \mathbb{R}$. The first statement then follows from the fact that any line $x_2 = cst > 0$ intersects S^+ in a neighborhood of zero, thus showing that if $u^0(x_2) = 0$ for all x_2 in a neighborhood of zero, then

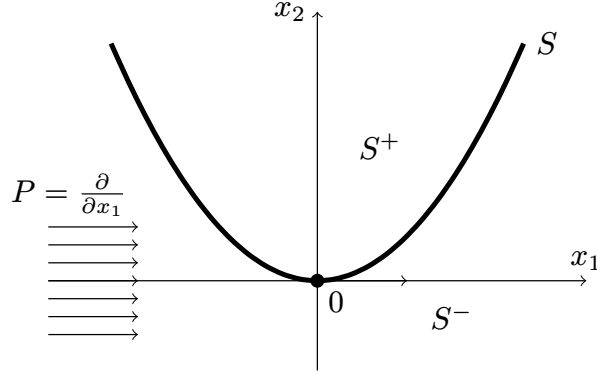


FIGURE 1. Constant vector field $P = \frac{\partial}{\partial x_1}$ and curved hypersurface $S = \{\Psi = 0\}$ with $\Psi(x_1, x_2) = x_2 - x_1^2$. Unique continuation holds at zero from $S^- = \{\Psi < 0\}$ to $S^+ = \{\Psi > 0\}$ but not from S^+ to S^- .

$u^0 = 0$. Choosing $u^0 \in C_c^\infty(\mathbb{R})$ such that $u^0(x_2) \neq 0$ on $0 > x_2 > -1$ and $u^0(x_2) = 0$ on $x_2 \geq 0$ yields the second statement.

About this example, note finally that the contact at order 2 of the vector field with S (here included in the curvature of the surface) is not necessary. Higher order contacts are allowed as long as the integral curve of $P = \frac{\partial}{\partial x_1}$ that is tangent to S at 0 remains in the side where $u = 0$ and only intersects S at 0. The latter conditions are however essentially necessary for *local* unique continuation to hold. Notice that one may construct situations in which *global* unique continuation holds whereas *local* unique continuation does not.

The above Examples 2 and 3 concerning first-order partial differential operators (namely, vector fields) show that geometrical conditions connecting the operator P and the hypersurface S are often needed for unique continuation to hold. Let us now discuss related properties for the wave operator, for which the situation is far more difficult.

1.2.2. Remarks on (non-)unique continuation for the flat/Minkowski wave operator.

– In this section, we collect known facts for the wave equation

$$(9) \quad (\partial_t^2 - \Delta)u = 0 \quad \text{on } \mathbb{R} \times \mathbb{R}^d,$$

in the flat space $\mathbb{R}^{1+d}$, that are related to unique continuation questions.

We start with the simpler case $d = 1$ and consider the wave operator $P = \partial_t^2 - \partial_x^2$ on $\mathbb{R}_t \times \mathbb{R}_x$. Then P factorizes as $P = (\partial_t + \partial_x)(\partial_t - \partial_x)$ and all solutions to $Pu = 0$ have the form $u(t, x) = f(x + t) + g(x - t) + C_0 t + C_1 x + C_2$, where f, g are functions and C_j constants. Take for instance $g = 0$, $C_j = 0$ and $f \in C^\infty(\mathbb{R})$ with $\text{supp}(f) = [0, 1]$. Then $u(t, x) = f(x + t)$ and the hypersurface $S = \{x + t = 0\}$ do not satisfy the unique continuation property (at any point of S). More precisely, up to a linear change of variables, this problem reduces to that of Examples 2–3

discussed above, and one sees that the only hyperplanes S not satisfying the unique continuation property (at any point) are $S_{\pm}^{\alpha} = \{x \pm t = \alpha\}$, for $\alpha \in \mathbb{R}$.

Let us now discuss the situation in higher dimensions $d \geq 2$, which is radically different. In this situation, the polynomial $\xi_t^2 - \sum_{j=1}^d \xi_{x_j}^2$ does not factorize in a product of polynomials of degree 1. This property explains that values of solutions to (9) are not “transported”. To see this, we can actually solve the wave Equation (9): for instance, in $\mathbb{R}^{1+3}$, the Kirchhoff formula (see, e.g., [22])

$$(10) \quad u(t, x) = \frac{1}{4\pi t} \int_{|y-x|=t} u_1(y) dS_t(y) = \frac{t}{4\pi} \int_{\mathbb{S}^2} u_1(x - t\sigma) dS_1(\sigma), \quad u(-t) = -u(t), \quad t > 0$$

gives the unique solution to (9) with $(u, \partial_t u)|_{t=0} = (0, u_1)$, $u_1 \in C^0(\mathbb{R}^3)$. In the first formula, the integration set is the (2 dimensional) sphere centered at x and of radius t ; in the second it is the unit sphere. The integration measure dS_t is the surface measure on the sphere of radius t (induced by the Euclidean measure dx on $\mathbb{R}^3$). As a consequence of this explicit solution, we see that if we choose u_1 such that, for some $r > 0$,

$$u_1 \in C_c^\infty(\mathbb{R}^3), \quad u_1 \geq 0 \text{ on } \mathbb{R}^3, \quad \text{supp } u_1 = \overline{B}(0, r), \quad \text{and } u_1 > 0 \text{ on } B(0, r),$$

then the associated solution u defined by (10) is smooth and satisfies $u \geq 0$ on $\mathbb{R}^{1+3}$. Moreover, notice that $u_1(x - t\sigma) = 0$ if and only if $x - t\sigma \notin B(0, r)$, and hence $u(t, x) = 0$ as soon as $t\mathbb{S}^2 \cap B(x, r) = \emptyset$. As a consequence, we have

$$(11) \quad \text{supp}(u) \cap \{t \geq 0\} = \{(t, x) \in \mathbb{R}^+ \times \mathbb{R}^3, t - r \leq |x| \leq t + r\}.$$

Several remarks are in order. The fact that the solution u at time t vanishes in the ball $|x| \leq t - r$ corresponds to the strong Huygens principle; this is strongly related to the fact that the dimension 3 of $\mathbb{R}^3$ is odd, the metric is flat, and the wave operator has no lower-order term. A contrario, the fact that the support of the solution at time t is contained in the ball $|x| \leq t + r$ translates the finite speed of propagation, discussed in more details in Theorem 4 below. Finally, (11) also tells us that any point in the annulus $t - r \leq |x| \leq t + r$ is actually in the support of $u(t, \cdot)$. This new piece of information is very important for what follows. It implies in particular that unique continuation cannot hold across an hyperplane tangent to the cone $|x| = t + r$. We recall in this flat geometric setting the finite speed of propagation for waves.

Theorem 4 (Finite speed of propagation for the wave equation). –

Let u be a $C^2(\mathbb{R}^{1+d})$ (real-valued) solution of (9). If $u|_{t=0}(x) = \partial_t u|_{t=0}(x) = 0$ for $|x| \leq r_0$, then $u = 0$ in the cone

$$C_{r_0} = \{(t, x) \in \mathbb{R}^{1+d} \text{ s.t. } t \in [0, r_0] \text{ and } |x| \leq r_0 - t\}.$$

We can infer an interesting consequence of Theorem 4 concerning the unique continuation property for the wave operator: unique continuation holds across the hypersurface $\{t = 0\}$ and actually, we have some nice local linear quantification of the unique continuation. This situation is actually a particular case of a more general

situation in which the differential operator P (here $\partial_t^2 - \Delta$) is said to be hyperbolic with respect to the hypersurface S (here, e.g., $\{t = 0\}$). We refer, e.g., to [88] or [22] for more precisions and usual proofs of finite propagation speed (which is also a consequence of Theorem 17 below, see Section 2.1.3).

1.2.3. *Differential operators.* – Recall that a function f on $\mathbb{R}^n$ is said homogeneous of degree $m > 0$ if

$$f(\lambda\xi) = \lambda^m f(\xi), \quad \text{for all } \lambda > 0 \text{ and } \xi \in \mathbb{R}^n.$$

Definition 5 (Classical differential operators). – Let $\Omega \subset \mathbb{R}^n$ be an open set and $m \in \mathbb{N}$. We say that P is a (linear) differential operator of order m on Ω if there are coefficients $a_\alpha \in C^\infty(\Omega)$ having all derivatives bounded uniformly on Ω , such that $P = \sum_{|\alpha| \leq m} a_\alpha(x) D^\alpha$ with $m = \max\{|\alpha|, a_\alpha \neq 0\}$. We denote by $\text{Diff}^m(\Omega)$ the set of differential operators of order m on Ω . We say that the function $p_m(x, \xi) = \sum_{|\alpha|=m} a_\alpha(x) \xi^\alpha$ is the principal symbol of P . It is a *homogeneous* polynomial of degree m in the variable ξ .

Example 6. – If $\Omega \subset \mathbb{R}^n$ and $a, b \in C^\infty(\Omega; \mathbb{C})$ have all derivatives bounded uniformly on Ω , then, for $j \in \{1, \dots, n\}$, $a(x)D_j + b(x) \in \text{Diff}^1(\Omega)$ with principal symbol $p_1(x, \xi) = a(x)\xi_j$, and $-\Delta + a(x)D_j + b(x) \in \text{Diff}^2(\Omega)$ with principal symbol $p_2(x, \xi) = |\xi|^2$.

The augmented set $\Omega \times \mathbb{R}^n$, in which the principal symbol p_m lives, may be seen as a “phase space” containing both the position variable x and the Fourier/frequency/momentum variable $\xi \in \mathbb{R}^n$. The latter is to be understood as a cotangent variable $\xi \in T_x^*\Omega$, as we shall see below.

1.2.4. *A general local unique continuation result in the analytic category.* – The first general unique continuation result of the form (8) is the Holmgren-John Theorem, stating that, for operators with analytic coefficients, unique continuation holds across any noncharacteristic hypersurface S .

Definition 7. – Let P be a differential operator of order m on Ω , $x_0 \in \Omega$ and S a local hypersurface passing through x_0 , that is $S = \{\Psi = 0\}$, $\Psi(x_0) = 0$ and $d\Psi(x_0) \neq 0$ with $\Psi \in C^1(\Omega)$. We say that S is characteristic (resp. non-characteristic) for P at x_0 if $p_m(x_0, d\Psi(x_0)) = 0$ (resp. $p_m(x_0, d\Psi(x_0)) \neq 0$).

Also, given a local hypersurface $S = \{\Psi = 0\}$, it has locally two sides which we write

$$S^\pm = \{x \in \Omega; \pm\Psi(x) > 0\}.$$

Theorem 8 (Holmgren-John Theorem). – Let P be a differential operator of order m on Ω , having all coefficients real analytic in a neighborhood of $x_0 \in \Omega$ and let $S \ni x_0$ be a local hypersurface. Assume that S is noncharacteristic for P at x_0 . Then, there exists a neighborhood V of x_0 so that every $u \in \mathcal{D}'(\Omega)$ satisfying $Pu = 0$ on Ω and $u = 0$ in the set S^+ vanishes identically in V .

Another usual way of writing the conclusion is to say that $x_0 \notin \text{supp}(u)$ (it is slightly weaker though, since the neighborhood in which it vanishes may then depend on the solution u itself). We refer, e.g., to [29, Theorem 5.3.1] for a proof of Theorem 8. Note that this unique continuation property does not take into account the orientation of the hypersurface S , i.e., it holds from S^+ to S^- as well as from S^- to S^+ .

The non-characteristicity condition is very weak, and in some sense optimal. Indeed, we saw in Example 2 in Section 1.2.1 for linear vector-fields that unique continuation holds for non-characteristic hypersurfaces, and does not hold for some characteristic hypersurfaces. We also saw in Section 1.2.2 for the wave operator that local uniqueness does not hold across some hypersurfaces that are tangent to the cone $|x| = t + r$. These are precisely characteristic hypersurfaces: the principal symbol of the wave operator $\partial_t^2 - \Delta$ is given by $p_2(t, x, \xi_t, \xi_x) = -\xi_t^2 + |\xi_x|^2$, and a hypersurface $\{\Psi(t, x) = 0\}$ tangent to $\{|x| = t + r\}$ at the point (t_0, x_0) has $|\partial_t \Psi(t_0, x_0)| = |d_x \Psi(t_0, x_0)|$. Remark however that the non-characteristicity condition is a “first-order condition”: it only concerns the tangent space of the hypersurface. We saw in Example 3 in Section 1.2.1 in the case of first-order differential operators a more subtle “second-order condition” (curvature condition) on the hypersurface that can yield unique continuation across a characteristic hypersurface. This is linked to the so-called strong pseudoconvexity condition (see, e.g., Definition 16 and Remark 19 below).

We recall that a function $f : \Omega \subset \mathbb{R}^n \rightarrow \mathbb{C}$ is *real analytic* if for every $y \in \Omega$, there is a convergence radius $R > 0$ and coefficients $a_\alpha \in \mathbb{C}^n$, $\alpha \in \mathbb{N}^n$ such that

$$f(x) = \sum_{\alpha \in \mathbb{N}^n} a_\alpha (x - y)^\alpha = \sum_{\alpha_1, \dots, \alpha_n \in \mathbb{N}} a_\alpha (x_1 - y_1)^{\alpha_1} \cdots (x_n - y_n)^{\alpha_n},$$

for all $x \in B(y, R) \subset \Omega$, where the series is absolutely convergent. For every compact set $K \Subset \Omega \subset \mathbb{R}^n$, such a function f can be extended to a complex neighborhood of K in $\mathbb{C}^n$ as a complex analytic function. Analyticity is a very demanding regularity assumption. In Theorem 8, we stress that *all* the coefficients of P should have this regularity. In most situations, however, this requirement is much too strong. As an example, even for the wave equation on a flat (and hence analytic) metric, this theorem does not permit for the addition of a C^∞ time independent potential $V(x)$. This is a very strong drawback to the result. Therefore, we would like to avoid the analyticity assumption on the coefficients. This will lead to consider stronger geometric assumption, of convexity type (see, e.g., Definition 16 below) and will be the object of Chapter 2. Then Chapter 3 will deal with an intermediate case where the analyticity is assumed with respect to only one variable (we will actually treat the simpler case where it is *independent* of one variable).

1.2.5. The general Carleman strategy. – We consider here Ω a bounded open subset of $\mathbb{R}^n$, P a differential operator on Ω , $x_0 \in \Omega$ a point, and a hypersurface $S = \{\Psi = 0\}$ containing x_0 . We aim to prove local unique continuation for P across S (say, a statement like (8)). In particular, we want to prevent the situation in which a smooth function u both solves $Pu = 0$ and vanishes (possibly “flatly,” in the sense that all its

derivatives vanish) on S . We thus need to “emphasize” the local behavior of functions close to the hypersurface S .

The general idea of Carleman is to consider weighted estimates of the form

$$(12) \quad \|e^{\tau\Phi}w\|_{L^2(\Omega)} \leq C \|e^{\tau\Phi}Pw\|_{L^2(\Omega)},$$

that hold:

- for some well-chosen weight function $\Phi : \bar{\Omega} \rightarrow \mathbb{R}$ (related to Ψ as discussed below);
- for all $w \in C_c^\infty(\Omega)$ (related to u , solution to $Pu = 0$ as discussed below);
- and *uniformly* for τ sufficiently large, i.e., $\tau \geq \tau_0$.

To prove the relevance/efficiency of this approach, two different points need to be exposed:

1. What is the link between Carleman estimates as in (12) and unique continuation properties like (8)?
2. How does one prove such Carleman estimates?

Let us first discuss point 1. Note that (12) implies directly that if $w \in C_c^\infty(\Omega)$ is solution of $Pw = 0$ on $\{\Phi \geq 0\}$, then the right-hand side tends to zero as τ tends to infinity. Therefore, the left-hand side converges to zero, which implies (see Lemma 9 below) that w is supported in $\{\Phi \leq 0\}$.

However, statements as in (8) that are useful in applications are not concerned with functions w having compact support. Moreover, in general, as we shall see, usual differential operators P do not admit solutions w to $Pw = 0$ having compact support!

The heart of the Carleman method to pass from Estimate (12) to the unique continuation statement (8) resides in applying (12) to $w = \chi u$, where u is the function for which unique continuation has to be proved (hence solving $Pu = 0$ in Ω and $u = 0$ on $\Psi \geq 0$), and $\chi \in C_c^\infty(\Omega)$ is a cut-off function (to be chosen) allowing one to apply (12).

Using that $P\chi u = \chi Pu + [P, \chi]u = [P, \chi]u$ (where $[P, \chi]$ denotes the commutator of P and the multiplication operator by χ), one obtains

$$(13) \quad \|e^{\tau\Phi}\chi u\|_{L^2(\Omega)} \leq C \|e^{\tau\Phi}[P, \chi]u\|_{L^2(\Omega)}.$$

We then notice that $\text{supp}[P, \chi] \subset \text{supp} \nabla \chi$. If we now *assume* (this can be achieved if Φ is a slight convexification of Ψ , see Figure 5 for the geometric context of this argument), that the functions Ψ, Φ, χ are chosen such that $\text{supp}(\nabla \chi) \cap \{\Psi \leq 0\} \subset \{\Phi \leq -\eta\}$, for some $\eta > 0$, then the support property of u (namely $u = 0$ on $\Psi \geq 0$) implies that $\text{supp}([P, \chi]u) \subset \{\Phi \leq -\eta\}$. We thus obtain

$$(14) \quad \|e^{\tau\Phi}\chi u\|_{L^2(\Omega)} \leq C_u e^{-\eta\tau}, \quad \text{for all } \tau \geq \tau_0.$$

The following lemma then implies that χu vanishes identically in $\{\Phi \geq -\eta\}$ which contains a neighborhood of the point x_0 .

Lemma 9. – Assume $w \in L^2(\Omega)$ satisfies $\|e^{\tau\Phi}w\|_{L^2(\Omega)} \leq C e^{-\eta\tau}$ for all $\tau \geq \tau_0$. Then we have $w = 0$ a.e. on $\{\Phi \geq -\eta\}$.

The proof of the lemma reduces first to the case $\eta = 0$ by changing Φ in $\Phi + \eta$. Then, it suffices to notice that if w does not vanish a.e. on $\{\Phi > 0\}$, there are $\varepsilon > 0$ and a compact set $E \subset \{\Phi > 0\}$ of positive measure such that $|w| \geq \varepsilon > 0$ a.e. on E . This yields

$$C^2 \geq \|e^{\tau\Phi} w\|_{L^2(\Omega)}^2 \geq \int_E e^{2\tau\Phi} |w|^2 \geq \varepsilon^2 \int_E e^{2\tau \min_E \Phi} = \varepsilon^2 |E| e^{2\tau \min_E \Phi} \xrightarrow{\tau \rightarrow +\infty} +\infty,$$

and hence a contradiction.

To conclude, this brief discussion of point 1 suggests that unique continuation (8) holds across $\{\Psi = 0\}$ provided the Carleman estimate (12) is true for some weight function Φ satisfying an appropriate *geometric convexity condition*.

As stated in point 2, the other issue is how to prove Carleman estimates, and, in particular, understand the conditions on Φ for which (12) can hold. As far as this analysis is concerned, the exponential weight is not convenient to work with. One might thus want to eliminate it by setting $v = e^{\tau\Phi} w$. Then (12) is equivalent to $\|v\|_{L^2(\Omega)} \leq C \|P_\Phi v\|_{L^2(\Omega)}$, where $P_\Phi = e^{\tau\Phi} P e^{-\tau\Phi}$ is the so-called conjugated operator. Note that again here, we slightly abuse notation and make the confusion between the function $e^{\tau\Phi}$ and the operator of multiplication by $e^{\tau\Phi}$. We are thus left to prove a lower bound for the operator P_Φ .

Writing $\partial_j(e^{-\tau\Phi} u) = e^{-\tau\Phi}(\partial_j u - \tau u \partial_j \Phi)$ implies that

$$(15) \quad e^{\tau\Phi} D_j e^{-\tau\Phi} = D_j + i\tau \partial_j \Phi.$$

The first effect of conjugation is that there is no exponential factor in the right-hand side, which is much more convenient. Second, the conjugation changes D_j into an operator having one derivative and one exponent of τ . We thus expect (and we will check) that for general differential operators $P = \sum_\alpha a_\alpha(x) D^\alpha$, the associated conjugated operator P_Φ will have as many derivatives as exponents of τ . Since we want to obtain estimates that are uniform for large τ , we have to think of τ as having the same weight as a derivative. We describe this calculus in the next section.

1.3. Operators depending on a large parameter τ . – In this section, we describe the setting in which Carleman estimates like (12) shall be proved (see Sections 2 and 3 below).

1.3.1. Differential operators depending on a large parameter τ . – We discuss the calculus for differential operators depending on a large parameter τ . One may think to τ as having the same weight as a derivative, i.e., as the Fourier variable ξ . Since τ is meant to be large, we will always assume $\tau \geq 1$ when dealing with estimates uniform in τ . We first define the H_τ^s norm of a function $u \in \mathcal{S}(\mathbb{R}^n)$ as

$$\|u\|_{H_\tau^s} = \|(|D|^2 + \tau^2)^{\frac{s}{2}} u\|_{L^2(\mathbb{R}^n)} = (2\pi)^{-n/2} \|(|\xi|^2 + \tau^2)^{\frac{s}{2}} \widehat{u}\|_{L^2(\mathbb{R}^n)}.$$

Note that for fixed τ , this norm is equivalent to the usual H^s norm, since, for $\tau \geq 1$, we have

$$|\xi|^2 + 1 \leq |\xi|^2 + \tau^2 \leq \tau^2(|\xi|^2 + 1).$$

That is to say that $\|u\|_{H^s(\mathbb{R}^n)} \leq \|u\|_{H_\tau^s} \leq \tau^s \|u\|_{H^s(\mathbb{R}^n)}$ for all $\tau \geq 1$. This is however not uniform as $\tau \rightarrow +\infty$. Note also that, as for usual Sobolev spaces, the definition of the H_τ^s norm has a uniformly equivalent definition in case $s = k \in \mathbb{N}$.

Lemma 10. – *Let $k \in \mathbb{N}$. Then, there is $C > 1$ such that for all $\tau \geq 1$ and all $u \in H^k(\mathbb{R}^n)$, we have*

$$C^{-1} \|u\|_{H_\tau^k(\mathbb{R}^n)}^2 \leq \sum_{|\alpha|+\beta \leq k} \tau^{2\beta} \|\partial^\alpha u\|_{L^2(\mathbb{R}^n)}^2 \leq C \|u\|_{H_\tau^k(\mathbb{R}^n)}^2.$$

In particular, we often use the case $k = 1$ for which

$$\|u\|_{H_\tau^1} \approx \|u\|_{H^1} + \tau \|u\|_{L^2}, \quad \text{uniformly for } \tau \geq 1.$$

Definition 11 (Differential operators depending on τ). – Let $m \in \mathbb{N}$ and $\Omega \subset \mathbb{R}^n$ be an open set. We denote $\text{Diff}_\tau^m(\Omega)$ the set of differential operators of the form $P = \sum_{|\alpha|+\beta \leq m} p_{\alpha,\beta}(x) \tau^\beta D^\alpha$ with $p_{\alpha,\beta} \in C^\infty(\Omega)$ such that all derivatives of $p_{\alpha,\beta}$ are bounded uniformly on Ω . We say that $p_m(x, \xi, \tau) = \sum_{|\alpha|+\beta=m} p_{\alpha,\beta}(x) \tau^\beta \xi^\alpha$ is its principal symbol (in the class $\text{Diff}_\tau^m(\Omega)$). It is homogeneous of degree m in (ξ, τ) , in the sense that

$$p_m(x, \lambda \xi, \lambda \tau) = \lambda^m p_m(x, \xi, \tau), \quad \text{for all } x \in \Omega, \xi \in \mathbb{R}^n, \tau \geq 0 \text{ and } \lambda > 0.$$

Recall that, the order m being fixed, smooth homogeneous functions of degree m in this sense identify (through a restriction map) to smooth functions on the half-sphere bundle over Ω , namely

$$\{(x, \xi, \tau) \in \Omega \times \mathbb{R}^n \times \mathbb{R}^+, |\xi|^2 + \tau^2 = 1\}.$$

Remark that Definition 11 is almost the same definition as Definition 5, except for the dependence on τ which changes the definition of the principal symbol.

Now, we describe the calculus of differential operators with a large parameter. This consists in explaining the properties of such operators with respect to usual operations (composition, commutators, taking the adjoint), their mapping properties (in τ dependent Sobolev spaces) and positivity properties. Moreover, we link such properties with those of the symbol of the operators. The philosophy is that we recover certain properties of the operators only from their principal symbols (which are simpler objects to manipulate, namely functions on the augmented space $\Omega_x \times \mathbb{R}_\xi^n \times \mathbb{R}_\tau^+$). The general Heuristic is that these differential operators act as if they were multiplication by $p_m(x, \xi, \tau)$, modulo lower-order terms. A rough summary of the calculus properties of operators of Diff_τ^m is as follows.

Proposition 12 (Symbolic calculus for differential operators). – *Let $\Omega \subset \mathbb{R}^n$ be an open set, $m, m_1, m_2 \in \mathbb{N}$ and $P \in \text{Diff}_\tau^m(\Omega)$, $A \in \text{Diff}_\tau^{m_1}(\Omega)$, $B \in \text{Diff}_\tau^{m_2}(\Omega)$ having respective principal symbol p, a, b , then*

- (1) (composition) $AB = A \circ B \in \text{Diff}_\tau^{m_1+m_2}(\Omega)$ with principal symbol ab ;

- (2) (commutators) $[A, B] = AB - BA \in \text{Diff}_\tau^{m_1+m_2-1}(\Omega)$ is of order $m_1 + m_2 - 1$ with principal symbol $\frac{1}{i}\{a, b\}$, where

$$(16) \quad \{a, b\} := \partial_\xi a \cdot \partial_x b - \partial_x a \cdot \partial_\xi b = \sum_{j=1}^n (\partial_{\xi_j} a \partial_{x_j} b - \partial_{x_j} a \partial_{\xi_j} b),$$

is the Poisson bracket.

- (3) (adjoint) P^* , the formal adjoint in L^2 (tested with functions in $C_c^\infty(\Omega)$) belongs to $\text{Diff}_\tau^m(\Omega)$ with principal symbol $\bar{p}$;
- (4) (action on Sobolev spaces) if $\Omega = \mathbb{R}^n$, P maps continuously H_τ^s into H_τ^{s-m} for all $s \in \mathbb{R}$, uniformly for $\tau \geq 1$.

Proofs are elementary as operators in Diff_τ^m are linear combinations of monomials $\tau^\beta a(x) D^\alpha$, $\beta \in \mathbb{N}, \alpha \in \mathbb{N}^n$ for which the properties can be checked by hand. We refer, e.g., to [53] for direct elementary proofs, and to [30] or [57] for the whole machinery of pseudodifferential calculus (with a large parameter).

1.3.2. *The conjugated operator.* – As described in Section 1.2.5, the introduction of the calculus with the large parameter τ is motivated by the conjugated operator $P_\Phi := e^{\tau\Phi} P e^{-\tau\Phi}$. We prove here that it belongs to the class Diff_τ^m , and compute its principal symbol.

Lemma 13 (Conjugated operator). – Let $P = \sum_{|\alpha| \leq m} p_\alpha(x) D^\alpha \in \text{Diff}^m(\Omega)$ be a (classical) differential operator with principal symbol p_m and let $\Phi \in C^\infty(\Omega)$ be real-valued and bounded as well as all its derivatives. Then, the operator P_Φ defined by $P_\Phi v = e^{\tau\Phi} P(e^{-\tau\Phi} v)$ satisfies $P_\Phi \in \text{Diff}_\tau^m(\Omega)$, and its principal symbol, denoted by $p_\Phi = p_{\Phi, m}$ (with a slight abuse of notation), is given by

$$p_\Phi(x, \xi, \tau) = p_m(x, \xi + i\tau d\Phi(x)) = \sum_{|\alpha|=m} p_\alpha(x) (\xi + i\tau d\Phi(x))^\alpha.$$

Roughly speaking, Lemma 13 says that p_Φ is obtained by replacing ξ by $\xi + i\tau d\Phi(x)$ in p_m . Note that it implies in particular that p_Φ has a complex-valued symbol if p_m is real-valued: conjugation turns selfadjoint operators into non-selfadjoint ones.

Proof. – As already checked in (15), we have $e^{\tau\Phi} D_j(e^{-\tau\Phi} u) = D_j u + i\tau(\partial_j \Phi)u$. In particular, the conjugated operator $e^{\tau\Phi} D_j e^{-\tau\Phi}$ lies in the class Diff_τ^1 with principal symbol $\xi_j + i\tau \partial_j \Phi$. We now write

$$e^{\tau\Phi} D_j^{\alpha_j} e^{-\tau\Phi} = (e^{\tau\Phi} D_j e^{-\tau\Phi})(e^{\tau\Phi} D_j e^{-\tau\Phi}) \cdots (e^{\tau\Phi} D_j e^{-\tau\Phi}) \quad (\alpha_j \text{ times}).$$

Therefore, using Proposition 12 $\alpha_j - 1$ times, we obtain that this is a differential operator depending on τ of order α_j with principal symbol $(\xi_j + i\tau \partial_j \Phi)^{\alpha_j}$. Since $D^\alpha = D_1^{\alpha_1} \cdots D_j^{\alpha_j} \cdots D_n^{\alpha_n}$, we obtain similarly that $e^{\tau\Phi} D^\alpha e^{-\tau\Phi} \in \text{Diff}_\tau^{|\alpha|}$, with principal symbol

$$\prod_{j=1}^n (\xi_j + i\tau \partial_j \Phi)^{\alpha_j} = (\xi + i\tau d\Phi)^\alpha.$$

Since p_α commutes with $e^{\tau\Phi}$ and $P = \sum_\alpha p_\alpha(x)D^\alpha$, this proves the sought result. $\square$

Example 14 (Second-order operators with real-valued principal symbol). –

In these notes, we are particularly interested in $P \in \text{Diff}^2(\Omega)$ with p_2 real-valued. The principal symbol of such operators reads $p_2(x, \xi) = \sum_{i,j=1}^n a^{ij}(x) \xi_i \xi_j$ with real coefficients a^{ij} . This encompasses the case of the Laplace operator and the wave operator. Notice that $\xi \mapsto p_2(x, \xi) = \sum_{i,j=1}^n a^{ij}(x) \xi_i \xi_j$ is a real quadratic form for all $x \in \Omega$. In particular, we have the canonical polar form:

$$p_2(x, \xi) = \sum_{i,j=1}^n a^{ij}(x) \xi_i \xi_j = \sum_{i,j=1}^n \frac{1}{2} (a^{ij}(x) + a^{ji}(x)) \xi_i \xi_j,$$

and we can thus assume that

$$(17) \quad \text{the matrix } (a^{ij}(x))_{i,j} \text{ is symmetric, i.e., } a^{ij}(x) = a^{ji}(x) \text{ for all } 1 \leq i, j \leq n.$$

Concerning the operator P , we then have

$$P - p_2(x, D) \in \text{Diff}^1(\Omega), \quad p_2(x, D) = \sum_{i,j=1}^n a^{ij}(x) D_i D_j = \sum_{i,j=1}^n D_i a^{ij}(x) D_j + R_1,$$

where $R_1 = -\sum_{i,j=1}^n D_i(a^{ij})D_j \in \text{Diff}^1(\Omega)$. The operator $\sum_{i,j=1}^n D_i a^{ij}(x) D_j$ is formally (i.e., tested with functions in $C_c^\infty(\Omega)$) selfadjoint in $L^2(\Omega, dx)$ (equivalently, one says that it is of divergence form with respect to the measure dx). This last form thus states in a clearer way that the operator is formally self-adjoint modulo $\text{Diff}^1(\Omega)$. Also, $\xi \mapsto p_2(x, \xi) = \sum_{i,j=1}^n a^{ij}(x) \xi_i \xi_j$ is a real quadratic form with (17), and thus Lemma 13 states that the principal symbol of the associated conjugated operator P_Φ is given by

$$p_\Phi(x, \xi, \tau) = p_2(x, \xi + i\tau d\Phi(x)) = p_2(x, \xi) - \tau^2 p_2(x, d\Phi(x)) + 2i\tau \tilde{p}_2(x, \xi, d\Phi(x)),$$

where $\tilde{p}_2(x, \xi, \eta) = \sum_{i,j=1}^n a^{ij}(x) \xi_i \eta_j$ is the polar bilinear form of the quadratic form $p_2(x, \xi)$.

1.3.3. A Gårding inequality for a class of operators with a large parameter. – In this section, we prove that operators having a real positive principal symbol are positive (referred to as a Gårding inequality). However, for the need of Carleman estimates, the class of differential operators is not quite sufficient. We need to consider a slightly larger class, that also includes the operator

$$(-\Delta + \tau^2)^{-1} = (|D|^2 + \tau^2)^{-1}, \quad \tau \geq 1,$$

defined as a Fourier multiplier:

$$\mathcal{F}((-\Delta + \tau^2)^{-1}u)(\xi) = (|\xi|^2 + \tau^2)^{-1}\hat{u}(\xi), \quad u \in \mathcal{S}(\mathbb{R}^n).$$

Note that, as opposed to differential operators, the operator $(-\Delta + \tau^2)^{-1}$ is non-local (in the sense that it does not satisfy $\text{supp}(Pu) \subset \text{supp}(u)$ for all $u \in C_c^\infty(\mathbb{R}^n)$). We write in this section a weak form of Gårding estimates for (almost-)differential operators of order 2, which is at the core of the Carleman method.

Proposition 15 (A local Gårding inequality for particular operators). –

Assume Ω is an open set with $0 \in \Omega$ and let P be an operator of the form

$$(18) \quad P = A + \sum_{i=1}^k B_i \circ (-\Delta + \tau^2)^{-1} \circ B_i$$

with $A, B_i \in \text{Diff}_\tau^2(\Omega)$ with real principal symbols $a_2(x, \xi, \tau)$ and $b_{2,i}(x, \xi, \tau)$ for some $k \in \mathbb{N}$. Define

$$(19) \quad p_2(x, \xi, \tau) = a_2(x, \xi, \tau) + \sum_{i=1}^k \frac{b_{2,i}^2(x, \xi, \tau)}{|\xi|^2 + \tau^2},$$

and assume that there is $C_0 > 0$ such that

$$(20) \quad p_2(0, \xi, \tau) \geq C_0(|\xi|^2 + \tau^2), \quad \text{for all } \xi \in \mathbb{R}^n, \tau \geq 0.$$

Then, there exist $r, C, \tau_0 > 0$ such that

$$\text{Re}(Pu, u)_{L^2} \geq C \|u\|_{H_\tau^1}^2, \quad \text{for all } u \in C_c^\infty(B(0, r)), \tau \geq \tau_0.$$

Note that formally, such operators P are “of order 2”. The “principal symbol,” defined in (19) is indeed a homogeneous function of degree 2. Inequality (20) is thus a homogeneous inequality, and it is sufficient to assume it on the half-sphere $\mathbb{S}_+^n := \{(\xi, \tau) \in \mathbb{R}^n \times \mathbb{R}_+, |\xi|^2 + \tau^2 = 1\}$. We refer, e.g., to [53] for a proof relying on freezing coefficients, together with the fact that the result is elementary for the Fourier multiplier $p_2(0, D, \tau)$.

2. Unique continuation under a convexity condition

This section is devoted to a classical unique continuation result under a pseudoconvexity condition, due to Hörmander [29].

2.1. Statement and examples

2.1.1. *Statement of Hörmander’s theorem.* – Given $a, b \in C^\infty(\Omega \times \mathbb{R}^n; \mathbb{C})$, the Poisson bracket $\{a, b\}$ is defined in (16) and its geometric content is recalled in Section 2.1.2 below. The geometric definition entering into the game is the following.

Definition 16 (Strongly pseudoconvex hypersurface for operators of order two with real principal symbols). –

Let $\Omega \ni x_0$ be an open set, $P \in \text{Diff}^2(\Omega)$ with real-valued principal symbol p_2 and $\Psi \in C^\infty(\Omega)$ real-valued. We say that the oriented hypersurface $S = \{\Psi = \Psi(x_0)\}$ is strongly pseudoconvex with respect to P at x_0 if it satisfies

$$(21) \quad p_2(x_0, \xi) = \{p_2, \Psi\}(x_0, \xi) = 0 \implies \{p_2, \{p_2, \Psi\}\}(x_0, \xi) > 0 \quad \text{for all } \xi \in \mathbb{R}^n \setminus \{0\}.$$

Note that $\{p, \Psi\}(x_0, \xi) = \partial_\xi p(x_0, \xi) \cdot \partial_x \Psi(x_0)$. One can check that Definition 16 is invariant if we change the defining function Ψ of S (see, e.g., Lemma 35 and [53]). That is why this is a geometric property of the oriented hypersurface solely. See Section 2.1.2 for an interpretation as convexity with respect to bicharacteristic curves. The geometric condition (21) has to be compared with that discussed for vector fields in Examples 2-3 in Section 1.2.1, see Remark 19 below. The following result is a particular case of the Hörmander theorem.

Theorem 17 (Unique continuation for real operators of order 2 under strong pseudoconvexity). –

Let $\Omega \ni x_0$ be an open set of $\mathbb{R}^n$, and let $P \in \text{Diff}^2(\Omega)$ with real principal symbol p_2 . Assume that the oriented hypersurface $S = \{\Psi = \Psi(x_0)\}$ is strongly pseudoconvex with respect to P at x_0 . Then, there exists a neighborhood V of x_0 so that for all $u \in H^1(\Omega)$, we have

$$(22) \quad \left\{ \begin{array}{l} Pu = 0 \text{ in } \Omega, \\ u = 0 \text{ in } \Omega \cap \{\Psi > \Psi(x_0)\} \end{array} \right\} \implies u = 0 \text{ in } V.$$

Another usual way of writing the conclusion is to say that $x_0 \notin \text{supp}(u)$ (it is slightly weaker though, since the neighborhood in which it vanishes may then depend on the solution u itself). Here, we have assumed that all coefficients of P are smooth for simplicity.

Remark 18 (Elliptic case). – Note now that in the particular case where the operator P is elliptic at x_0 , i.e., $p_2(x_0, \xi) \geq c|\xi|^2$, then the condition $p_2(x_0, \xi) = 0$ is never fulfilled when $\xi \neq 0$ and the left-hand side of (21) is empty. Local unique continuation thus holds across *any* hypersurface.

Theorem 17 will be proved in Section 2.3.3. Before this, let us describe the underlying geometry of the condition (21) together with typical examples of application of this result.

2.1.2. *Geometric content of the strong pseudoconvexity condition (21).*– Introduce the Hamiltonian flow of the symbol p_2 . Recall that $\{p_2, \cdot\}$ (defined in (16)) is a derivation on $C^\infty(\Omega \times \mathbb{R}^n)$ and can thus be identified with the vector field

$$(23) \quad H_{p_2}(x, \xi) = \partial_\xi p_2(x, \xi) \cdot \partial_x - \partial_x p_2(x, \xi) \cdot \partial_\xi = \sum_{j=1}^n \partial_{\xi_j} p_2(x, \xi) \frac{\partial}{\partial x_j} - \partial_{x_j} p_2(x, \xi) \frac{\partial}{\partial \xi_j}$$

on $\Omega \times \mathbb{R}^n$. We denote by χ_s the associated flow, defined by

$$(24) \quad \begin{cases} \frac{d}{ds} \chi_s(x_0, \xi_0) = H_{p_2}(\chi_s(x_0, \xi_0)), \\ \chi_0(x_0, \xi_0) = (x_0, \xi_0), \end{cases}$$

and called the Hamiltonian flow of p_2 . Remark that $H_{p_2}(p_2) = \{p_2, p_2\} = 0$ so that p_2 is preserved along the flow: $p_2 \circ \chi_s = p_2$. Note also that the flow χ_s is (at

least) locally defined for (s, x, ξ) in a neighborhood of $(0, x_0, \xi_0)$ according to the Cauchy-Lipschitz theorem.

If we now denote by $(x_s, \xi_s) = \chi_s$, that is $\chi_s(x_0, \xi_0) = (x_s(x_0, \xi_0), \xi_s(x_0, \xi_0))$ with (23), (24) now reads

$$(25) \quad \begin{cases} \frac{d}{ds} x_s(x_0, \xi_0) = \partial_\xi p_2(\chi_s(x_0, \xi_0)), \\ \frac{d}{ds} \xi_s(x_0, \xi_0) = -\partial_x p_2(\chi_s(x_0, \xi_0)), \\ (x_s(x_0, \xi_0), \xi_s(x_0, \xi_0))|_{s=0} = (x_0, \xi_0). \end{cases}$$

With these definitions in hand, we can now reformulate the strong pseudoconvexity condition of Definition 16. Namely, note that we have

$$\begin{aligned} \{p_2, \Psi\}(x_0, \xi) &= H_{p_2}(\Psi)(x_0, \xi) = \frac{d}{ds} \Psi \circ x_s(x_0, \xi)|_{s=0}, \\ \{p_2, \{p_2, \Psi\}\}(x_0, \xi) &= H_{p_2}(H_{p_2}(\Psi))(x_0, \xi) = \frac{d^2}{ds^2} \Psi \circ x_s(x_0, \xi)|_{s=0}. \end{aligned}$$

Now, if for $\xi \in \mathbb{R}^n$ we define $c_\xi(s) = \Psi \circ x_s(x_0, \xi)$, then (21) is equivalent to:

$$\text{For all } \xi \in \mathbb{R}^n \setminus \{0\}, \text{ we have: } p_2(x_0, \xi) = 0 \text{ and } \dot{c}_\xi(0) = 0 \implies \ddot{c}_\xi(0) > 0.$$

Suppose $\xi \in \mathbb{R}^n \setminus \{0\}$,

- if ξ is noncharacteristic ($p_2(x_0, \xi) \neq 0$), this point is not considered in (21);
- if (ξ is characteristic and) the (projected) Hamiltonian curve $x_s(x_0, \xi)$ is not tangent to $S = \{\Psi = \Psi(x_0)\}$ at $s = 0$, this point is not considered in (21), see Figure 2 left;
- if ξ is characteristic and the curve $x_s(x_0, \xi)$ is tangent to $S = \{\Psi = \Psi(x_0)\}$, then it should have non-vanishing second derivative (tangency at order 2) and the curve $(x_s(x_0, \xi))_{s \in (-\varepsilon, \varepsilon)}$ should stay in $\{\Psi \geq \Psi(x_0)\}$ (for $\varepsilon > 0$ small enough), see Figure 2 right.

This excludes the situations where tangent characteristic curves stay in $S^- = \{\Psi \leq \Psi(x_0)\}$ (see Figure 3 left) or the case of contacts of higher order with the tangent space of S at x_0 (see Figure 3 right).

Finally, we stress the fact that for the wave operator, $x_s(x_0, \xi)$ denote *time-space* curves.

Remark 19. – Let us now briefly shed another light on this strong pseudoconvexity condition in view of Examples 2 and 3 of Section 1.2.1. Assume $\mathcal{X} = \sum_{j=1}^n \mathcal{X}_j(x) \frac{\partial}{\partial x_j}$ is a non-singular real vectorfield at x_0 , i.e.,

$$\mathcal{X}(x_0) = (\mathcal{X}_1, \dots, \mathcal{X}_n)(x_0) \neq 0,$$

and let $P = \frac{1}{i} \mathcal{X}$, that is $P = \sum_{j=1}^n \mathcal{X}_j(x) D_{x_j}$.

Then $P \in \text{Diff}^1$ with real principal symbol $p_1(x, \xi) = \xi \cdot \mathcal{X}(x)$ (where $\cdot$ denotes the duality bracket between $T^*\mathbb{R}^n$ and $T\mathbb{R}^n$) and the vector field H_{p_1} in (23) is

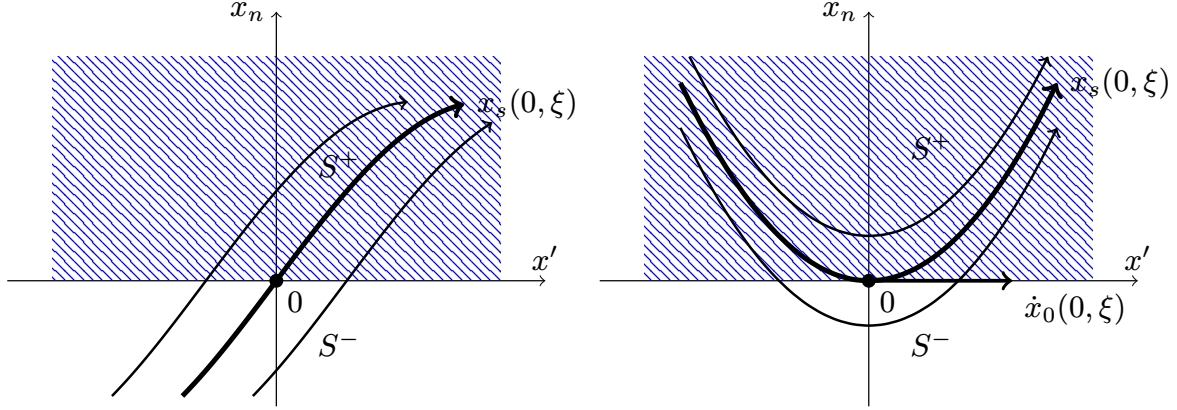


FIGURE 2. Picture with $x_0 = 0$ and $\Psi(x', x_n) = x_n$, hence $S = \{x_n = 0\}$ (which can always be assumed when straightening the oriented surface). Here $u = 0$ on the blue striped zone $S^+ = \{x_n > 0\}$. Each picture illustrates the (projection of) hamiltonian curve $x_s(0, \xi)$ for a given ξ . We also draw nearby (projections of) hamiltonian curves. If null bicharacteristic curves belong to one of these two cases for all $\xi \in \mathbb{R}^n \setminus \{0\}$, then Assumption (21) is satisfied, and Theorem 17 yields unique continuation from the blue striped region to a neighborhood of zero.

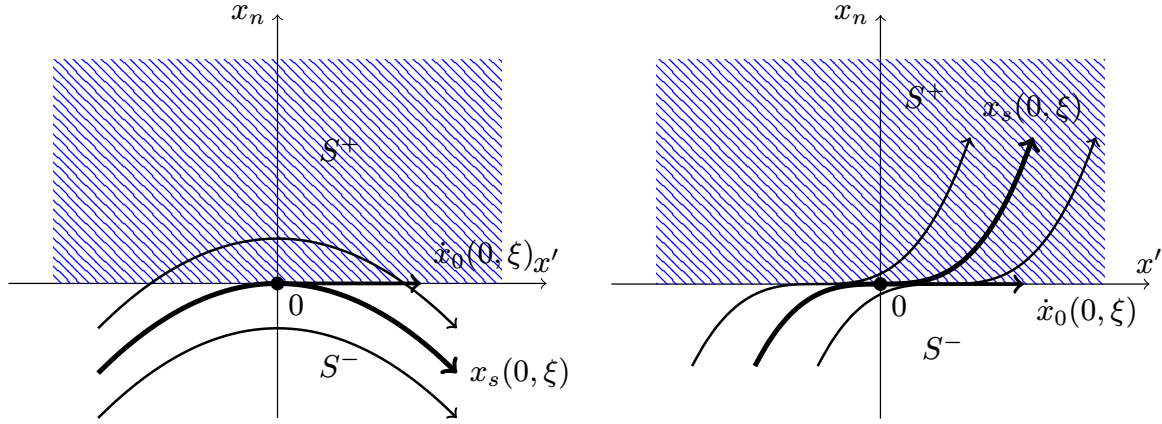


FIGURE 3. Picture with $\Psi(x', x_n) = x_n$, and $S = \{x_n = 0\}$. Again $u = 0$ on the blue striped zone $S^+ = \{x_n > 0\}$. Each picture illustrates the (projection of) hamiltonian curve $x_s(0, \xi)$ for a given ξ . We also draw nearby (projections of) hamiltonian curves. Left: a tangent bicharacteristic curve stays in $S^- = \{\Psi < 0\}$. Right: contact of higher order with the tangent space of S at 0. In both cases, (21) is not satisfied.

$H_{p_1} = \mathcal{X} + \mathcal{V}$ with $\mathcal{V} = -\sum_{j=1}^n \partial_{x_j}(\xi \cdot \mathcal{X}(x)) \partial_{\xi_j}$. The strong pseudoconvexity condition (21) reads

$$\xi \cdot \mathcal{X}(x_0) = \mathcal{X}(\Psi)(x_0) = 0 \implies \mathcal{X}^2(\Psi)(x_0) = \mathcal{X}(\mathcal{X}(\Psi))(x_0) > 0 \quad \text{for all } \xi \in \mathbb{R}^n \setminus \{0\},$$

itself equivalent to the (ξ -independent) condition

$$(26) \quad \mathcal{X}(\Psi)(x_0) = 0 \implies \mathcal{X}^2(\Psi)(x_0) > 0.$$

Hence, for non-singular real vectorfields, the strong pseudoconvexity condition is precisely that described (in two dimensions only) in Examples 2 and 3. Although Theorem 17 does not apply per se for first order operators, Condition (26) can be proven to imply local unique continuation directly in the x -space, exactly as in Examples 2 and 3. Note that perturbation of the vectorfield by a zero order term does not modify the unique continuation result. Note finally that in this context, Figure 1 (where the vector field is straightened) corresponds exactly to Figure 2 (where the surface is straightened).

Remark 20. – Note also that, as a consequence of the above discussion, the fact that $\{\Psi = 0\}$ satisfies the strong pseudoconvexity condition at x_0 implies in particular that the set $\{\Psi > 0\}$ “controls geometrically” a whole neighborhood of x_0 , in the sense of [74, 9]: any null bicharacteristic curve of P passing close enough to x_0 intersects $\{\Psi > 0\}$. In particular if P has smooth coefficients in Ω and $u \in \mathcal{D}'(\Omega)$ satisfies $Pu = 0$ in $\mathcal{D}'(\Omega)$ and $u = 0$ in $\{\Psi > 0\}$, then propagation of singularities [30, Theorem 23.2.9] together with the fact that $\{\Psi > 0\}$ satisfies this local form of geometric control condition imply that $u \in C^\infty$ in a neighborhood of x_0 , and Theorem 17 applies.

2.1.3. Examples

Example 21 (Operators with constant coefficients). – Consider here the simple case where $P = AD \cdot D$ where A is a constant real symmetric matrix (for instance $A = \text{diag}(-1, 1, \dots, 1)$ for the flat/Minkowski wave operator). Then $p_2(x, \xi) = A\xi \cdot \xi$. We have

$$\{p_2, \Psi\}(x, \xi) = 2A\xi \cdot \nabla \Psi(x) = d\Psi(x)(2A\xi)$$

$$\text{and } \{p_2, \{p_2, \Psi\}\}(x, \xi) = \{A\xi \cdot \xi, 2A\xi \cdot \nabla \Psi(x)\} = 4 \text{Hess } \Psi(x)(A\xi, A\xi).$$

Condition (21) reads

$$A\xi \cdot \xi = 0 \quad \text{and} \quad d\Psi(x_0)(A\xi) = 0 \implies \text{Hess } \Psi(x_0)(A\xi, A\xi) > 0 \quad \text{for all } \xi \in \mathbb{R}^n \setminus \{0\}.$$

Notice that, in case A is invertible, this cotangent formulation can be equivalently replaced by the following tangent one:

$$(27) \quad A^{-1}X \cdot X = 0 \quad \text{and} \quad d\Psi(x_0)(X) = 0 \implies \text{Hess } \Psi(x_0)(X, X) > 0 \quad \text{for all } X \in \mathbb{R}^n \setminus \{0\}.$$

Example 22 (Pseudo-Riemannian metric and operator). – As in Example 21, in the general case of operators $P = D_i a^{ij}(x) D_j$ (plus lower-order terms) with variable coefficients but nondegenerate cometric $a^{ij}(x_0)$, one may want to rephrase the cotangent formulation of strong pseudoconvexity (21) on the tangent space at x_0 . Note that this covers the wave operator in case the signature of $a^{ij}(x_0)$ is $(-1, 1, \dots, 1)$ (i.e., the metric is Lorentzian). Introducing the pre-dual metric $a(x) = a_{ij}(x) = (a^{ij}(x))^{-1}$

and setting $X = a(x)^{-1}\xi$, i.e., $X^i = a^{ij}\xi_j$ (where the Einstein summation convention is used), one can check that $\{p_2, \Psi\}(x_0, \xi) = 2d\Psi(x_0)(X) = 2(\nabla_a \Psi, X)_a$, where ∇_a is the usual pseudo-Riemannian gradient and $(Y, X)_a = a_{ij}(x)Y^i X^j$. Moreover, one can also check that $\{p_2, \{p_2, \Psi\}\}(x_0, \xi) = 4\text{Hess}_a \Psi(x_0)(X, X)$, where $\text{Hess}_a \Psi(X, Y) = (D_X d\Psi)(Y)$ is the usual pseudo-Riemannian Hessian of Ψ (and D denotes the Levi-Civita connection associated to the metric a , see, e.g., [25, Chapter II Section B]). Condition (21) reads (recall that in this expression x denotes the full time-space variable).

(28)

$$(X, X)_a = 0 \text{ and } d\Psi(x_0)(X) = 0 \implies \text{Hess}_a \Psi(x_0)(X, X) > 0 \text{ for all } X \in T_{x_0}\Omega \setminus \{0\}.$$

Example 23 (The flat/Minkowski wave operator). – We discuss here the case of the wave operator with constant coefficients, which is a particular case of the above examples with $A = \text{diag}(-1, 1, \dots, 1)$, $P = \partial_t^2 - \Delta$, $p_2 = -\xi_t^2 + |\xi_x|^2$. We may now write the strong pseudoconvexity condition (21) specialized in the point $(t, x) = (t_0, x_0) =: \mathbf{x}_0$, in different situations.

- Spacelike hypersurface: if $|\partial_t \Psi(\mathbf{x}_0)| > |\nabla_x \Psi(\mathbf{x}_0)|$: the hypersurface $\{\Psi = \Psi(\mathbf{x}_0)\}$ is called spacelike (its normal vector $\nabla_{t,x} \Psi$ is timelike). The first two conditions imply

$$|\xi_t \partial_t \Psi(\mathbf{x}_0)| = |\xi_x \cdot \partial_x \Psi(\mathbf{x}_0)| \leq |\xi_x| |\partial_x \Psi(\mathbf{x}_0)| = |\xi_t| |\partial_x \Psi(\mathbf{x}_0)| < |\xi_t| |\partial_t \Psi(\mathbf{x}_0)|.$$

This is a contradiction, and hence the left-hand side of Condition (21) is empty.

Any spacelike hypersurface satisfies the unique continuation property. This is natural since the Cauchy problem is hyperbolic and thus locally wellposed for any spacelike hypersurface (like for instance the wave equation posed with initial data at $t = 0$, see Theorem 4).

- Time-invariant hypersurface: in the applications discussed in Section 1.1, a typical unique continuation result needed is rather across hypersurfaces of the form $\Psi(t, x) = \varphi(x)$. The strong pseudoconvexity condition at the point $(t, x) = (t_0, x_0)$ then reads

$$(29) \quad \xi_t^2 = |\xi_x|^2 \text{ and } \xi_x \cdot \nabla_x \varphi(x_0) = 0 \implies \text{Hess}_x(\varphi)(x_0)(\xi_x, \xi_x) > 0 \quad \forall \xi \in \mathbb{R}^n \setminus \{0\}.$$

Typically, if $\Psi(t, x) = \varphi(x) = |x|^2$, the condition (29) holds at any point $\mathbf{x}_0 = (t_0, x_0)$ since $\text{Hess}_x(\varphi)(x_0)(\xi_x, \xi_x) = 2|\xi_x|^2 > 0$ if $\xi_t^2 = |\xi_x|^2$ and $\xi \neq 0$. Theorem 17 thus implies unique continuation from the exterior of the cylinder $\{\Psi > \Psi(\mathbf{x}_0)\} = \{(t, x) \in \mathbb{R}^{1+d}, |x| > |x_0|\}$ towards the interior of the cylinder near $\mathbf{x}_0$. If one chooses instead $\Psi(t, x) = \varphi(x) = -|x|^2$, then the condition (29) holds at no point $\mathbf{x}_0 = (t_0, x_0)$: indeed, taking any $\xi_x \neq 0$ such that $\xi_x \cdot x_0 = 0$ and $\xi_t := \pm|\xi_x|$, we have $\text{Hess}_x(\varphi)(x_0)(\xi_x, \xi_x) = -2|\xi_x|^2 \leq 0$, which contradicts (29).

Note also that for the $1D$ wave equation, the constraint $\xi_x \cdot \nabla_x \varphi(x_0) = 0$ is much more demanding and implies $\xi_x = 0$ and $\xi = 0$ if $\xi_t^2 = |\xi_x|^2$. This is natural since time and space variables play the same role. Hence, finite speed of

propagation essentially implies the local unique continuation property across any noncharacteristic hypersurface.

- More generally, if one considers the function $\Psi(t, x) = |x - x_1|^2 - \gamma^2 t^2$, see Figure 4, then we have, for any $(t_0, x_0) \in \mathbb{R}^{1+d}$,

$$\text{Hess}(\Psi)(t_0, x_0)(X, X) = 2(|X_x|^2 - \gamma^2 |X_t|^2).$$

As a consequence, on the tangent cone $|X_x| = |X_t|, X \neq 0$, we have

$$\text{Hess}(\Psi)(t_0, x_0)(X, X) = 2|X_x|^2(1 - \gamma^2)$$

with $X_x \neq 0$. That is to say that the strong pseudoconvexity condition (27) is satisfied at any point (t_0, x_0) as soon as $\gamma \in [0, 1)$. Then, Theorem 17 implies that the local unique continuation property holds at any point (t_0, x_0) of the one sheet hyperboloid of revolution $\{\Psi = \Psi(t_0, x_0)\}$ from the exterior of the hyperboloid $\{\Psi > \Psi(t_0, x_0)\}$ towards the interior $\{\Psi < \Psi(t_0, x_0)\}$. We refer to Figure 4 for a representation of the level sets of Ψ .

Using a compactness argument and letting γ close to one, one may deduce the following global result.

Proposition 24. – Assume $\mathcal{M}$ is the closure of a bounded open set in $\mathbb{R}^d$, let ω be an open neighborhood of $\partial \mathcal{M}$ in $\mathcal{M}$, and fix any $x_1 \in \mathbb{R}^d$. Then, assuming

$$T > \sup\{|x - x_1|, x \in \mathcal{M} \setminus \omega\},$$

and letting $q_0, q_1 \in L^\infty((-T, T) \times \mathcal{M}; \mathbb{C})$, $q_2 \in L^\infty((-T, T) \times \mathcal{M}; \mathbb{C}^d)$, any solution to

$$\begin{cases} (\partial_t^2 - \Delta)u + q_0 u + q_1 \partial_t u + q_2 \cdot \nabla_x u = 0 & \text{in } (-T, T) \times \text{Int}(\mathcal{M}), \\ u \in H^1((-T, T) \times \mathcal{M}), \\ u = 0 & \text{in } (-T, T) \times \omega \end{cases}$$

satisfies

$$(u, \partial_t u)|_{t=0} = (0, 0) \text{ in } \text{Int}(\mathcal{M}).$$

Let us give a short sketch of the proof. We extend the u solution to $\mathbb{R}^d$ by zero outside of $\mathcal{M}$, and, with a slight abuse of notation, we denote by ω the set $\mathbb{R}^d \setminus (\mathcal{M} \setminus \omega)$. Setting $S = \sup\{|x - x_1|, x \in \mathcal{M} \setminus \omega\}$, we notice that $|x - x_1| > S$ implies $x \in \omega$. For $\gamma \in [0, 1)$ and $c_0 > 0$, we aim to apply the local unique continuation result of Theorem 17 at each point of $\{\Psi = c_0\} \cap \{t \in (-T, T)\}$. The only problem is at the times $t = T$ and $t = -T$. To be able to propagate along the full hypersurface, we need to ensure that $\{\Psi = c_0\} \cap \{t \in \{-T, T\}\}$ is contained in ω . This is satisfied if $c_0 + \gamma^2 T^2 > S^2$. The unique continuation result of Theorem 17 and a compactness argument prove that

$$u = 0 \text{ a.e. in } (-T, T) \times \mathcal{M} \cap \{(t, x) \in \mathbb{R}^{1+d}, |x - x_1|^2 - \gamma^2 t^2 \geq c\}$$

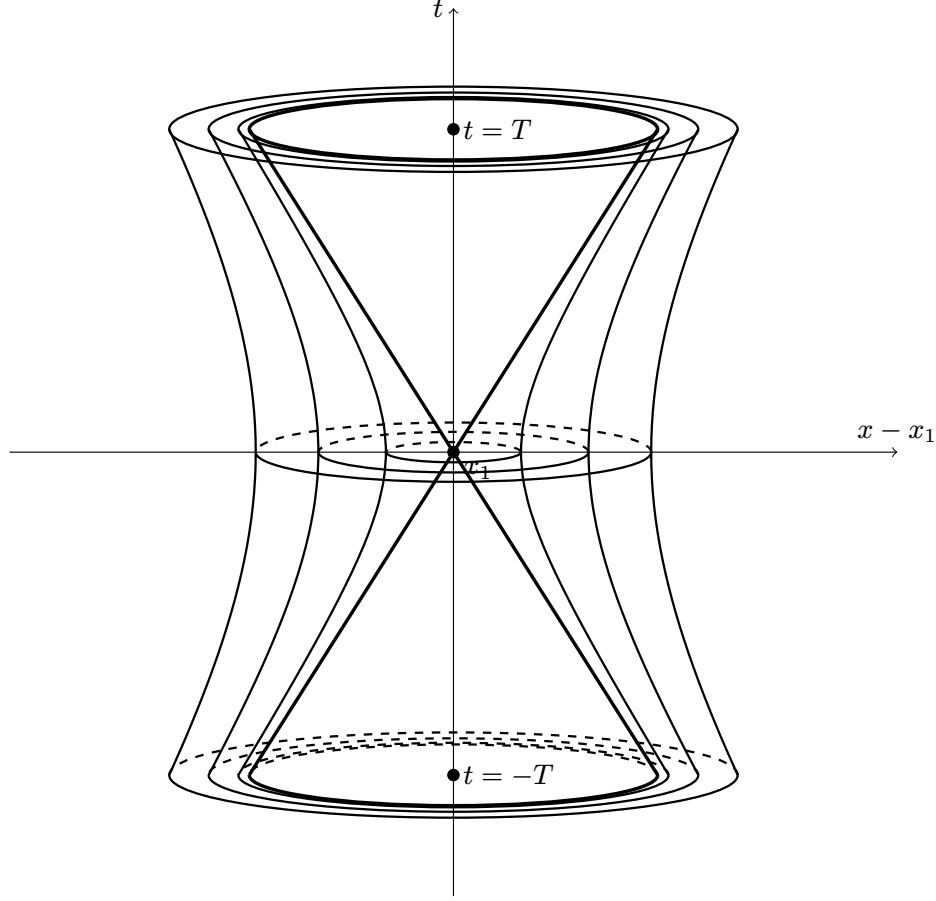


FIGURE 4. The level sets of the function $\Psi(t, x) = |x - x_1|^2 - \gamma^2 t^2$ for $\gamma < 1$ fixed (one sheet hyperboloids of revolution). The limiting levelset $\{\Psi = 0\}$ is the central cone. The Hörmander Theorem 17 applies to the flat/Minkowski wave operator $P = \partial_t^2 - \Delta$ at each point of the *oriented* surface $\{\Psi = c_0\}$, from the exterior of the hyperboloid $\{\Psi > c_0\}$ towards the interior $\{\Psi < c_0\}$, for any $c_0 > 0$. The assumption (29) (or rather its tangent form (28)) of Theorem 17 can be verified on the figure at a point $(t_0, x_0) \in \{\Psi = c_0\}$ by checking that tangent vectors X to $\{\Psi = c_0\}$ belonging to the characteristic cone $|X_t|^2 = |X_x|^2$ give rise to geodesic curves that remain outside of the hyperboloid $\{\Psi = c_0\}$ (and the hyperboloid has nonvanishing curvature). We recover here the situation of Figure 2.

for any $c > 0$ with $c > S^2 - \gamma^2 T^2$. The assumption $T > S$ and the fact that it is true for any $\gamma \in [0, 1)$ implies that $u = 0$ in a neighborhood of any point in $\{0\} \times (\mathcal{M} \setminus \{x_1\})$. Then, regularity of the wave equation implies that $u \in C^0((-T, T); H_{\text{loc}}^1(\mathcal{M})) \cap C^1((-T, T); L_{\text{loc}}^2(\mathcal{M}))$, so that $(u, \partial_t u)|_{t=0} \in H_{\text{loc}}^1(\mathcal{M}) \times L_{\text{loc}}^2(\mathcal{M})$ have $\text{supp}((u, \partial_t u)|_{t=0}) \subset \{x_1\}$ and thus vanish identically.

2.2. Carleman estimate. – Here, we recall that, $P \in \text{Diff}^2(\Omega)$ (with principal symbol $p_2(x, \xi)$) and $\Phi \in C^\infty(\Omega; \mathbb{R})$ being given, the conjugated operator is $P_\Phi = e^{\tau\Phi} P e^{-\tau\Phi} \in \text{Diff}_\tau^2(\Omega)$ and its principal symbol is $p_\Phi(x, \xi, \tau) = p_2(x, \xi + i\tau d\Phi(x))$ (computed in Lemma 13 and Example 14). We write $p_\Phi = \text{Re}(p_\Phi) + i \text{Im}(p_\Phi)$.

2.2.1. Carleman estimate under subellipticity condition. – As we have seen in Section 1.2.5, the key intermediate step for proving Theorem 17 is to obtain estimates of the type of (12). In this section, we prove a Carleman estimate (Theorem 25) under a symbolic condition usually called “Hörmander subellipticity condition” (see (30) below). The next sections connect this condition to the strong pseudoconvexity condition (21).

Theorem 25 (Local Carleman estimate). – *Let Ω be an open subset of $\mathbb{R}^n$ and $x_0 \in \Omega$. Let $P \in \text{Diff}^2(\Omega)$ be a (classical) differential operator with real-valued principal symbol p_2 and $\Phi \in C^\infty(\bar{\Omega}; \mathbb{R})$. Assume there exist $C_1, C_2 > 0$ such that for all $(\xi, \tau) \in \mathbb{R}^n \times \mathbb{R}_+^*$,*

$$(30) \quad \frac{C_1}{|\xi|^2 + \tau^2} [(\text{Re } p_\Phi)^2 + (\text{Im } p_\Phi)^2] + \frac{1}{\tau} \{\text{Re } p_\Phi, \text{Im } p_\Phi\} \geq C_2 (|\xi|^2 + \tau^2),$$

with symbols evaluated at (x_0, ξ, τ) . Then, there exist $C, r, \tau_0 > 0$ such that

$$(31) \quad \tau^3 \|e^{\tau\Phi} u\|_{L^2}^2 + \tau \|e^{\tau\Phi} \nabla u\|_{L^2}^2 \leq C \|e^{\tau\Phi} P u\|_{L^2}^2, \quad \text{for all } u \in C_c^\infty(B(x_0, r)), \tau \geq \tau_0.$$

Remark 26. – Given $r > 0$, Inequality (31) is equivalent to the existence of $C, \tau_0 > 0$ such that

$$(32) \quad \tau \|v\|_{H_\tau^1}^2 \leq C \|P_\Phi v\|_{L^2}^2, \quad \text{for all } v \in C_c^\infty(B(x_0, r)), \tau \geq \tau_0.$$

The equivalence between (31) and (32) comes from the change of unknown $v = e^{\tau\Phi} u$. This yields $P_\Phi v = e^{\tau\Phi} P e^{-\tau\Phi} v = e^{\tau\Phi} P u$.

Moreover, we have $\nabla u = \nabla(e^{-\tau\Phi} v) = e^{-\tau\Phi} (\nabla v - \tau v \nabla \Phi)$ so that

$$\tau^2 \|e^{\tau\Phi} u\|_{L^2}^2 + \|e^{\tau\Phi} \nabla u\|_{L^2}^2 \leq \tau^2 \|v\|_{L^2}^2 + 2 \|\nabla v\|_{L^2}^2 + 2 \|\tau v \nabla \Phi\|_{L^2}^2 \leq C \|v\|_{H_\tau^1}^2,$$

and thus (32) implies (31). Conversely, we have $\nabla v = \nabla(e^{\tau\Phi} u) = e^{\tau\Phi} (\nabla u + \tau u \nabla \Phi)$ so that

$$\|v\|_{H_\tau^1}^2 = \|\nabla v\|_{L^2}^2 + \tau^2 \|v\|_{L^2}^2 \leq C \left(\tau^2 \|e^{\tau\Phi} u\|_{L^2}^2 + \|e^{\tau\Phi} \nabla u\|_{L^2}^2 \right),$$

and (31) implies (32).

Notice that $\frac{\text{Im } p_\Phi}{\tau} = 2\tilde{p}_2(x, \xi, d\Phi(x))$ (see Example 14) is smooth, so dividing the Poisson bracket by τ in (30) makes sense, even as $\tau \rightarrow 0^+$. Before proceeding to the proof of this result, several comments are in order. First, the statement (31) is useful for applications to unique continuation, see (12) and the discussion in Section 1.2.5. The statement (32) is only a reformulation in terms of the conjugated operator, which belongs to Diff_τ^2 , and is thus analyzable with the tools developed in Section 1.3. The statement (30), as opposed to (32)–(31), is a “symbolic estimate,” concerning only

the principal symbol of the conjugated operator. The interest of this result is that it reduces the problem of proving a Carleman estimate to a property that one can check on the principal symbol of the conjugated operator. The question of rephrasing the condition (30) in geometric terms is addressed in Sections 2.2.2 and 2.3 below. A converse to Theorem 25 *almost* holds, which indicates the limit of this classical Carleman approach. This point is slightly more technical and we refer the reader to [33, Theorems 28.2.1 and 28.2.1'] for a precise statement and proof.

Proof. – Notice that the result of Lemma 13 only depends on the leading-order terms of the operator P . More precisely, if $\tilde{P} \in \text{Diff}^2(\Omega)$ has the same principal symbol as P , then $P - \tilde{P} \in \text{Diff}^1(\Omega)$ and $P_\Phi - \tilde{P}_\Phi = R \in \text{Diff}_\tau^1(\Omega)$. Henceforth, if one assumes that the Carleman inequality (32) holds for $\tilde{P}_\Phi$,

$$(33) \quad \tau \|v\|_{H_\tau^1}^2 \leq C \left\| \tilde{P}_\Phi v \right\|_{L^2}^2,$$

then

$$\tau \|v\|_{H_\tau^1}^2 \leq C \|(P_\Phi - R)v\|_{L^2}^2 \leq C \|P_\Phi v\|_{L^2}^2 + C \|Rv\|_{L^2}^2 \leq C \|P_\Phi v\|_{L^2}^2 + \check{C} \|v\|_{H_\tau^1}^2$$

with Part 4 in Proposition 12. Then, for τ large enough, we have $\tau - \check{C} \geq \tau/2$, and the last term can then be absorbed in the left-hand-side, yielding the sought Carleman inequality (32) for P_Φ , with different constants C and τ_0 .

Since the operator P has real principal symbol p_2 , we choose $\tilde{P} = \frac{P+P^*}{2}$, which is selfadjoint and has the same principal symbol p_2 . Note that we may write $P = \sum_{i,j=1}^n a^{ij}(x) D_i D_j$ with $a^{ij} = a^{ji}$ real-valued (see Example 17), and we have $\tilde{P} = \sum_{i,j=1}^n D_i a^{ij}(x) D_j$ up to a (selfadjoint) first-order operator. We may thus focus on $\tilde{P}_\Phi = e^{\tau\Phi} \tilde{P} e^{-\tau\Phi}$ and prove (33). To this aim, we decompose the operator $\tilde{P}_\Phi$ as

$$(34) \quad \tilde{P}_\Phi = Q_R + iQ_I, \quad \text{with} \quad Q_R = \frac{\tilde{P}_\Phi + \tilde{P}_\Phi^*}{2}; \quad Q_I = \frac{\tilde{P}_\Phi - \tilde{P}_\Phi^*}{2i}$$

Note that both Q_R and Q_I are formally selfadjoint ($Q_R^* = Q_R$ and $Q_I^* = Q_I$), and, according to Part 3 in Proposition 12, we have $Q_R, Q_I \in \text{Diff}_\tau^2$ with principal symbols (see Example 14) given by

$$\begin{aligned} q_R(x, \xi, \tau) &= \frac{p_\Phi + \overline{p_\Phi}}{2}(x, \xi, \tau) = \text{Re } p_\Phi(x, \xi, \tau) = p_2(x, \xi) - \tau^2 p_2(x, d\Phi(x)), \\ q_I(x, \xi, \tau) &= \frac{p_\Phi - \overline{p_\Phi}}{2i}(x, \xi, \tau) = \text{Im } p_\Phi(x, \xi, \tau) = 2\tau \tilde{p}_2(x, \xi, d\Phi(x)). \end{aligned}$$

Moreover (this is a key point), $\tilde{P}_\Phi = \sum_{i,j=1}^n (D_i + i\tau \partial_i \Phi) a^{ij}(x) (D_j + i\tau \partial_j \Phi)$ and we may hence write $\tilde{P}_\Phi = \tilde{P} + \tau M$ for some $M \in \text{Diff}_\tau^1(\Omega)$. Since $\tilde{P}$ is selfadjoint, this implies

$$(35) \quad Q_I = \frac{\tilde{P}_\Phi - \tilde{P}_\Phi^*}{2i} = \frac{\tau M - \tau M^*}{2i} = \tau \tilde{Q}_I, \quad \text{with} \quad \tilde{Q}_I = \frac{M - M^*}{2i} \in \text{Diff}_\tau^1(\Omega),$$

i.e., τ may be factorized in the skewadjoint part of the operator.

Using (34), the central computation is now as follows, for $v \in C_c^\infty(\Omega)$,

$$\begin{aligned}
 \left\| \tilde{P}_\Phi v \right\|_{L^2}^2 &= \left(\tilde{P}_\Phi v, \tilde{P}_\Phi v \right)_{L^2} = ((Q_R + iQ_I)v, (Q_R + iQ_I)v)_{L^2} \\
 &= (Q_R v, Q_R v)_{L^2} + (iQ_I v, iQ_I v)_{L^2} + (Q_R v, iQ_I v)_{L^2} + (iQ_I v, Q_R v)_{L^2} \\
 (36) \quad &= \|Q_R v\|_{L^2}^2 + \|Q_I v\|_{L^2}^2 + (i[Q_R, Q_I]v, v)_{L^2}.
 \end{aligned}$$

There are two kinds of terms in (36):

- the one with $\|Q_R v\|_{L^2}^2 = (Q_R^2 v, v)_{L^2}$ where $Q_R^2 \in \text{Diff}^4(\Omega)$ with principal symbol $(\text{Re } p_\Phi)^2$ (resp. $\|Q_I v\|_{L^2}^2 = (Q_I^2 v, v)_{L^2}$ where $Q_I^2 \in \text{Diff}^4(\Omega)$ with principal symbol $(\text{Im } p_\Phi)^2$);
- the one with $i[Q_R, Q_I]$ which is of order $2 + 2 - 1 = 3$ and with principal symbol $\{\text{Re } p_\Phi, \text{Im } p_\Phi\}$ by Part 2 of Proposition 12.

The first two operators, Q_R^2 and Q_I^2 , are of order 4. Yet their symbols can vanish. They are therefore not sufficient to obtain a “coercivity” estimate. The idea is thus to use the commutator in the case where both q_R^2 and q_I^2 vanish. However, to compare these terms, we need to bring them to the same order and “sacrifice” this main order 4. More precisely, let $C_1 > 0$ be as in Assumption (30) (that this is the right constant will appear in (38) below). For $\tau \geq C_1$, we have

$$\frac{1}{\tau^{1/2}} \geq \frac{C_1^{1/2}}{\tau} \geq \frac{C_1^{1/2}}{(|\xi|^2 + \tau^2)^{1/2}} \quad \text{for all } \xi \in \mathbb{R}^n.$$

This implies (using again the Plancherel Theorem)

$$\begin{aligned}
 (37) \quad \frac{1}{\tau} \|Q_R v\|_{L^2}^2 &= \left\| \frac{Q_R v}{\tau^{1/2}} \right\|_{L^2}^2 \geq \left\| C_1^{1/2} (-\Delta + \tau^2)^{-1/2} Q_R v \right\|_{L^2}^2 \\
 &= C_1 (Q_R (-\Delta + \tau^2)^{-1} Q_R v, v)_{L^2}.
 \end{aligned}$$

The same estimate applies to Q_I . Combining (36) with (37), we have now proved

$$(38) \quad \frac{1}{\tau} \left\| \tilde{P}_\Phi v \right\|_{L^2}^2 \geq (Lv, v)_{L^2},$$

with

$$L = C_1 \left(Q_R (-\Delta + \tau^2)^{-1} Q_R + Q_I (-\Delta + \tau^2)^{-1} Q_I \right)_{L^2} + i \left[Q_R, \frac{Q_I}{\tau} \right].$$

By (35), $Q_I = \tau \tilde{Q}_I$ with $\tilde{Q}_I \in \text{Diff}_\tau^1(\Omega)$. This implies that $\left[Q_R, \frac{Q_I}{\tau} \right] \in \text{Diff}_\tau^2$ as well. The operator L is thus precisely of the form of that in Proposition 15, is moreover selfadjoint, and has principal symbol (in the sense of Proposition 15)

$$\frac{C_1}{|\xi|^2 + \tau^2} ((\text{Re } p_\Phi)^2 + (\text{Im } p_\Phi)^2) + \left\{ \text{Re } p_\Phi, \frac{\text{Im } p_\Phi}{\tau} \right\},$$

which satisfies (30). Hence, the Gårding inequality of Proposition 15 applies and yields the existence of $C, \tau_0, r > 0$ such that

$$(Lv, v)_{L^2} \geq C \|v\|_{H_\tau^1}^2, \quad \text{for all } v \in C_c^\infty(B(x_0, r)), \quad \tau \geq \tau_0,$$

which, in view of (38), yields (33) and concludes the proof of the Carleman estimate (32). $\square$

Note that in (37), since Q_R is only defined on Ω , and since $(-\Delta + \tau^2)^{-1}Q_R v \notin C_c^\infty(\Omega)$, the expression $Q_R(-\Delta + \tau^2)^{-1}Q_R v$ is not well-defined. However, its pairing with the function $v \in C_c^\infty(B(x_0, r))$ is well defined (e.g., as $(\chi Q_R \tilde{\chi}(-\Delta + \tau^2)^{-1}Q_R v, v)$ with $\chi \in C^\infty(\Omega)$ such that $\chi = 1$ on a neighborhood of $B(x_0, r)$, and $\tilde{\chi} \in C^\infty(\Omega)$ with $\tilde{\chi} = 1$ on a neighborhood of $\text{supp}(\chi)$).

Remark 27 (Lower-order terms). – As seen in the proof, an important feature of the Carleman estimates (31) is its insensitivity with respect to lower-order terms. More precisely, if (31) is satisfied for an operator P , then it also holds for $P' := P + \sum_{k=1}^n b_k(x)D_k + c(x)$ as soon as $b_k, c \in L^\infty(\Omega)$. Indeed, applying (31) for P yields

$$\begin{aligned} \tau^3 \|e^{\tau\Phi} u\|_{L^2}^2 + \tau \|e^{\tau\Phi} \nabla u\|_{L^2}^2 &\leq C \left\| e^{\tau\Phi} \left(P' - \sum_{k=1}^n b_k(x)D_k + c(x) \right) u \right\|_{L^2}^2 \\ &\leq C \|e^{\tau\Phi} P'\|_{L^2}^2 + C \|e^{\tau\Phi} \nabla u\|_{L^2}^2 + C \|e^{\tau\Phi} u\|_{L^2}^2, \end{aligned}$$

and the last two terms can be absorbed in the left-hand side for τ large enough. Note in particular that no regularity is required on the lower-order terms when proceeding that way.

To conclude the section, notice that the Carleman estimate (31) (or equivalently (32)) is stated in Theorem 25 for functions $u \in C_c^\infty(B(x_0, r))$. It can be generalized to all functions such that all terms in the inequality make sense.

Lemma 28 (Carleman estimate for finite regularity functions). – *Let $P \in \text{Diff}^2$, $r, \tau_0 > 0$ and assume that the Carleman estimate (31) is satisfied for all $u \in C_c^\infty(B(x_0, r))$. Then it is also satisfied (with the same constant) for all functions*

$$(39) \quad u \in H^1(\Omega) \quad \text{such that} \quad \text{supp}(u) \subset B(x_0, r) \quad \text{and} \quad Pu \in L^2(\Omega)$$

(in the sense of distributions).

The proof of this fact relies on a classical regularization argument.

Proof. – According to Remark 27, we may assume that the operator is of divergence form and has no lower-order term, that is to say, $P = \sum_{1 \leq j, k \leq n} D_j a^{ij}(x) D_i$. We consider an approximation of the identity ρ_δ , $\delta > 0$, defined by $\rho_\delta(x) = \frac{1}{\delta^n} \rho(\frac{x}{\delta})$ with $\rho \in C_c^\infty(\mathbb{R}^n; \mathbb{R}^+)$, $\text{supp}(\rho) \subset B(0, 1)$ and $\int_{\mathbb{R}^n} \rho(x) dx = 1$. Given u as in (39), we set $u_\delta := u * \rho_\delta$ and there exists $\delta_0 > 0$ such that this function satisfies $u_\delta \in C_c^\infty(B(x_0, r))$ for all $\delta \in (0, \delta_0)$. The Carleman estimate (31) applies to u_δ . Since $u \in H^1(\Omega)$, $u_\delta \rightarrow u$ in H^1 as $\delta \rightarrow 0^+$ and hence the left-hand side of (31) for u_δ converges to the left-hand side of (31) for u . Concerning now the right-hand side of (31), we need to prove that assuming (39), we have $Pu_\delta \rightarrow Pu$ in $L^2(\Omega)$. To prove this, we first write $Pu_\delta - Pu = Pu_\delta - (Pu)_\delta + (Pu)_\delta - Pu$, where $w_\delta = w * \rho_\delta$. Since

$Pu \in L^2(\Omega)$ with compact support, we have $(Pu)_\delta - Pu \rightarrow 0$ in $L^2(\Omega)$. We now claim that

$$(40) \quad \text{for all } a \in C^1(\Omega), v \in L^2_{\text{comp}}(\Omega), \quad av_\delta - (av)_\delta \rightarrow 0 \text{ in } H^1(\Omega).$$

Assume first that the claim (40) is satisfied. Using the divergence form of P , we can write

$$Pu_\delta - (Pu)_\delta = \sum_{1 \leq j, k \leq n} D_j(a^{ij}(D_i u)_\delta - (a^{ij} D_i u)_\delta),$$

where $D_i u \in L^2_{\text{comp}}(\Omega)$. The claim (40) thus implies that $Pu_\delta - (Pu)_\delta \rightarrow 0$ in $L^2(\Omega)$, concluding the proof of the regularization argument, and thus of the lemma.

It remains now to prove the claim (40). Using the definition of the convolution product, we have

$$(av_\delta - (av)_\delta)(x) = \int_{\mathbb{R}^n} (a(x) - a(x - \delta y))v(x - \delta y)\rho(y)dy,$$

which converges to zero in $L^2(\Omega)$ as $\delta \rightarrow 0^+$. We fix now $j \in \{1, \dots, n\}$ and compute

$$\begin{aligned} \partial_{x_j}(av_\delta - (av)_\delta)(x) &= \int_{\mathbb{R}^n} (\partial_{x_j} a(x) - \partial_{x_j} a(x - \delta y))v(x - \delta y)\rho(y)dy \\ &\quad + \int_{\mathbb{R}^n} (a(x) - a(x - \delta y))\partial_{x_j} v(x - \delta y)\rho(y)dy, \end{aligned}$$

where the second integral is understood as a duality bracket between

$$y \mapsto \partial_{x_j} v(x - \delta y)\rho(y) \in H^{-1}_{\text{comp}}(\Omega)$$

and

$$y \mapsto (a(x) - a(x - \delta y)) \in C^1(\Omega) \subset H^1_{\text{loc}}(\Omega).$$

The first integral converges to zero in $L^2(\Omega)$ and it only remains to study the second one. The latter reads

$$\begin{aligned} R_\delta(x) &= \int_{\mathbb{R}^n} (a(x) - a(x - \delta y))\partial_{x_j} v(x - \delta y)\rho(y)dy \\ &= \int_{\mathbb{R}^n} (a(x) - a(x - \delta y))\rho(y) \left(-\frac{\partial_{y_j}}{\delta} \right) v(x - \delta y)dy = R_\delta^1(x) + R_\delta^2(x), \end{aligned}$$

after an integration by parts, where

$$\begin{aligned} R_\delta^1(x) &= \int_{\mathbb{R}^n} (a(x) - a(x - \delta y))v(x - \delta y)\frac{\partial_{y_j}}{\delta}\rho(y)dy \\ &= \int_{\mathbb{R}^n} \frac{a(x) - a(x - \delta y)}{\delta}v(x - \delta y)\partial_{y_j}\rho(y)dy, \\ R_\delta^2(x) &= \int_{\mathbb{R}^n} \frac{\partial_{y_j}}{\delta}(a(x) - a(x - \delta y))v(x - \delta y)\rho(y)dy \\ &= \int_{\mathbb{R}^n} \partial_{x_j} a(x - \delta y)v(x - \delta y)\rho(y)dy. \end{aligned}$$

We notice that $R_\delta^2 \rightarrow \partial_{x_j} a v$ in $L^2(\Omega)$. As for R_δ^1 , we have with the Taylor formula

$$R_\delta^1(x) = \int_{\mathbb{R}^n} \int_0^1 y \cdot \nabla a(x - t\delta y) dt v(x - \delta y) \partial_{y_j} \rho(y) dy.$$

The dominated convergence theorem and the Cauchy-Schwarz inequality allow to prove that, in $L^2(\Omega)$,

$$\begin{aligned} R_\delta^1(x) &\rightarrow \int_{\mathbb{R}^n} \int_0^1 y \cdot \nabla a(x) dt v(x) \partial_{y_j} \rho(y) dy = v(x) \int_{\mathbb{R}^n} \nabla a(x) \cdot y \partial_{y_j} \rho(y) dy \\ &= -v(x) \int_{\mathbb{R}^n} \partial_{y_j} (\nabla a(x) \cdot y) \rho(y) dy = -v(x) \partial_{x_j} a(x) \int_{\mathbb{R}^n} \rho(y) dy = -v(x) \partial_{x_j} a(x), \end{aligned}$$

after an integration by parts. As a consequence, $R_\delta = R_\delta^1 + R_\delta^2 \rightarrow 0$ in $L^2(\Omega)$. This concludes the proof of the claim (40), and hence of the lemma. $\square$

2.2.2. Carleman estimate for pseudoconvex Carleman weights. – We now reduce the quantitative symbolic Assumption (30) of the Carleman estimate to a qualitative convexity condition on the weight function Φ (with respect to the symbol p_2).

Definition 29 (Pseudoconvex Carleman weights). – Let $\Omega \ni x_0$ be an open set, $P \in \text{Diff}^2(\Omega)$ be a (classical) differential operator with real-valued principal symbol p_2 and $\Phi \in C^\infty(\Omega)$ real-valued. We say that Φ is a pseudoconvex Carleman weight for P at x_0 if it satisfies

$$(41) \quad \{p_2, \{p_2, \Phi\}\}(x_0, \xi) > 0, \text{ if } p_2(x_0, \xi) = 0 \text{ and } \xi \neq 0;$$

$$(42) \quad \frac{1}{i\tau} \{\overline{p_\Phi}, p_\Phi\}(x_0, \xi, \tau) > 0, \text{ if } p_\Phi(x_0, \xi, \tau) = 0 \text{ and } \tau > 0,$$

where $p_\Phi(x, \xi, \tau) = p_2(x, \xi + i\tau d\Phi(x))$. A “pseudoconvex Carleman weight” is sometimes called a “strongly pseudoconvex function” in the literature, see [87, Definition 2.4], [86] or [49, Definition 2.1]. The latter terminology however might lead to confusion with the (different but related) notion of strongly pseudoconvex surfaces (Definitions 16 and 34 below) and we thus do not use it in these notes.

Lemma 30 shows that (41) is the limit of (42) as $\tau \rightarrow 0^+$.

Lemma 30. – Let p be a real-valued smooth function on $\Omega \times \mathbb{R}^n$ polynomial in the second variable and set

$$p_\Phi(x, \xi, \tau) = p(x, \xi + i\tau d\Phi(x))$$

for $(x, \xi) \in \Omega \times \mathbb{R}^n, \tau \geq 0$. Then, we have $\lim_{\tau \rightarrow 0^+} \frac{1}{i\tau} \{\overline{p_\Phi}, p_\Phi\}(x, \xi, \tau) = 2 \{p, \{p, \Phi\}\}(x, \xi)$ for all $(x, \xi) \in \Omega \times \mathbb{R}^n$.

Lemma 30 is used for $p = p_2$. A proof is given below. We now state the equivalence between Definition 29 and the Hörmander subellipticity condition (30).

Proposition 31. – Let $\Omega \ni x_0$ be an open set, $P \in \text{Diff}^2(\Omega)$ with real-valued principal symbol p_2 and $\Phi \in C^\infty$ real-valued. The subellipticity condition (30) is satisfied at x_0 if and only if Φ is a pseudoconvex Carleman weight for P at x_0 .

In particular, if Φ is a pseudoconvex Carleman weight for P at x_0 in the sense of Definition 29, the Carleman estimate of Theorem 25 holds with weight Φ . The proof uses the following (elementary but very useful) lemma.

Lemma 32. – Let K be a compact set and f, g two continuous real-valued functions on K . Assume that $f \geq 0$ on K , and $g > 0$ on $\{f = 0\}$. Then, there exists $A_0, C > 0$ such that for all $A \geq A_0$, we have $g + Af \geq C$ on K .

A proof of this elementary lemma can be found, e.g., in [53]. We now prove Proposition 31 from Lemmata 30 and 32.

Proof of Proposition 31. – Note first that since $\{f, f\} = 0$ and $\{f, g\} = -\{g, f\}$ for any f and g , we have

$$(43) \quad \begin{aligned} \frac{1}{i\tau} \{\overline{p_\Phi}, p_\Phi\} &= \frac{1}{i\tau} \{\text{Re } p_\Phi - i \text{Im } p_\Phi, \text{Re } p_\Phi + i \text{Im } p_\Phi\} \\ &= \frac{1}{\tau} \{\text{Re } p_\Phi, \text{Im } p_\Phi\} - \frac{1}{\tau} \{\text{Im } p_\Phi, \text{Re } p_\Phi\} = \frac{2}{\tau} \{\text{Re } p_\Phi, \text{Im } p_\Phi\}. \end{aligned}$$

Moreover, we recall that $\frac{\text{Im } p_\Phi}{\tau} = 2\tilde{p}_2(x_0, \xi, d\Phi(x_0))$ is smooth. We notice that all terms in (30) are homogeneous in (ξ, τ) of order 2 and continuous thanks to the previous remark. Therefore, it is enough to prove (30) on the set $K = \{(\xi, \tau), |\xi|^2 + \tau^2 = 1; \tau \geq 0\}$. On this compact set, the result is a consequence of Lemma 32 with $f = (\text{Re } p_\Phi)^2 + (\text{Im } p_\Phi)^2$ and $g = 2\{\text{Re } p_\Phi, \frac{\text{Im } p_\Phi}{\tau}\}$. Lemma 30 then proves that the first assumption in Definition 29 is the limit of the second one on the set $\{\tau = 0\}$. Hence, we have $g > 0$ on $\{f = 0\}$ on the whole K , up to the set $\{\tau = 0\} \cap \{|\xi|^2 + \tau^2 = 1\}$. Lemma 32 then concludes the proof of the subellipticity condition (30).

Conversely, according to (43), the inequality (30) can be rewritten as

$$\frac{C_1}{|\xi|^2 + \tau^2} [(\text{Re } p_\Phi)^2 + (\text{Im } p_\Phi)^2](x_0, \xi, \tau) + \frac{1}{2i\tau} \{\overline{p_\Phi}, p_\Phi\}(x_0, \xi, \tau) \geq C_2 (|\xi|^2 + \tau^2),$$

for all $(\xi, \tau) \in \mathbb{R}^n \times \mathbb{R}_+$, $(\xi, \tau) \neq (0, 0)$. This implies (42) for $\tau > 0$. Now taking $\xi \neq 0$, recalling that

$$p_\Phi(x, \xi, \tau) = p_2(x, \xi + i\tau d\Phi(x)) = p_2(x, \xi) - \tau^2 p_2(x, d\Phi(x)) + i\tau \tilde{p}_2(x, \xi, d\Phi(x)),$$

and letting $\tau \rightarrow 0^+$ in this inequality yields

$$\frac{C_1}{|\xi|^2} p_2(x_0, \xi)^2 + \{p_2, \Phi\}(x_0, \xi) \geq C_2 |\xi|^2,$$

after an application of Lemma 30. This implies (41) and hence Φ is a pseudoconvex Carleman weight for P at x_0 in the sense of Definition 29. $\square$

Proof of Lemma 30. – We first notice that for $\tau = 0$, $\{\overline{p_\Phi}, p_\Phi\} = \{\overline{p}, p\}$ so since p is real, $\{\overline{p_\Phi}, p_\Phi\} = 0$ for $\tau = 0$. The definition of the derivative in $\tau = 0$ then yields

$$(44) \quad \lim_{\tau \rightarrow 0} \frac{1}{\tau} \{\overline{p_\Phi}, p_\Phi\} = \frac{\partial}{\partial \tau} \{\overline{p_\Phi}, p_\Phi\}|_{\tau=0}.$$

Also, we have $\partial_\tau(\{\overline{p_\Phi}, p_\Phi\}) = \{\partial_\tau \overline{p_\Phi}, p_\Phi\} + \{\overline{p_\Phi}, \partial_\tau p_\Phi\}$.

But since p is real, $\overline{p_\Phi} = p(x, \xi - i\tau d\Phi(x))$, so that

$$\begin{aligned} \partial_\tau p_\Phi(x, \xi, \tau) &= id\Phi \cdot \partial_\xi p(x, \xi + i\tau d\Phi) = i\{p_\Phi, \Phi\}(x, \xi, \tau), \\ \partial_\tau \overline{p_\Phi}(x, \xi, \tau) &= -id\Phi \cdot \partial_\xi p(x, \xi - i\tau d\Phi) = -i\{\overline{p_\Phi}, \Phi\}(x, \xi, \tau). \end{aligned}$$

We obtain $\partial_\tau(\{\overline{p_\Phi}, p_\Phi\}) = -i\{\{\overline{p_\Phi}, \Phi\}, p_\Phi\} + i\{\overline{p_\Phi}, \{p_\Phi, \Phi\}\}$, which, specified to $\tau = 0$, writes

$$\frac{\partial}{\partial \tau} \{\overline{p_\Phi}, p_\Phi\}|_{\tau=0} = -i\{\{p, \Phi\}, p\} + i\{p, \{p, \Phi\}\} = 2i\{p, \{p, \Phi\}\}.$$

Together with (44), this concludes the proof of the lemma. $\square$

Let us draw a provisional conclusion concerning pseudoconvex Carleman weights as introduced in Definition 29. On the one hand, the fact that Φ is pseudoconvex Carleman weight for P at x_0 is equivalent to the subellipticity condition (30), and hence it implies (and is *almost* equivalent to) the Carleman estimate of Theorem 25 with weight $e^{\tau\Phi}$. That is to say, “a pseudoconvex Carleman weight Φ for P is an admissible weight for a Carleman estimate”. On the other hand, an important drawback to Definition 29 is that it is not a geometric assumption: it is not only dependent on the level set of the function Φ , but also on the “convexity with respect to the level sets” (in general, $g''(x_0)$ is a geometric quantity only if $g'(x_0) = 0$). This is in strong contrast with strongly pseudoconvex surfaces in Definition 16 (that is invariant by diffeomorphisms), which is the appropriate geometric assumption for the unique continuation result of Theorem 17. We shall now relate pseudoconvex Carleman weights (Definition 29) to strongly pseudoconvex surfaces (Definition 16). Before that, let us stress an important stability feature of the pseudoconvexity assumption of Definition 29.

2.2.3. Stability of the pseudoconvexity assumption. – We prove that the pseudoconvexity condition of Definition 29 is stable by small C^2 perturbations of the weight function Φ . This will be very useful for perturbing the hypersurface across which to prove unique continuation.

Proposition 33 (Stability and Geometric convexification). – *Let $\Omega \ni x_0$ such that $\overline{\Omega}$ is compact. Assume $P \in \text{Diff}^2(\Omega)$ has real-valued principal symbol, and $\Phi \in C^\infty$ is a pseudoconvex Carleman weight for P at x_0 (in the sense of Definition 29). Then, there exists $\varepsilon_0 > 0$ so that any $\Phi_\varepsilon \in C^2(\overline{\Omega})$ with $\|\Phi - \Phi_\varepsilon\|_{C^2(\overline{\Omega})} < \varepsilon_0$ is a pseudoconvex Carleman weight for P at x_0 .*

Note that modifying Φ allows to slightly change its level sets. For instance, taking $\Phi_\varepsilon(x) = \Phi(x) - \varepsilon|x - x_0|^2$ (which shall be very useful for applications to unique continuation), the level set $\{\Phi_\varepsilon = 0\}$ is slightly bent (except at x_0) into the set $\{\Phi > 0\}$

(where u will be assumed to be zero). This slight change will be crucial for the proof of the unique continuation theorem.

Proof. – Definition 29 is equivalent to

$$c_\Phi(\xi, \tau) + C_1 \frac{|p_\Phi(x_0, \xi, \tau)|^2}{|\xi|^2 + \tau^2} \geq C_2(|\xi|^2 + \tau^2), \quad C_2 > 0,$$

uniformly for (ξ, τ) with $|\xi|^2 + \tau^2 = 1$, $\tau \geq 0$, where

$$c_\Phi(\xi, \tau) = \frac{1}{i\tau} \{\overline{p_\Phi}, p_\Phi\}(x_0, \xi, \tau), \text{ for } \tau > 0 \quad \text{and} \quad c_\Phi(\xi, 0) = 2\{p_2, \{p_2, \Phi\}\}(x_0, \xi).$$

See the proof of Proposition 31 (relying on Lemma 32 combined with Lemma 30 for the limit $\tau \rightarrow 0^+$). We then remark that all quantities in this inequality only involve derivatives of Φ of order at most 2 (as a consequence of Lemma 30) at the point x_0 . It is therefore stable by the addition of a function small for the C^2 -norm around x_0 . $\square$

2.3. Strongly pseudoconvex hypersurfaces. – Until this point, we have proved a Carleman estimate with weight $e^{\tau\Phi}$ provided Φ is a pseudoconvex Carleman weight for P (Definition 29). The main purpose of this section is to provide a geometric characterization of hypersurfaces S for which we can find a function Φ having S as a level set and being appropriate for the Carleman estimate (i.e., satisfying Definition 29). We shall eventually prove that the sought geometric condition on the hypersurface S is (21). We first introduce a seemingly stronger condition.

Definition 34 (Strongly pseudoconvex hypersurface). – Let $\Omega \ni x_0$ be an open set, $P \in \text{Diff}^2(\Omega)$ with real-valued principal symbol p_2 and $\Psi \in C^\infty(\Omega)$ real-valued. We say that the *oriented hypersurface* $S = \{\Psi = \Psi(x_0)\} \ni x_0$ is strongly pseudoconvex with respect to P at x_0 if

$$(45) \quad \{p_2, \{p_2, \Psi\}\}(x_0, \xi) > 0, \text{ if } p_2(x_0, \xi) = \{p_2, \Psi\}(x_0, \xi) = 0 \text{ and } \xi \neq 0;$$

$$(46) \quad \frac{1}{i\tau} \{\overline{p_\Psi}, p_\Psi\}(x_0, \xi, \tau) > 0, \text{ if } p_\Psi(x_0, \xi, \tau) = \{p_\Psi, \Psi\}(x_0, \xi, \tau) = 0 \text{ and } \tau > 0,$$

where $p_\Psi(x, \xi, \tau) = p_2(x, \xi + i\tau d\Psi(x))$.

Note that the first condition (45) is precisely (21) in Definition 16. We shall eventually prove that for differential operators of order two with real principal symbols, (45) implies (46), hence Definitions 16 and 34 are equivalent, which justifies that we do not introduce two different terminologies. Note that the definition seems to depend on the defining function Ψ for the hypersurface S , and not only on the oriented hypersurface S itself. Lemma 35 (see, e.g., [53] for a proof) shows this is not the case, and hence justifies the definition.

Lemma 35. – Assume $S = \{\Psi_1 = \Psi_1(x_0)\} = \{\Psi_2 = \Psi_2(x_0)\}$ with $d\Psi_j(x_0) \neq 0$, $j = 1, 2$ and $d\Psi_2(x_0) = \lambda d\Psi_1(x_0)$ for some $\lambda > 0$ (same orientation). Then Ψ_1 satisfies (45) if and only if Ψ_2 satisfies (45), and Ψ_1 satisfies (46) if and only if Ψ_2 satisfies (46).

Remark that Definition 34 looks very similar to Definition 29. It is just slightly weaker because the positivity condition is assumed only under the additional conditions $\{p_2, \Phi\} = 0$ and $\{p_\Phi, \Phi\} = 0$. In particular, the level sets of a pseudoconvex Carleman weight for P are pseudoconvex oriented hypersurfaces. This is however not useful since Definition 29 is not geometric (but rather linked to Carleman estimates).

The importance of Definition 34 is twofold:

- It is a purely geometric definition: this comes from Lemma 35 and the fact that Conditions (45)-(46) are invariant by diffeomorphisms.
- Once Ψ satisfies this geometric condition, one can produce a function Φ having the same levelsets (hence keeping the geometry unchanged), and that satisfies the stronger pseudoconvexity condition of Definition 29. This is the goal of the next section.

Note that, once again, Condition (45) (on the real domain) is the limit as $\tau \rightarrow 0^+$ of Condition (46) (on the complex domain). This follows both from Lemma 30 and the fact that

$$\begin{aligned} \{p_\Psi, \Psi\}(x, \xi, \tau) &= \partial_\xi(p(x, \xi + i\tau d\Psi(x))) \cdot \partial_x \Psi(x) = (\partial_\xi p)(x, \xi + i\tau d\Psi(x)) \cdot \partial_x \Psi(x) \\ (47) \quad &= \{p_2, \Psi\}(x, \xi + i\tau d\Psi(x)) \rightarrow \{p_2, \Psi\}(x, \xi), \quad \text{as } \tau \rightarrow 0^+. \end{aligned}$$

2.3.1. (Analytic) convexification

Proposition 36 (Analytic convexification). – *Let $\Omega \ni x_0$ be an open set, $P \in \text{Diff}^2(\Omega)$ with real-valued principal symbol p_2 and $\Psi \in C^\infty(\Omega)$ real-valued. Assume the oriented hypersurface $S = \{\Psi = \Psi(x_0)\}$ is strongly pseudoconvex with respect to P at x_0 (Definition 34). Then there exists $\lambda_0 > 0$ such that for all $\lambda \geq \lambda_0$, the function $\Phi = e^{\lambda\Psi}$ is a pseudoconvex Carleman weight for P at x_0 (Definition 29).*

Hence, the Carleman estimate of Theorem 25 holds with weight Φ . Note that the geometry of the level-sets of Φ and Ψ are actually the same: only the values of the level sets of Φ are stretched. Here, for any strongly pseudoconvex oriented hypersurface $S = \{\Psi = \Psi(x_0)\}$, this proposition produces an admissible Carleman weight (that is, a pseudoconvex Carleman weight for P) Φ having exactly the same level sets. In order to simplify the notation for the proof, we recall that x_0 is fixed and remark that changing the function Ψ by a constant does not change the assumption. We may thus assume that

$$(48) \quad \Psi(x_0) = 0, \quad \text{and hence} \quad \Phi(x_0) = 1 \quad \text{and} \quad d\Phi(x_0) = \lambda d\Psi(x_0).$$

We also denote

$$(49) \quad c_\Psi(\xi, \tau) = \frac{1}{i\tau} \{\overline{p_\Psi}, p_\Psi\}(x_0, \xi, \tau), \text{ for } \tau > 0 \quad \text{and} \quad c_\Psi(\xi, 0) = 2\{p_2, \{p_2, \Psi\}\}(x_0, \xi),$$

with a similar definition for $c_\Phi(\xi, \tau)$. According to Lemma 30, $c_\Psi(\xi, \tau)$ and $c_\Phi(\xi, \tau)$ are continuous on the whole $\mathbb{R}^n \times \mathbb{R}^+$. The proof of Proposition 36 is then based on the following computation.

Lemma 37. – Assume $\Phi = e^{\lambda\Psi}$. For all $(\xi, \tau) \in \mathbb{R}^n \times \mathbb{R}^+$ and all $\lambda > 0$, we have

$$c_\Phi(\xi, \tau) = \lambda c_\Psi(\xi, \lambda\tau) + 2\lambda^2 |\{p_\Psi, \Psi\}(x_0, \xi, \lambda\tau)|^2.$$

We first prove the proposition from the lemma and then prove the lemma.

Proof of Proposition 36 from Lemma 37. – Using Lemma 32 (combined with Lemma 30 and (47) in the limit $\tau \rightarrow 0^+$), Properties (45)-(46) imply the existence of $C_1, C_2 > 0$ so that

$$c_\Psi(\xi, \tau) + C_1 |\{p_\Psi, \Psi\}(x_0, \xi, \tau)|^2 + C_1 \frac{|p_\Psi(x_0, \xi, \tau)|^2}{|\xi|^2 + \tau^2} \geq C_2(|\xi|^2 + \tau^2),$$

for any $\tau \geq 0$, $|\xi|^2 + \tau^2 = 1$ (note that this takes into account the limit $\tau \rightarrow 0^+$). Replacing τ by $\lambda\tau$ for $\lambda \geq 1$ and using homogeneity, this can be reformulated as

$$(50) \quad c_\Psi(\xi, \lambda\tau) + C_1 |\{p_\Psi, \Psi\}(x_0, \xi, \lambda\tau)|^2 + C_1 \frac{|p_\Psi(x_0, \xi, \lambda\tau)|^2}{|\xi|^2 + \lambda^2\tau^2} \geq C_2(|\xi|^2 + \lambda^2\tau^2),$$

for any $(\xi, \tau) \neq (0, 0)$ with $\tau \geq 0$. Moreover, using Lemma 37 and noticing (see (48)) that

$$p_\Psi(x_0, \xi, \lambda\tau) = p_2(x_0, \xi + i\lambda\tau d\Psi(x_0)) = p_2(x_0, \xi + i\tau d\Phi(x_0)) = p_\Phi(x_0, \xi, \tau),$$

we obtain

$$\begin{aligned} c_\Phi(\xi, \tau) + C_1 \lambda \frac{|p_\Phi(x_0, \xi, \tau)|^2}{|\xi|^2 + \tau^2} \\ = \lambda \left(c_\Psi(\xi, \lambda\tau) + 2\lambda |\{p_\Psi, \Psi\}(x_0, \xi, \lambda\tau)|^2 + C_1 \frac{|p_\Psi(x_0, \xi, \lambda\tau)|^2}{|\xi|^2 + \tau^2} \right). \end{aligned}$$

Now taking $\lambda \geq \max\{C_1/2, 1\}$ and using (50) yields

$$\begin{aligned} c_\Phi(\xi, \tau) + C_1 \lambda \frac{|p_\Phi(x_0, \xi, \tau)|^2}{|\xi|^2 + \tau^2} \\ \geq \lambda \left(c_\Psi(\xi, \lambda\tau) + C_1 |\{p_\Psi, \Psi\}(x_0, \xi, \lambda\tau)|^2 + C_1 \frac{|p_\Psi(x_0, \xi, \lambda\tau)|^2}{|\xi|^2 + \lambda^2\tau^2} \right) \\ \geq C_2 \lambda (|\xi|^2 + \lambda^2\tau^2) \geq C_2 \lambda (|\xi|^2 + \tau^2). \end{aligned}$$

When recalling the definition of c_Φ , this readily implies (42), and also (41) in the limit $\tau \rightarrow 0^+$ (with Lemma 30). This concludes the proof that Φ is pseudoconvex for P at x_0 in the sense of Definition 29. $\square$

Proof of Lemma 37. – We compute

$$\partial_j \Phi = \lambda \partial_j \Psi e^{\lambda\Psi}, \quad \partial_{j,k} \Phi = \lambda \partial_{j,k} \Psi e^{\lambda\Psi} + \lambda^2 (\partial_j \Psi)(\partial_k \Psi) e^{\lambda\Psi},$$

which we write in a shorter way as

$$d\Phi = \lambda e^{\lambda\Psi} d\Psi, \quad \text{Hess}(\Phi)(\xi, \eta) = \lambda \text{Hess}(\Psi)(\xi, \eta) e^{\lambda\Psi} + \lambda^2 (\xi \cdot \partial_x \Psi)(\eta \cdot \partial_x \Psi) e^{\lambda\Psi}.$$

Taken at the point x_0 , and recalling (48), this implies

(51)

$$d\Phi(x_0) = \lambda d\Psi(x_0), \quad \text{Hess}(\Phi)(x_0) = \lambda \text{Hess}(\Psi)(x_0) + \lambda^2 d\Psi(x_0) \otimes d\Psi(x_0).$$

We now compute

$$\begin{aligned} c_\Phi(\xi, \tau) &= \frac{1}{i\tau} \{\overline{p_\Phi}, p_\Phi\}(x_0, \xi, \tau) = \frac{1}{i\tau} \partial_\xi p(x_0, \xi - i\tau d\Phi(x_0)) \cdot \partial_x p(x_0, \xi + i\tau d\Phi(x_0)) \\ &\quad + \text{Hess}(\Phi)(x_0) [\partial_\xi p(x_0, \xi - i\tau d\Phi(x_0)), \partial_\xi p(x_0, \xi + i\tau d\Phi(x_0))] \\ &\quad - \frac{1}{i\tau} \partial_x p(x_0, \xi - i\tau d\Phi(x_0)) \cdot \partial_\xi p(x_0, \xi + i\tau d\Phi(x_0)) \\ &\quad + \text{Hess}(\Phi)(x_0) [\partial_\xi p(x_0, \xi - i\tau d\Phi(x_0)), \partial_\xi p(x_0, \xi + i\tau d\Phi(x_0))] \\ &= \frac{2}{\tau} \text{Im} [\partial_\xi p(x_0, \xi - i\tau d\Phi(x_0)) \cdot \partial_x p(x_0, \xi + i\tau d\Phi(x_0))] \\ &\quad + 2 \text{Hess}(\Phi)(x_0) [\partial_\xi p(x_0, \xi - i\tau d\Phi(x_0)), \partial_\xi p(x_0, \xi + i\tau d\Phi(x_0))]. \end{aligned}$$

Using now (51), and dropping from the notation the fact that Ψ and its derivatives are evaluated at x_0 , this reads

$$\begin{aligned} c_\Phi(\xi, \tau) &= \frac{2}{\tau} \text{Im} [\partial_\xi p(x_0, \xi - i\tau \lambda d\Psi) \cdot (\partial_x p(x_0, \xi + i\tau \lambda d\Psi))] \\ &\quad + 2\lambda \text{Hess}(\Psi) [\partial_\xi p(x_0, \xi - i\tau \lambda d\Psi), \partial_\xi p(x_0, \xi + i\tau \lambda d\Psi)] \\ &\quad + 2\lambda^2 (\partial_\xi p(x_0, \xi - i\tau \lambda d\Psi) \cdot \partial_x \Psi) (\partial_\xi p(x_0, \xi + i\tau \lambda d\Psi) \cdot \partial_x \Psi) \\ &= \lambda c_\Psi(\xi, \lambda\tau) + 2\lambda^2 |\{p, \Psi\}(x_0, \xi + i\tau \lambda d\Psi)|^2 \\ &= \lambda c_\Psi(\xi, \lambda\tau) + 2\lambda^2 |\{p_\Psi, \Psi\}(x_0, \xi, \lambda\tau)|^2, \end{aligned}$$

proving the lemma. $\square$

2.3.2. Reducing the strong pseudoconvexity assumption to the condition on the real space. – In the particular case of differential operators of order two, with real principal symbol, Condition (45) on the real space implies Condition (46) in the complex space (this is no longer the case if one of these two conditions is not satisfied, see [33, 65]).

Proposition 38. – *Let $\Omega \ni x_0$ be an open set, $P \in \text{Diff}^2(\Omega)$ with real-valued principal symbol p_2 and $\Psi \in C^\infty(\Omega)$ real-valued. Assume that the oriented hypersurface $S = \{\Psi = \Psi(x_0)\}$ satisfies Condition (45) at x_0 . Then it also satisfies Condition (46).*

In other words, Definitions 16 and 34 are equivalent for differential operators of order two with real principal symbol. For this reason, we only introduced one terminology for the two Definitions 16 and 34, namely “strongly pseudoconvex surfaces”. For general differential operators with real principal symbol (or more generally for principally normal differential operators), the appropriate notion of strongly pseudoconvex surface is that of Definition 34, see [33, 65].

We split the proof of Proposition 38 into two lemmata, concerned with the non-characteristic case, that is $p_2(x_0, d\Psi(x_0)) \neq 0$, and the characteristic case, that is $p_2(x_0, d\Psi(x_0)) = 0$, respectively.

Lemma 39. – Assume p_2 is a real symbol of order two near x_0 , and Ψ is such that $p_2(x_0, d\Psi(x_0)) \neq 0$. Then, for any $\xi \in \mathbb{R}^n$ we have

$$(52) \quad p_\Psi(x_0, \xi, \tau) = \{p_\Psi, \Psi\}(x_0, \xi) = 0 \implies \tau = 0.$$

In this case, Assumption (46) holds trivially.

Lemma 40. – Assume p_2 is a real symbol of order two near x_0 , and Ψ is such that $p_2(x_0, d\Psi(x_0)) = 0$. Assume also (45) for all $\xi \in \mathbb{R}^n \setminus \{0\}$. Then we also have (46).

Both proofs of Lemmata 39 and 40 rely on the fact that for fixed $\xi \in \mathbb{R}^n$,

$$f(z) = p_2(x_0, \xi + zd\Psi(x_0)) = p_2(x_0, \xi) + z^2 p_2(x_0, d\Psi(x_0)) + 2z\tilde{p}_2(x_0, \xi, d\Psi(x_0)),$$

is a *second-order polynomial* in the variable $z \in \mathbb{C}$, with *real coefficients*. Moreover, the assumption of (52) (resp. of (46)) implies that

$$\begin{aligned} f(i\tau) &= p_2(x_0, \xi + i\tau d\Psi(x_0)) = p_\Psi(x_0, \xi, \tau) = 0 \quad \text{and} \\ f'(i\tau) &= \partial_\xi p_2(x_0, \xi + i\tau d\Psi(x_0)) \cdot \partial_x \Psi(x_0) \\ &= \{p_2, \Psi\}(x_0, \xi + i\tau d\Psi(x_0)) = \{p_\Psi, \Psi\}(x_0, \xi) = 0, \end{aligned}$$

that is to say, $z = i\tau$ ($\tau \in \mathbb{R}^+$) is a double root of the polynomial f .

Proof of Lemma 39. – Since the coefficient in front of z^2 , namely $p_2(x_0, d\Psi(x_0))$, is non-zero, the polynomial f has two complex roots which are either both in $\mathbb{R}$, or complex conjugate. That $z = i\tau$ ($\tau \in \mathbb{R}^+$) is a double root of the polynomial f implies $\tau = 0$. $\square$

The proof of Lemma 40 relies on tedious computations, and we refer the reader to [53]. Note that so far, we have given a complete proof of Theorem 17 under the additional non-characteristicity condition $p_2(x_0, d\Psi(x_0)) \neq 0$.

2.3.3. Unique continuation: end of proof of Theorem 17. – In this section, we conclude the proof of Theorem 17. After a geometric convexification procedure, it consists essentially in using Lemma 9.

Proof of Theorem 17. – We first remark that we may assume that $\Psi(x_0) = 0$ (up to changing Ψ into $\Psi - \Psi(x_0)$, which does not change the assumption), so that $S = \{\Psi = 0\}$. Assume first that u is a C^∞ solution of $Pu = 0$ in Ω so that $u = 0$ on $\Omega \cap \{\Psi > 0\}$ (the general case $u \in H^1$ is treated afterwards). The hypersurface $S = \{\Psi = 0\}$ being strongly pseudoconvex at x_0 , Proposition 36 shows that for λ large enough (but fixed) $\Phi := e^{\lambda\Psi} - 1$ is a pseudoconvex Carleman weight for P with $\{\Phi = 0\} = \{\Psi = 0\}$, $\{\Phi > 0\} = \{\Psi > 0\}$ and $\{\Phi < 0\} = \{\Psi < 0\}$. Proposition 33 yields the existence of $\varepsilon > 0$, such that $\Phi_\varepsilon = \Phi - \varepsilon|x - x_0|^2$ remains a pseudoconvex

Carleman weight for P (Definition 29). As a consequence of Proposition 31 and Theorem 25, it therefore satisfies the following properties (see Figure 5 for the geometry):

(1) there exist $R > 0$, $C > 0$ and $\tau_0 > 0$ so that we have the following estimate

$$(53) \quad \tau^3 \|e^{\tau\Phi_\varepsilon} w\|_{L^2}^2 + \tau \|e^{\tau\Phi_\varepsilon} \nabla w\|_{L^2}^2 \leq C \|e^{\tau\Phi_\varepsilon} Pw\|_{L^2}^2,$$

for any $w \in C^\infty(B(x_0, R))$ and $\tau \geq \tau_0$.

(2) there exists $\eta > 0$ so that $\Phi_\varepsilon(x) \leq -\eta$ for $x \in \{\Phi \leq 0\} \cap \{|x - x_0| \geq R/2\}$,

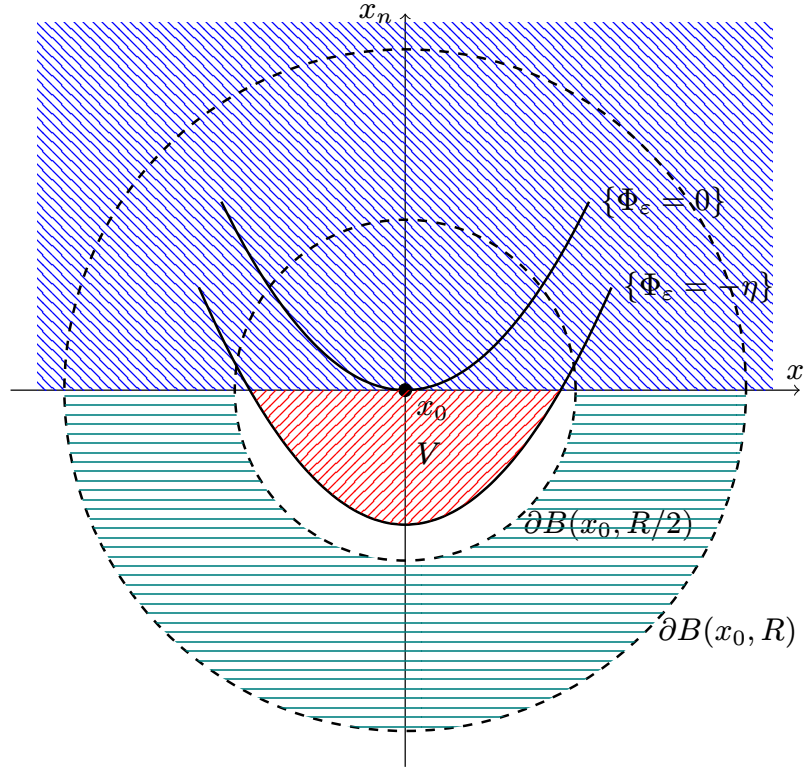


FIGURE 5. Picture of the geometric context of the proof of Theorem 17 with $\Psi(x', x_n) = x_n$, and $S = \{x_n = 0\}$. Again $u = 0$ on the blue striped zone $S^+ = \{x_n > 0\} = \{\Phi > 0\}$. The red striped zone $V := \{\Phi_\varepsilon > -\eta\} \cap \{\Phi \leq 0\}$ is the region where nullity/unique continuation is propagated. The term $\text{supp}([P, \chi]u)$ is supported in the horizontally light blue striped region and a key geometric feature in the proof is that $\text{supp}([P, \chi]u) \subset \text{supp}(\nabla \chi) \cap \text{supp}(u) \subset \{R/2 \leq |x| \leq R\} \cap \{\Phi \leq 0\} \subset \{\Phi_\varepsilon \leq -\eta\}$.

Property 1 is a consequence of Theorem 25, and R is fixed by that theorem. Property 2 is true thanks to the parameter ε in the geometric convexification. Indeed, for $|x - x_0| \geq R/2$, we have $\Phi_\varepsilon(x) \leq \Phi(x) - \varepsilon R^2/4$. If $\Phi(x) \leq 0$, this implies $\Phi_\varepsilon(x) \leq -\varepsilon R^2/4$, so that we can take $\eta = -\varepsilon R^2/4$. From this point forward, if

$u \in C^\infty(\Omega)$, we take $\chi \in C_c^\infty(B(x_0, R))$ such that $\chi = 1$ on $B(x_0, R/2)$, and we follow the strategy described in Section 1.2.5 to conclude the proof of Theorem 17: we apply the Carleman estimate (53) to $w = \chi u$, leading to (13) (in which Φ has to be replaced by Φ_ε). This implies (14) (still with Φ replaced by Φ_ε) and Lemma 9 finally implies that χu vanishes identically in $\{\Phi_\varepsilon \geq -\eta\}$, which contains a neighborhood of x_0 .

Note finally that, in order for the result to hold for all $u \in H^1(\Omega)$ such $Pu = 0$, we first use Lemma 28, which yields (53) for all $w \in H^1(\Omega)$, with $\text{supp } w \subset B(x_0, R)$ and $Pw \in L^2(\Omega)$. Second, we check that the argument in Section 1.2.5 works with no change: setting $w = \chi u$ we have $w \in H^1(\Omega)$ with $\text{supp}(w) \subset \text{supp}(\chi) \subset B(x_0, R)$ and $Pw = 0 + [P, \chi]u \in L^2(\Omega)$ since $[P, \chi] \in \text{Diff}^1(\Omega)$ and $u \in H^1(\Omega)$. Hence, the strategy described in Section 1.2.5 applies and the remainder of the proof of Theorem 17 follows in this case. $\square$

2.4. Necessity of strong pseudoconvexity for stable unique continuation. – In this section, we discuss optimality/limitations of Hörmander’s theorem for the wave operator, via two counterexamples due to Alinhac and Alinhac-Baouendi, respectively. We recall (see Remark 27) that Hörmander’s theorem is insensitive to the addition of lower-order terms to the operator. The following result is a particular case of [4, Théorème 2].

Theorem 41 (Alinhac). – *Let Ω be an open subset of $\mathbb{R}^n$, let $x_0 \in \Omega$, and let $P \in \text{Diff}^2(\Omega)$ with real principal symbol $p_2(x, \xi) = a^{ij}(x)\xi_i\xi_j$. Assume a^{ij} is a smooth symmetric real-valued matrix defined in a neighborhood of x_0 and such that $(a^{ij}(x_0))$ is non-degenerate. Let $\Psi \in C^\infty(\Omega; \mathbb{R})$ be such that $\Psi(x_0) = 0$ and $p_2(x_0, d\Psi(x_0)) \neq 0$, and assume that there exists $\xi_0 \in \mathbb{R}^n \setminus \{0\}$ such that*

$$(54) \quad p_2(x_0, \xi_0) = \{p_2, \Psi\}(x_0, \xi_0) = 0 \quad \text{and} \quad \{p_2, \{p_2, \Psi\}\}(x_0, \xi_0) < 0.$$

Then, there exist $U \subset \Omega$ a neighborhood of x_0 and $q, u \in C^\infty(U; \mathbb{C})$ such that

$$Pu + qu = 0 \text{ in } U, \quad \text{and} \quad \text{supp}(u) \cap U = \{\Psi \geq 0\} \cap U.$$

In particular, under Assumption (54) on the oriented hypersurface $\{\Psi = 0\}$, unique continuation from $\{\Psi > 0\}$ to a neighborhood of x_0 does not hold for the operator $P + q$. This applies to the wave operator. Condition (54) is a strong negation of strong pseudoconvexity (Definition 16) and thus Theorem 41 is an almost converse to Theorem 17 if one considers “stable unique continuation” for P , that is to say unique continuation for all zero-order perturbations of P .

Note that Assumption (54) can be reformulated on the tangent space as in Example 22. If we denote by $a(x) = a_{ij}(x) = (a^{ij}(x))^{-1}$ the pre-dual (pseudo-Riemannian) metric as in Example 22, then Assumption (54) is equivalent to: there exists $X_0 \in T_{x_0}\Omega \setminus \{0\}$ such that

$$(X_0, X_0)_a = 0, \quad d\Psi(x_0)(X_0) = 0 \quad \text{and} \quad \text{Hess}_a \Psi(x_0)(X_0, X_0) < 0.$$

The following result is another counterexample to stable unique continuation in the limit case where $\{p_2, \{p_2, \Psi\}\}(x_0, \xi_0) = 0$. It is a particular case of [5, Théorème 2], [6, Theorem].

Theorem 42 (Alinhac-Baouendi). – Assume $d \geq 2$ and consider

$$P := D_t^2 - D_{x_1}^2 - D_{x_2}^2 - \cdots - D_{x_d}^2$$

near the point $0 \in \mathbb{R}^{1+d}$. There exist an open set $U \subset \mathbb{R}^{1+d}$ with $0 \in U$ and $q, u \in C^\infty(U; \mathbb{C})$ such that

$$Pu + qu = 0 \text{ in } U, \quad \text{and} \quad \text{supp}(u) \cap U = \{x_1 \geq 0\} \cap U.$$

A similar counterexample with an additional first-order term is already obtained in [29, Theorem 8.9.4]. This implies that unique continuation fails for the operator $P + q$ across the hypersurface $\{x_1 = 0\}$, even though P has constant coefficients (hence the Holmgren-John Theorem 8 applies to P across $\{x_1 = 0\}$) and the perturbation q is of lower-order and smooth. Note that the principal symbol of $P + q$ is $p_2(t, x, \xi_t, \xi_x) = \xi_t^2 - \xi_{x_1}^2 - \xi_{x_2}^2 - \cdots - \xi_{x_d}^2$ and the hypersurface $\{x_1 = 0\}$ barely fails to be strongly pseudoconvex (see Definition 16). Indeed, we have $\{p_2, x_1\} = -2\xi_{x_1}$ and $\{p_2, \{p_2, x_1\}\} = 0$ so that, if one chooses $\xi_0 := (1, 0, 1, 0, \dots, 0)$ (that is to say $\xi_t = \xi_{x_2} = 1$, $\xi_{x_1} = 0$ and $\xi_{x_j} = 0$ for $j \in \{3, \dots, d\}$), we have

$$p_2(0, \xi_0) = 0, \quad \{p_2, x_1\}(0, \xi_0) = 0 \quad \text{and} \quad \{p_2, \{p_2, x_1\}\}(x_0, \xi_0) = 0.$$

In some sense, this is a weaker form of violation of strong pseudoconvexity (see Definition 16) compared to (54). Note also the regularity in Theorem 42 has been pushed by Hörmander: he proved in [35] that the potential q and the solution u in Theorem 42 can be taken in the Gevrey class G^α for any $\alpha > 1$ (the case $\alpha = 1$ corresponding to analytic functions), or even in more general Denjoy-Carleman classes of functions.

Finally, a family of semiglobal counterexamples has been constructed by Kumano-go [47] (inspired by a construction by John [42]) and Takase [84]. Kumano-go [47] proved existence of $u, g, q \in C^\infty(\mathbb{R}^{1+2}; \mathbb{R})$ real-valued solving (for the Euclidean metric) the damped wave equation $(\partial_t^2 - \Delta + g\partial_t + q)u = 0$ in $\mathbb{R}^{1+2}$ with $\text{supp}(u) = \mathbb{R} \times (\mathbb{R}^2 \setminus B(0, 1))$. Takase [84], proved existence of $u, q \in C^\infty(\mathbb{R}^{1+2}; \mathbb{C})$ solving (for the Euclidean metric) $(\partial_t^2 - \Delta + q)u = 0$ in $\mathbb{R}^{1+2}$ and $\text{supp}(u) = \mathbb{R} \times (\mathbb{R}^2 \setminus B(0, 1))$.

3. Unique continuation for time-independent wave operators

To summarize the discussion so far, if one considers a general wave operator $P = D_i a^{ij} D_j +$ lower-order terms, we have on the one hand the Holmgren-John Theorem 8: we assume analyticity of all coefficients and unique continuation holds across any noncharacteristic hypersurface. The latter geometric condition appears to be the appropriate one in applications (see Sections 3.4 and 3.5 below) and is essentially optimal, whereas analyticity is a very demanding condition. On the other hand the Hörmander Theorem 17 assumes only C^∞ regularity of the principal

part of the operator and L^∞ regularity of the lower-order terms (these regularity assumptions can even be relaxed, see [29], but we do not discuss this issue here) and yields unique continuation across any strongly pseudoconvex hypersurface. The regularity assumption is suitable for applications, but the geometric pseudoconvexity condition is extremely demanding (see, e.g., the geometric discussion in Section 2.1.2 and the examples in Section 2.1.3). As explained in Section 2.4, it is however optimal if one considers stable unique continuation.

In the present chapter, we explore further the case where all coefficients of the wave operator are *time-independent* with only finite regularity in space. The result we present has a long history with several ancestors and descendants. A historical account is provided in Section 4.2. One important fact noticed along the way is the role of time analyticity (which obviously holds in the time-independent case). Here, we focus our attention to the operator $\partial_t^2 + Q$ where $Q = q(x, D_x)$ is a positive elliptic operator. The result presented in this chapter has been proved in [85] and our presentation is inspired by [34].

3.1. Setting and statement of the unique continuation result. – In what follows, we denote the generic variable by $\mathbf{x} = (t, x) \in \mathbb{R}^{1+d} = \mathbb{R}^n$ (note the slight difference with the notation in Section 2, where $x \in \mathbb{R}^n$ includes the time variable) with dual variable $\xi = (\xi_t, \xi_x) \in \mathbb{R}^{1+d}$. The main theorem of this section is as follows.

Theorem 43 (Wave type operator with time-independent coefficients). –

Let $T > 0$ and Ω_x be an open set of $\mathbb{R}^d$. Set $\Omega =]-T, T[\times \Omega_x$. Let

$$Q = \sum_{i,j=1}^d g^{ij}(x) D_i D_j + \sum_{k=1}^d b_k(x) D_k + c(x)$$

be a differential operator of order 2 with $g^{ij} \in C^\infty(\Omega_x)$ real-valued, $b_k, c \in L^\infty(\Omega_x)$. Assume also that Q is positive elliptic, i.e., there exists $C > 0$ so that

$$(55) \quad q(x, \xi_x) := \sum_{i,j=1}^d g^{ij}(x) \xi_i \xi_j \geq C |\xi_x|^2, \text{ for all } (x, \xi_x) \in \Omega_x \times \mathbb{R}^d.$$

Define $P = \partial_t^2 + Q$ on Ω and set $p_2(t, x, \xi_t, \xi_x) := -\xi_t^2 + q(x, \xi_x)$. Let $\mathbf{x}_0 = (t_0, x_0) \in \Omega$ and suppose $\Psi \in C^2(\Omega)$ with $d\Psi(\mathbf{x}_0) \neq 0$ such that $p_2(\mathbf{x}_0, d\Psi(\mathbf{x}_0)) \neq 0$, i.e.,

$$(\partial_t \Psi(\mathbf{x}_0))^2 \neq \sum_{i,j} g^{ij}(x_0) (\partial_i \Psi(\mathbf{x}_0)) (\partial_j \Psi(\mathbf{x}_0)).$$

Then, there exists a neighborhood V of $\mathbf{x}_0$ so that for any $u \in H^1(\Omega)$,

$$(56) \quad \begin{cases} Pu = 0 & \text{in } \Omega, \\ u = 0 & \text{in } \Omega \cap \{\Psi > \Psi(\mathbf{x}_0)\} \end{cases} \implies u = 0 \text{ on } V.$$

Note that there is no connection between the strong pseudoconvexity condition in Definition 16 and the non-characteristicity condition in Theorem 43. Many useful hypersurfaces are non-characteristic but not strongly pseudoconvex (see Section 3.4 below), but one may also construct characteristic hypersurfaces that are strongly pseudoconvex (Indeed, the former condition is a first-order condition whereas the latter is a second-order condition). The result of Theorem 43 actually holds under the weaker condition that the hypersurface $\{\Psi = 0\}$ is strongly pseudoconvex in $\xi_t = 0$ (see Definition 47 below). However, in all applications we have in mind, only the condition of being non-characteristic is useful to deduce optimal results. See Sections 3.4 and 3.5 below.

Remark 44. – As in Example 22 concerning the pseudoconvexity condition, the non-characteristicity condition $p_2(\mathbf{x}_0, d\Psi(\mathbf{x}_0)) \neq 0$ has a tangent counterpart. Denoting again by $a = (a_{ij})$ the metric on the tangent space $T((-T, T) \times \Omega_x)$, the hypersurface $S = \{\Psi = 0\}$ is non-characteristic at $\mathbf{x}_0 \in S$ if and only if $(\nabla_a \Psi, \nabla_a \Psi)_a(\mathbf{x}_0) \neq 0$. Notice that under the assumptions of Theorem 43, the metric a is assumed of the (particular) form $a(X, Y) = (X, Y)_a = -X_t Y_t + g(X_x, Y_x)$ where g is a time-independent Riemannian metric and $X = (X_t, X_x), Y = (Y_t, Y_x)$.

The proof of Theorem 43 relies on an inequality of Carleman type, but with an additional weight in the Fourier variable. Namely, we let $e^{-\varepsilon \frac{|D_t|^2}{2\tau}}$ be the Fourier multiplier defined naturally by

$$\mathcal{F} \left(e^{-\varepsilon \frac{|D_t|^2}{2\tau}} u \right) (\xi) = e^{-\varepsilon \frac{|\xi_t|^2}{2\tau}} \widehat{u}(\xi), \quad u \in \mathcal{S}(\mathbb{R}^{1+d}),$$

where ξ_t is the Fourier variable corresponding to the variable t and $\xi = (\xi_t, \xi_x)$. Note that this amounts to solving the heat equation with t as a “spatial” variable, during a “time” $\frac{\varepsilon}{2\tau}$. Using the explicit expression of the Fourier transform of the Gaussian $e^{-\varepsilon \frac{|\xi_t|^2}{2\tau}}$, this may be rewritten as a convolution with a heat kernel:

$$(57) \quad \left(e^{-\varepsilon \frac{|D_t|^2}{2\tau}} u \right) (t, x) = \left(\frac{\tau}{2\pi\varepsilon} \right)^{1/2} \int_{\mathbb{R}} e^{-\frac{\tau}{2\varepsilon}(t-s)^2} u(s, x) ds.$$

This operator has several interesting features: it localizes close to $D_t = 0$ (i.e., in low frequencies w.r.t. the time variable t), in an analytic way (the function $e^{-\varepsilon \frac{|D_t|^2}{2\tau}} u$ produced is an entire function in the t -variable). However (and consequently), note that $e^{-\frac{\varepsilon}{2\tau}|D_t|^2}$ is not local; in particular, $e^{-\varepsilon \frac{|D_t|^2}{2\tau}} u$ is not compactly supported, even if u is.

For a smooth real-valued weight function Φ (later on, we will assume that it is polynomial of order 2), the Carleman estimate below will make use of the operator

$$Q_{\varepsilon, \tau}^{\Phi} u := e^{-\varepsilon \frac{|D_t|^2}{2\tau}} e^{\tau \Phi} u.$$

The following is an analogue of the Definition 29, under which the Carleman estimate of Theorem 25 holds. Here, the condition is weaker for it is only restricted to $\xi_t = 0$.

Definition 45 (Pseudoconvex Carleman weight in $\xi_t = 0$). – With the above assumptions for P , let Φ be smooth and real-valued. We say that Φ is a pseudoconvex Carleman weight for P in $\xi_t = 0$ at $\mathbf{x}_0$ if

$$(58) \quad \{p_2, \{p_2, \Phi\}\}(\mathbf{x}_0, \xi) > 0, \text{ if } p_2(\mathbf{x}_0, \xi) = 0, \quad \xi_t = 0, \quad \xi \neq 0;$$

$$(59) \quad \frac{1}{i\tau} \{\bar{p}_\Phi, p_\Phi\}(\mathbf{x}_0, \xi, \tau) > 0, \text{ if } p_\Phi(\mathbf{x}_0, \xi, \tau) = 0, \quad \xi_t = 0, \quad \tau > 0,$$

where $p_\Phi(z, \xi, \tau) = p_2(z, \xi + i\tau d\Phi(z))$.

Theorem 46 (Carleman estimate for wave type operators with coefficients constant in time). –

Suppose P is as in Theorem 43. Let Φ be a quadratic real-valued polynomial such that Φ is a pseudoconvex Carleman weight for P in $\xi_t = 0$ at $\mathbf{x}_0$, in the sense of Definition 45. Then, there exist $r, \varepsilon, d, C, \tau_0 > 0$ such that for all $\tau \geq \tau_0$ and $w \in H_{\text{comp}}^1(B(\mathbf{x}_0, r))$, we have

$$(60) \quad \tau \|Q_{\varepsilon, \tau}^\Phi w\|_{H_\tau^1}^2 \leq C \|Q_{\varepsilon, \tau}^\Phi Pw\|_{L^2}^2 + Ce^{-d\tau} \|e^{\tau\Phi} w\|_{H_\tau^1}^2.$$

Note that for $\varepsilon = 0$, this would be a classical Carleman estimate. Yet, the role of the Fourier multiplier $e^{-\varepsilon \frac{|D_t|^2}{2\tau}}$ is to truncate the “high frequencies” (in comparison with τ) in the variable t . Consequently, we just need to consider small frequencies in ξ_t , say $|\xi_t| \leq \varsigma\tau$. Note that the set $|\xi_t| \geq \varsigma\tau$ yields a contribution of order $e^{-\varepsilon\varsigma^2\tau/2}$ to the estimate, which is an admissible remainder in view of (60). This explains why the pseudoconvexity assumption is only made in $\xi_t = 0$.

3.2. Proving unique continuation using the Carleman estimate. – In this section, we assume that Theorem 46 holds and we prove Theorem 43. Part of the argument is similar to the classical case: constructing an appropriate pseudoconvex Carleman weight Φ in $\xi_t = 0$ at $\mathbf{x}_0$ from the function Ψ defining the hypersurface $S = \{\Psi = \Psi(\mathbf{x}_0)\}$. The main differences are the following:

- pseudoconvexity holds only on $\xi_t = 0$, which requires a small adaptation of the convexification procedure. Moreover, Φ is required to be quadratic.
- the Carleman estimate implies an additional Fourier multiplier $(e^{-\varepsilon \frac{|D_t|^2}{2\tau}})$. This leads to changes in the proof of unique continuation. The additional difficulty comes from the fact that the Carleman estimate (60) only dominates low t -frequencies of the function u .

3.2.1. *Convexification.* – Similarly to the classical case studied in Section 2, the natural assumption for the unique continuation Theorem 43 is a strong pseudoconvexity condition similar to that of Definition 34, but restricted to the set $\{\xi_t = 0\}$. We define this notion, and then check that any noncharacteristic hypersurface is strongly pseudoconvex in $\xi_t = 0$.

Definition 47 (Strongly pseudoconvex hypersurface in $\xi_t = 0$). –

Let $\Omega \ni \mathbf{x}_0$ be an open set, $P \in \text{Diff}^2(\Omega)$ with real-valued principal symbol p_2 and $\Psi \in C^\infty(\Omega)$ real-valued. We say that the *oriented hypersurface* $S = \{\Psi = \Psi(\mathbf{x}_0)\} \ni \mathbf{x}_0$ is strongly pseudoconvex with respect to P at $\mathbf{x}_0$ in $\xi_t = 0$ if

$$(61) \quad \{p_2, \{p_2, \Psi\}\}(\mathbf{x}_0, \xi) > 0, \text{ if } p_2(\mathbf{x}_0, \xi) = \{p_2, \Psi\}(\mathbf{x}_0, \xi) = \xi_t = 0 \text{ and } \xi \neq 0;$$

$$(62) \quad \frac{1}{i\tau} \{\overline{p_\Psi}, p_\Psi\}(\mathbf{x}_0, \xi, \tau) > 0, \text{ if } p_\Psi(\mathbf{x}_0, \xi, \tau) = \{p_\Psi, \Psi\}(\mathbf{x}_0, \xi, \tau) = \xi_t = 0 \text{ and } \tau > 0,$$

where $p_\Psi(z, \xi, \tau) = p_2(z, \xi + i\tau d\Psi(z))$.

The next lemma explains that the noncharacteristicity condition assumed in Theorem 43 is a particular case of Definition 47.

Lemma 48 (Noncharacteristicity implies strong pseudoconvexity in $\xi_t = 0$). –

Suppose Ω, P are as in Theorem 43. If the hypersurface $S = \{\Psi = \Psi(\mathbf{x}_0)\} \ni \mathbf{x}_0$ is noncharacteristic for P at $\mathbf{x}_0$ ($p_2(\mathbf{x}_0, d\Psi(\mathbf{x}_0)) \neq 0$), then it is strongly pseudoconvex with respect to P at $\mathbf{x}_0$ in $\xi_t = 0$.

Proof. – The principal symbol of P is $p_2(t, x, \xi_t, \xi_x) = -\xi_t^2 + q(x, \xi_x)$ where $q(x, \xi_x) = \sum_{i,j} a^{ij}(x) \xi_i \xi_j$. We first notice that for $\xi_t = 0$, we have $p_2(t, x, 0, \xi_x) = q(x, \xi_x)$. Since q is assumed to be elliptic, the assumption $p(\mathbf{x}_0, \xi) = \xi_t = 0$ implies $\xi = 0$ and therefore Condition (61) holds trivially. Second, we have proved in Lemma 39 that (62) holds true if $p_2(\mathbf{x}_0, d\Psi(\mathbf{x}_0)) \neq 0$, which concludes the proof. $\square$

Next, we follow the same convexification procedure as Section 2.3.1.

Proposition 49 (Analytic convexification). – Suppose Ω, P are as in Theorem 43. Assume that the hypersurface $S = \{\Psi = \Psi(\mathbf{x}_0)\}$ is strongly pseudoconvex with respect to P at $\mathbf{x}_0$ in $\xi_t = 0$, in the sense of Definition 47. Then, there exists $\lambda_0 > 0$ such that for all $\lambda \geq \lambda_0$, the function $\Phi = e^{\lambda\Psi}$ is a pseudoconvex Carleman weight for P in $\xi_t = 0$ at $\mathbf{x}_0$, in the sense of Definition 45.

Note however that, as opposed to the classical case, the Carleman estimate of Theorem 46 does not apply to the weight function Φ since it is not quadratic. This issue is carried out in the following proposition.

Proposition 50 (Geometric convexification). – Let Φ be a pseudoconvex Carleman weight for P in $\xi_t = 0$ at $\mathbf{x}_0$, in the sense of Definition 45 with $\Phi(\mathbf{x}_0) = 0$. Then, there exists a function φ such that

$$(1) \quad \varphi \text{ is a pseudoconvex Carleman weight for } P \text{ in } \xi_t = 0 \text{ at } \mathbf{x}_0,$$

- (2) φ is a quadratic polynomial,
 (3) $\varphi(\mathbf{x}_0) = 0$ and there exists $R_0 > 0$ such that for any $0 < R < R_0$, there exists $\eta > 0$ so that $\varphi(\mathbf{x}) \leq -\eta$ for $\mathbf{x} \in \{\Phi \leq 0\} \cap \{R/2 \leq |\mathbf{x} - \mathbf{x}_0| \leq R\}$.

Proof of Proposition 49. – The proof is very similar to that of Proposition 36. Again, assume that $\Psi(\mathbf{x}_0) = 0$ for simplicity, and use the notation $c_\Psi(\xi, \tau)$, $c_\Phi(\xi, \tau)$ as in (49). Lemma 30 applies and $c_\Psi(\xi, \tau)$ and $c_\Phi(\xi, \tau)$ are both continuous on the whole $\mathbb{R}^n \times \mathbb{R}^+$. Then, using Lemma 32, Definition 47 may be equivalently reformulated as the existence of constants $C_1, C_2 > 0$ so that

$$c_\Psi(\xi, \tau) + C_1 \left[|\{p_\Psi, \Psi\}(\mathbf{x}_0, \xi, \tau)|^2 + \frac{|p_\Psi(\mathbf{x}_0, \xi, \tau)|^2}{|\xi|^2 + \tau^2} + |\xi_t|^2 \right] \geq C_2(|\xi|^2 + \tau^2).$$

By Lemma 37, the argument in the Proof of Proposition 36 gives

$$c_\Phi(\xi, \tau) + \tilde{C}_1 \left[\frac{|p_\Phi(\mathbf{x}_0, \xi, \tau)|^2}{|\xi|^2 + \tau^2} + |\xi_t|^2 \right] \geq \tilde{C}_2(|\xi|^2 + \tau^2),$$

for λ large enough. This implies the sought result. $\square$

Proof of Proposition 50. – For $\delta > 0$, set $\varphi(\mathbf{x}) = \Phi_T(\mathbf{x}) - \delta|\mathbf{x} - \mathbf{x}_0|^2$, where

$$\Phi_T(\mathbf{x}) = \sum_{|\alpha| \leq 2} \frac{1}{\alpha!} (\partial^\alpha \Phi)(\mathbf{x}_0) (\mathbf{x} - \mathbf{x}_0)^\alpha$$

is the Taylor expansion of Φ at order 2. First, we notice that the pseudoconvexity condition only involves derivative up to order 2 at $\mathbf{x}_0$. Hence, Φ_T is also a pseudoconvex Carleman weight in $\xi_t = 0$ at $\mathbf{x}_0$. Moreover, the same stability argument performed in the proof of Proposition 33 applies. For δ small enough, φ is also a pseudoconvex Carleman weight in $\xi_t = 0$ at $\mathbf{x}_0$. Fix $\delta > 0$ sufficiently small. It remains to prove the geometric properties of Point (3). Since Φ_T is the Taylor expansion of Φ at order 2, there exists R_0 small enough so that $|\Phi_T - \Phi| \leq |\mathbf{x} - \mathbf{x}_0|^2 \delta/2$ for $|\mathbf{x} - \mathbf{x}_0| \leq R_0$. Now, take $R \leq R_0$. Let $\mathbf{x} \in \{\Phi \leq 0\} \cap \{R/2 \leq |\mathbf{x} - \mathbf{x}_0| \leq R\}$. Since $\Phi(\mathbf{x}) \leq 0$, we have $\Phi_T(\mathbf{x}) \leq |\mathbf{x} - \mathbf{x}_0|^2 \delta/2$. Therefore, $\varphi(\mathbf{x}) \leq -\delta|\mathbf{x} - \mathbf{x}_0|^2/2$. Since $|\mathbf{x} - \mathbf{x}_0|^2 \geq R^2/4$, we get $\varphi(\mathbf{x}) \leq -\delta R^2/8$ and Point (3) holds for $\eta = \delta R^2/8$. $\square$

3.2.2. Unique continuation. – In this section, we conclude the proof of the unique continuation Theorem 43 assuming that the Carleman estimate of Theorem 46 holds.

Proof of Theorem 43. – Let u be a solution of $Pu = 0$ in Ω such that $u = 0$ on $\Omega \cap \{\Psi > 0\}$. The hypersurface $S = \{\Psi = \Psi(\mathbf{x}_0)\}$ is strongly pseudoconvex at $\mathbf{x}_0$ in $\xi_t = 0$. Propositions 49 and 50 allow one to produce a quadratic function Φ (it is the function called φ in Proposition 50, which we now call Φ) that satisfies the pseudoconvexity for functions at $\mathbf{x}_0$ in $\xi_t = 0$. With this weight function, Theorem 46 applies. We therefore obtain:

- (1) there exist $R, \varepsilon, \mathbf{d}, C, \tau_0 > 0$ such that the Carleman estimate (60) holds for all $\tau \geq \tau_0$ and $w \in H_{\text{comp}}^1(B(\mathbf{x}_0, r))$,

- (2) $\Phi(\mathbf{x}_0) = 0$ and there exists $\eta > 0$ so that $\Phi(\mathbf{x}) \leq -\eta$
for $\mathbf{x} \in \{\Psi \leq 0\} \cap \{|\mathbf{x} - \mathbf{x}_0| \geq R/2\}$,
- (3) $\Phi(\mathbf{x}) \leq d/4$ in $B(\mathbf{x}_0, R)$.

The third property follows from a continuity argument (holding up to reducing R) using $\Phi(\mathbf{x}_0) = 0$. Now we pick $\chi \in C_c^\infty(B(\mathbf{x}_0, R))$ so that $\chi = 1$ on $B(\mathbf{x}_0, R/2)$. We apply the Carleman estimate (60) to $w = \chi u \in H_{\text{comp}}^1(B(\mathbf{x}_0, R))$, solution of $Pw = \chi Pu + [P, \chi]u = [P, \chi]u$. Again, $[P, \chi]$ is a classical differential operator of order 1 with coefficients supported in the set $\{\frac{R}{2} \leq |\mathbf{x} - \mathbf{x}_0| \leq R\}$. Moreover, $\text{supp}(u) \subset \{\Psi \leq 0\}$, and thus $[P, \chi]u$ is supported in $\{\Psi \leq 0\} \cap \{\frac{R}{2} \leq |\mathbf{x} - \mathbf{x}_0| \leq R\}$, where $\Phi \leq -\eta$. This gives $\|Q_{\varepsilon, \tau}^\Phi Pw\|_{L^2} \leq \|e^{\tau\Phi} Pw\|_{L^2} \leq Ce^{-\tau\eta} \|u\|_{H^1}$. As for the second term in the right-hand side of (60), we use Item 3 to deduce

$$e^{-d\tau} \|e^{\tau\Phi} w\|_{H_\tau^1}^2 \leq e^{-d\tau} e^{d\tau/2} \|w\|_{H_\tau^1}^2 \leq e^{-d\tau/2} \tau^2 \|w\|_{H^1}^2 \leq Ce^{-d\tau/4} \|u\|_{H^1}^2.$$

From (60), we have obtained that there exist $C, \delta, \tau_0 > 0$ so that

$$(63) \quad \|Q_{\varepsilon, \tau}^\Phi w\|_{L^2} \leq Ce^{-\delta\tau}, \quad \text{for all } \tau \geq \tau_0.$$

The following lemma is an analogue to Lemma 9 in the present setting.

Lemma 51. – *Let $\Phi \in C^\infty(\Omega)$ be a real-valued function such that $d\Phi \neq 0$ on Ω . Let $v \in L_{\text{comp}}^2(\Omega)$ and assume there exists $C_0, \tau_0, \varepsilon > 0$ such that*

$$(64) \quad \|Q_{\varepsilon, \tau}^\Phi v\|_{L^2} \leq C_0 \quad \text{for all } \tau \geq \tau_0.$$

Then, v is supported in $\{\Phi \leq 0\}$.

To apply the lemma, we rewrite (63) as $\|Q_{\varepsilon, \tau}^\Phi e^{\delta\tau} w\|_{L^2} \leq C$, i.e., $\|Q_{\varepsilon, \tau}^{\Phi+\delta} w\|_{L^2} \leq C$. Lemma 51 applies to the function $\Phi + \delta$ and implies that w is supported in the set $\{\Phi + \delta \leq 0\} = \{\Phi \leq -\delta\}$. Since $\Phi(\mathbf{x}_0) = 0$ and $\chi = 1$ on $B(\mathbf{x}_0, R/2)$, the set $V = B(\mathbf{x}_0, R/2) \cap \{\Phi > -\delta\}$ is a neighborhood of $\mathbf{x}_0$ on which $u = w = 0$, concluding the proof of the theorem. $\square$

For the proof to be complete, it remains to prove Lemma 51. Note that if we had $\varepsilon = 0$, this is precisely Lemma 9. We first give a sketch of its proof to present the main new ideas, recalling that $n = 1 + d$:

- (1) Proving that $\text{supp}(v) \subset \{\Phi \leq 0\}$ is equivalent to proving that $(\chi \circ \Phi)v$ vanishes a.e. on $\mathbb{R}^n$ for all test function $\chi \in C^\infty(\mathbb{R})$, such that $\text{supp}(\chi) \subset [0, +\infty)$. This may be reformulated equivalently in a weak form as

$$\int_{\mathbb{R}^n} f(\mathbf{x})v(\mathbf{x})\chi(\Phi(\mathbf{x}))d\mathbf{x} = 0, \quad \text{for all } f \in \mathcal{S}(\mathbb{R}^n), \chi \in C^\infty(\mathbb{R}), \text{supp}(\chi) \subset [0, +\infty).$$

- (2) We change slightly the point of view and, considering f fixed, see this quantity as a distribution on $\mathbb{R}$, with χ as test function:

$$(65) \quad \langle h_f, \chi \rangle_{\mathcal{E}'(\mathbb{R}), C^\infty(\mathbb{R})} := \langle f v, \chi(\Phi) \rangle_{\mathcal{E}'(\mathbb{R}^n), C^\infty(\mathbb{R}^n)} = \int_{\mathbb{R}^n} f(\mathbf{x})v(\mathbf{x})\chi(\Phi(\mathbf{x}))d\mathbf{x}.$$

This corresponds to defining the distribution $h_f = \Phi_*(fv)$. Heuristically, $h_f(s)$ is the integral of fv on the level set $\{\Phi(x) = s\}$. According to the first point, $\text{supp}(v) \subset \{\Phi \leq 0\}$ is now equivalent to $\text{supp}(h_f) \subset (-\infty, 0]$.

(3) We shall see that the Fourier transform of h_f is

$$\widehat{h_f}(\zeta) = \langle h_f, s \mapsto e^{-i\zeta s} \rangle_{\mathcal{E}'(\mathbb{R}), C^\infty(\mathbb{R})} = \int_{\mathbb{R}^n} f(\mathbf{x})v(\mathbf{x})e^{-i\zeta\Phi(\mathbf{x})}d\mathbf{x},$$

and can be extended to the complex domain if v is compactly supported (which is assumed here). In particular, for $\zeta \in i\mathbb{R}^+$, $\zeta = i\tau$, we have $\widehat{h_f}(i\tau) = \langle f, ve^{\tau\Phi} \rangle$. The assumption (64) gives an information on the norm of $e^{\tau\Phi}v$ for τ large which can be translated in a uniform bound on $|\widehat{h_f}|$ on the upper imaginary axis. A Phragmén-Lindelöf type argument allows one to transfer this uniform bound on $|\widehat{h_f}|$ to the whole upper half complex plan.

(4) From the bound $|\widehat{h_f}| \leq C$ on the whole upper half plan, a Paley-Wiener theorem (roughly saying $\text{supp}(g) \subset (-\infty, 0] \iff |\hat{g}| \leq C$ uniformly on the upper half complex plane) allows one to conclude that $\text{supp}(h_f) \subset (-\infty, 0]$ for all f , which is the sought result according to the second point.

Let us now proceed with the details of the proof.

Proof of Lemma 51. – Let $f \in \mathcal{S}(\mathbb{R}^n)$ have its Fourier transform $\widehat{f}$ compactly supported in $B(0, R)$ for R large. We define the distribution $h_f \in \mathcal{E}'(\mathbb{R})$ by (65). Note that h_f is a distribution of order zero since

$$|\langle h_f, \chi \rangle_{\mathcal{E}'(\mathbb{R}), C^\infty(\mathbb{R})}| \leq \int_{\mathbb{R}^n} |f(\mathbf{x})v(\mathbf{x})||\chi(\Phi(\mathbf{x}))|d\mathbf{x} \leq \|f\|_{L^2} \|v\|_{L^2} \sup_{\Phi(\text{supp}(v))} |\chi|,$$

and is indeed compactly supported because

$$\text{supp}(h_f) \subset \Phi(\text{supp}(v)) = \{\Phi(\mathbf{x}); \mathbf{x} \in \text{supp}(v)\},$$

which is compact. Since $h_f \in \mathcal{E}'(\mathbb{R})$, the Fourier transform of h_f can be computed for $\zeta \in \mathbb{R}$ by

$$\widehat{h_f}(\zeta) = \langle h_f, e^{-is\zeta} \rangle_{\mathcal{E}'(\mathbb{R}_s), C^\infty(\mathbb{R}_s)} = \langle fv, e^{-i\zeta\Phi} \rangle_{\mathcal{E}'(\mathbb{R}^n), C^\infty(\mathbb{R}^n)} = \int_{\mathbb{R}^n} f(\mathbf{x})v(\mathbf{x})e^{-i\zeta\Phi(\mathbf{x})}d\mathbf{x}.$$

We notice that this formula still defines a function for $\zeta \in \mathbb{C}$ satisfying the bound (66)

$$|\widehat{h_f}(\zeta)| \leq \int_{\text{supp}(v)} |f(\mathbf{x})v(\mathbf{x})|e^{\text{Im}(\zeta)\Phi(\mathbf{x})}d\mathbf{x} \leq e^{C_1|\text{Im}(\zeta)|} \|f\|_{L^2} \|v\|_{L^2}, \quad C_1 = \max_{\text{supp}(v)} |\Phi|.$$

Holomorphy of the integrand with respect to ζ implies that $\widehat{h_f}(\zeta)$ is holomorphic on the whole $\mathbb{C}$. The Cauchy-Schwarz inequality (66) yields the general bound

$$|\widehat{h_f}(\zeta)| \leq \|f\|_{L^2} \|v\|_{L^2} = C_{f,v}, \quad \text{for } \zeta \in \mathbb{R}.$$

Now, we use the assumption of the lemma, namely (64) to obtain a bound on the upper imaginary axis. Let $\zeta = i\tau$, and $\tau \geq \tau_0$, (64) implies

$$\begin{aligned}
|\widehat{h_f}(i\tau)| &= |\langle f v, e^{\tau\Phi} \rangle_{\mathcal{E}'(\mathbb{R}^n), C^\infty(\mathbb{R}^n)}| = |\langle f, v e^{\tau\Phi} \rangle_{\mathcal{S}'(\mathbb{R}^n), \mathcal{S}(\mathbb{R}^n)}| \\
&= \left| \langle e^{\varepsilon \frac{|D_t|^2}{2\tau}} f, e^{-\varepsilon \frac{|D_t|^2}{2\tau}} v e^{\tau\Phi} \rangle_{\mathcal{S}'(\mathbb{R}^n), \mathcal{S}(\mathbb{R}^n)} \right| \\
&\leq \left\| e^{\varepsilon \frac{|D_t|^2}{2\tau}} f \right\|_{L^2(\mathbb{R}^n)} \|Q_{\varepsilon, \tau}^\Phi v\|_{L^2(\mathbb{R}^n)} \\
&\leq \left\| e^{\varepsilon \frac{|D_t|^2}{2\tau}} \right\|_{L^\infty(\text{supp}(\widehat{f}))} \|f\|_{L^2(\mathbb{R}^n)} \|Q_{\varepsilon, \tau}^\Phi v\|_{L^2(\mathbb{R}^n)} \\
&\leq C_0 e^{\frac{\varepsilon R^2}{2\tau}} \|f\|_{L^2(\mathbb{R}^n)} \leq C_0 e^{\frac{\varepsilon R^2}{2\tau_0}} \|f\|_{L^2(\mathbb{R}^n)} = C_{\varepsilon, f, \tau_0} C_0.
\end{aligned}$$

Note that the operator $e^{\varepsilon \frac{|D_t|^2}{2\tau}}$ is harmless as the Fourier transform of f is assumed compactly supported in $B(0, R)$ (in general, $e^{\varepsilon \frac{|D_t|^2}{2\tau}} f$ has no meaning, even for $f \in \mathcal{S}(\mathbb{R}^n)$; this is the reason for working by duality). Moreover, for $\tau \in [0, \tau_0]$, the estimate $|\widehat{h_f}(i\tau)| \leq C$ follows by compactness and continuity, with some appropriate constant C independent on τ .

Now, $|\widehat{h_f}|$ has a uniform bound on $\mathbb{R} \cup i\mathbb{R}_+$, as well as an a priori subexponential growth (66). We are thus in position to transfer the uniform bounds to the whole upper half plane by the Phragmén-Lindelöf Theorem.

Lemma 52 (Phragmén-Lindelöf Theorem). – *Let g be a holomorphic function in $Q_1 = \{x + iy; x > 0, y > 0\}$, continuous in $\bar{Q}_1$. Assume that there exist $C > 0$ such that*

$$\begin{cases} |g(z)| \leq C e^{C|z|}, & \text{for all } z \in Q_1, \\ |g(z)| \leq 1, & \text{for all } z \in \partial Q_1 = \mathbb{R}_+ \cup i\mathbb{R}_+. \end{cases}$$

Then, $|g(z)| \leq 1$ for all $z \in Q_1$.

We refer, e.g., to [82, Theorem 3.4] for a proof. Applying this result to the function $g = \widehat{h_f}$ on both Q_1 and the quarter plane $\{x + iy; x < 0, y > 0\}$, we obtain that

$$|\widehat{h_f}(\zeta)| \leq C \quad \text{for all } \zeta \in \mathbb{C}, \text{Im}(\zeta) \geq 0.$$

We may now apply the following version of the Paley-Wiener theorem to h_f .

Theorem 53 (Paley-Wiener-Schwartz). – *Suppose that $g \in \mathcal{E}'(\mathbb{R})$ is of order zero. Then the following two statements are equivalent:*

- $\text{supp}(g) \subset (-\infty, 0]$,
- $\widehat{g}$ can be extended continuously as an entire function which is uniformly bounded in the closed upper half-plane

$$\mathbb{C}^+ = \{x + iy; x \in \mathbb{R}, y \geq 0\}.$$

This is a particular case of the general Paley-Wiener-Schwartz theorem, see, e.g., [31, Theorem 7.3.1]. Applying this result to the function h_f gives $\text{supp}(h_f) \subset]-\infty, 0]$. Therefore, we have proved that for $\chi \in C^\infty(\mathbb{R})$,

$$\text{supp}(\chi) \subset [0, +\infty) \implies 0 = \langle h_f, \chi \rangle = \langle f v, \chi(\Phi) \rangle = \langle f, \chi(\Phi) v \rangle.$$

Since this is true for a subset of function f dense in $\mathcal{S}$ (those having compactly supported Fourier transform), this means that the function $\chi(\Phi)v$ is identically zero on $\mathbb{R}^n$ as soon as $\text{supp}(\chi) \subset [0, +\infty)$. That is to say $v = 0$ a.e. on $\Phi > 0$ or, equivalently $\text{supp}(v) \subset \{\Phi \leq 0\}$, which concludes the proof of the lemma. $\square$

3.3. Carleman estimate. – To complete the proof of Theorem 43, we are now left with proving the Carleman estimate of Theorem 46.

3.3.1. The “conjugated operator”. – As in the classical case, we first compute the “conjugated operator”. Yet, care is needed since $e^{\varepsilon \frac{|D_t|^2}{2\tau}}$ is not well defined on any Sobolev space, not even on $\mathcal{S}$. With $v = e^{\tau\Phi} w$, (60) reads

$$\tau \|e^{-\varepsilon \frac{|D_t|^2}{2\tau}} v\|_{H_\tau^1}^2 \leq C \left\| e^{-\varepsilon \frac{|D_t|^2}{2\tau}} P_\Phi v \right\|_{L^2}^2 + C e^{-d\tau} \|v\|_{H_\tau^1}^2.$$

The operator P commutes with $e^{-\varepsilon \frac{|D_t|^2}{2\tau}}$ since its coefficients are independent on t . Yet, the operator $P_\Phi = e^{\tau\Phi} P e^{-\tau\Phi}$ now depends on t through Φ . We take advantage of the fact that Φ is quadratic, hence the coefficients of the operator P_Φ only involves derivative of Φ of order at most 1 and therefore are linear in t . We first prove the following simple lemma.

Lemma 54. – Let $u \in \mathcal{S}(\mathbb{R}^{1+d})$, then

$$e^{-\varepsilon \frac{|D_t|^2}{2\tau}}(tu) = \left(t + i\varepsilon \frac{D_t}{\tau}\right) e^{-\varepsilon \frac{|D_t|^2}{2\tau}} u.$$

Proof. – We first recall that $\mathcal{F}(tv)(\xi) = i\partial_{\xi_t} \widehat{v}(\xi)$ and hence

$$\begin{aligned} \mathcal{F}\left(e^{-\varepsilon \frac{|D_t|^2}{2\tau}}(tu)\right)(\xi) &= e^{-\varepsilon \frac{|\xi_t|^2}{2\tau}} \widehat{(tu)}(\xi) = e^{-\varepsilon \frac{|\xi_t|^2}{2\tau}} i\partial_{\xi_t} \widehat{u}(\xi) \\ &= i\partial_{\xi_t} \left(e^{-\varepsilon \frac{|\xi_t|^2}{2\tau}} \widehat{u}(\xi)\right) + i \frac{\varepsilon \xi_t}{\tau} e^{-\varepsilon \frac{|\xi_t|^2}{2\tau}} \widehat{u}(\xi) \\ &= \mathcal{F}\left(te^{-\varepsilon \frac{|D_t|^2}{2\tau}} u + i \frac{\varepsilon D_t}{\tau} e^{-\varepsilon \frac{|D_t|^2}{2\tau}} u\right)(\xi), \end{aligned}$$

which proves the lemma. $\square$

Remark 55. – Lemma 54 can be iterated to deduce that

$$e^{-\varepsilon \frac{|D_t|^2}{2\tau}}(t^k u) = \left(t + i\varepsilon \frac{D_t}{\tau}\right)^k e^{-\varepsilon \frac{|D_t|^2}{2\tau}} u,$$

where the exponent k is meant in the sense of composition. For f polynomial in t , we obtain

$$e^{-\varepsilon \frac{|D_t|^2}{2\tau}} (f(t)u) = f\left(t + i\varepsilon \frac{D_t}{\tau}\right) e^{-\varepsilon \frac{|D_t|^2}{2\tau}} u.$$

This means that the “formal” conjugated operator of $f(t)$ by $e^{-\varepsilon \frac{|D_t|^2}{2\tau}}$ is a differential operator, whose order is given by the degree of the polynomial f . For a general real-analytic function $f(t)$, giving a meaning to $f\left(t + i\varepsilon \frac{D_t}{\tau}\right)$ is one of the difficulties in [85, 77, 34, 87].

We now want to understand how $Q_{\varepsilon, \tau}^\Phi$ “commutes” with an operator P . To this aim, let us first consider the simplest case in which $P = D_j$. We have the following key lemma.

Lemma 56. – *Assume Φ is a real-valued quadratic function. For all $k \in \{0, \dots, d\}$ (with the convention $\mathbf{x}_0 = t$, $D_0 = D_t$, $\mathbf{x}_k = x_k$ and $D_k = D_{x_k}$ for $1 \leq k \leq d$)*

$$Q_{\varepsilon, \tau}^\Phi D_k = (D_k)_{\Phi, \varepsilon} Q_{\varepsilon, \tau}^\Phi,$$

where

$$(D_k)_{\Phi, \varepsilon} = D_k + i\tau \partial_k \Phi(\mathbf{x}) - \varepsilon \Phi''_{t, \mathbf{x}_k} D_t, \text{ with } \Phi''_{t, \mathbf{x}_k} = \partial_t \partial_{\mathbf{x}_k} \Phi.$$

Note that since Φ is quadratic in the variable t , the quantity $\Phi''_{t, \mathbf{x}_j}$ is actually constant in t ! In particular, the principal symbol of $(D_k)_{\Phi, \varepsilon}$ is $\xi_k + i\tau \partial_k \Phi - \varepsilon \Phi''_{t, \mathbf{x}_k} \xi_t$.

Proof. – Since Φ is quadratic in the variable t , $\partial_k \Phi$ is a polynomial of degree 1 in t and can be written as $\partial_k \Phi = f_1(x) + t f_0$, where $f_1(x)$ (resp. f_0) is polynomial in x of order 1 (resp. a constant). In particular, Lemma 54 gives for $e^{\tau \Phi} D_k e^{-\tau \Phi} = D_k + i\tau \partial_k \Phi$

$$\begin{aligned} e^{-\varepsilon \frac{|D_t|^2}{2\tau}} [(D_k + i\tau \partial_k \Phi)u] &= e^{-\varepsilon \frac{|D_t|^2}{2\tau}} [(D_k + i\tau(f_1(x) + t f_0))u] \\ &= \left[D_k + i\tau \left(f_1(x) + \left(t + i\varepsilon \frac{D_t}{\tau} \right) f_0 \right) \right] e^{-\varepsilon \frac{|D_t|^2}{2\tau}} u \\ &= (D_k + i\tau \partial_k \Phi - \varepsilon f_0 D_t) e^{-\varepsilon \frac{|D_t|^2}{2\tau}} u. \end{aligned}$$

To get an intrinsic expression, we notice that $f_0 = \partial_t \partial_k \Phi$, so $f_0 D_t$ reads $\partial_t \partial_k \Phi D_t$. This concludes the proof of the lemma. $\square$

This lemma allows one to compute the principal symbol of the “conjugated operator” of general differential operators (with coefficients independent of t).

Corollary 57 (The “conjugated operator”). – *Let $\Omega \subset \mathbb{R}^{1+d} = \mathbb{R}_t \times \mathbb{R}_x$ and $P \in \text{Diff}^m(\Omega)$ be a (classical) differential operator with principal symbol p_m . Assume also that all its coefficients are independent on t (that is $p_\alpha(\mathbf{x}) = p_\alpha(x)$ for all $|\alpha| \leq m$). Let Φ be a real-valued quadratic function. Then, for any $\varepsilon > 0$, there exists a unique $P_{\Phi, \varepsilon} \in \text{Diff}_\tau^m(\Omega)$ so that*

$$Q_{\varepsilon, \tau}^\Phi P = P_{\Phi, \varepsilon} Q_{\varepsilon, \tau}^\Phi.$$

Moreover, the principal symbol of $P_{\Phi,\varepsilon}$ is

$$p_{\Phi,\varepsilon}(\mathbf{x}, \xi, \tau) = p_m(\mathbf{x}, \xi + i\tau d\Phi(\mathbf{x}) - \varepsilon \Phi''_{t,\mathbf{x}} \xi_t),$$

where we use the notation $\Phi''_{t,\mathbf{x}} \xi_t = \text{Hess}(\Phi)((\xi_t, 0, \dots, 0), \cdot) = \xi_t V$ with V the constant vector with coefficients $V_k = (\partial_t \partial_{\mathbf{x}_k} \Phi)$ (using the convention of Lemma 56).

We stress the fact that all coefficients of P should be independent of t : this is not an assumption on the principal part of the operator only. The proof is similar to that of Lemma 13, using Lemma 56, Item 1 of Proposition 12 together with the fact that the coefficients of P commute with $Q_{\varepsilon,\tau}^\Phi$.

Remark 58. – In the case of a second-order operator P (with coefficients independent of t), with real symbol p_2 , we have (denoting by $\tilde{p}_2$ the polar symmetric bilinear form of p_2),

$$\begin{aligned} p_{\Phi,\varepsilon}(\mathbf{x}, \xi, \tau) &= p_2(\mathbf{x}, \xi + i\tau d\Phi(\mathbf{x}) - \varepsilon \Phi''_{t,\mathbf{x}} \xi_t) \\ &= p_2(\mathbf{x}, \xi - \varepsilon \Phi''_{t,\mathbf{x}} \xi_t) - \tau^2 p_2(\mathbf{x}, d\Phi(\mathbf{x})) + 2i\tau \tilde{p}_2(\mathbf{x}, \xi - \varepsilon \Phi''_{t,\mathbf{x}} \xi_t, d\Phi(\mathbf{x})). \end{aligned}$$

As in the classical case, an important point here is that

$$\text{Im}(p_{\Phi,\varepsilon}(\mathbf{x}, \xi, \tau)) = 2\tau \tilde{p}_2(\mathbf{x}, \xi - \varepsilon \Phi''_{t,\mathbf{x}} \xi_t, d\Phi(\mathbf{x}))$$

and divided by τ it yields a smooth operator.

An important feature of Corollary 57 is that the principal symbol of $P_{\Phi,\varepsilon}$ is actually close to the principal symbol of P_Φ if ε is small. So, we can expect that it satisfies the same subelliptic estimates.

3.3.2. A first subelliptic estimate. – We first write the following lemma on p_Φ , that we have actually already used and proved in Proposition 49, using Lemma 32 and homogeneity, so we skip the proof.

Lemma 59. – Suppose Ω, P are as in Theorem 43. Assume that the function Φ is a pseudoconvex Carleman weight for P in $\xi_t = 0$ at $\mathbf{x}_0$, in the sense of Definition 45. Then there exist $C_1, C_2 > 0$ such that for any $(\xi, \tau) \in \mathbb{R}^n \times \mathbb{R}^+$, we have

$$\frac{1}{i\tau} \{\overline{p_\Phi}, p_\Phi\}(\mathbf{x}_0, \xi, \tau) + C_1 \left[\frac{|p_\Phi(\mathbf{x}_0, \xi, \tau)|^2}{|\xi|^2 + \tau^2} + |\xi_t|^2 \right] \geq C_2(|\xi|^2 + \tau^2),$$

where we have extended $\frac{1}{i\tau} \{\overline{p_\Phi}, p_\Phi\}(\mathbf{x}_0, \xi, \tau)$ by continuity at $\tau = 0$ with the value $2\{p, \{p, \Phi\}\}(\mathbf{x}_0, \xi)$.

By perturbation, we obtain a similar conclusion for the perturbed operator.

Lemma 60. – Suppose Ω, P are as in Theorem 43. Assume that the function Φ is a pseudoconvex Carleman weight for P in $\xi_t = 0$ at $\mathbf{x}_0$, in the sense of Definition 45. Then, there exists $\varepsilon_0 > 0$ so that for any $0 \leq \varepsilon < \varepsilon_0$, there exist $C_1, C_2 > 0$ such that for any $(\xi, \tau) \in \mathbb{R}^n \times \mathbb{R}^+$, we have

$$\frac{1}{i\tau} \{\overline{p_{\Phi, \varepsilon}}, p_{\Phi, \varepsilon}\}(\mathbf{x}_0, \xi, \tau) + C_1 \left[\frac{|p_{\Phi, \varepsilon}(\mathbf{x}_0, \xi, \tau)|^2}{|\xi|^2 + \tau^2} + |\xi_t|^2 \right] \geq C_2(|\xi|^2 + \tau^2),$$

where the quantity $\frac{1}{i\tau} \{\overline{p_{\Phi, \varepsilon}}, p_{\Phi, \varepsilon}\}(\mathbf{x}_0, \xi, \tau)$ is extended by continuity at $\tau = 0$.

Proof. – The lemma mainly follows from the fact that $p_{\Phi, \varepsilon}$ is a perturbation of p_{Φ} and using Lemma 59. Yet, care is needed because of the factor $\frac{1}{\tau}$. Noticing again that $\frac{1}{i\tau} \{\overline{p_{\Phi, \varepsilon}}, p_{\Phi, \varepsilon}\} = \frac{2}{\tau} \{\operatorname{Re} p_{\Phi, \varepsilon}, \operatorname{Im} p_{\Phi, \varepsilon}\}$ and using Remark 58 we can write $\operatorname{Im} p_{\Phi, \varepsilon} = \tau \widetilde{p_{\Phi, \varepsilon}}^i$. Moreover, $\widetilde{p_{\Phi, \varepsilon}}^i$ and all its derivatives are continuous in ε . Hence, $\frac{1}{i\tau} \{\overline{p_{\Phi, \varepsilon}}, p_{\Phi, \varepsilon}\} = 2\{\operatorname{Re} p_{\Phi, \varepsilon}, \widetilde{p_{\Phi, \varepsilon}}^i\}$, which may therefore be extended by continuity to $\tau = 0$. The result then follows by a perturbation of Lemma 59. $\square$

We are now in position to prove a first subelliptic estimate that will be crucial for the final proof of Theorem 46.

Proposition 61. – Suppose Ω, P are as in Theorem 43. Assume that the function Φ is real-valued, quadratic and is a pseudoconvex Carleman weight for P in $\xi_t = 0$ at $\mathbf{x}_0$, in the sense of Definition 45. Then, there exist $\varepsilon, r, C, \tau_0 > 0$ so that we have the estimate

$$(67) \quad \tau \|v\|_{H_\tau^1}^2 \leq C \|P_{\Phi, \varepsilon} v\|_{L^2}^2 + C\tau \|D_t v\|_{L^2}^2,$$

for any $v \in C_c^\infty(B(\mathbf{x}_0, r))$ and $\tau \geq \tau_0$.

Note that the parameter $\varepsilon > 0$ is fixed by this proposition (in fact, by Lemma 60). This estimate is very close to the usual Carleman estimate (32) of Theorem 25. The only difference is the last term $\tau \|D_t v\|_{L^2}^2$ on the right-hand side. This term comes from the fact that the pseudoconvexity assumption (and hence the symbolic estimate of Lemma 60) is made on $\xi_t = 0$ only, i.e., on $D_t = 0$ only. Also, remark that this additional term has precisely the same strength as the term $\tau \|v\|_{H_\tau^1}^2$ on the left-hand side of the estimate.

Proof. – The proof is as well very similar to that of Theorem 25. A little care is needed to factorize the skew-adjoint part of the operator. Note first that the form of Estimate (67) remains unchanged under addition to P of a (classical) differential operator in $\operatorname{Diff}^1(\Omega)$, the coefficients of which do not depend on the variable t . Indeed, after conjugation, the latter perturbation will yield a perturbation of $P_{\Phi, \varepsilon}$ being in $\operatorname{Diff}_\tau^1(\Omega)$, which, applied to v , is bounded by $\|v\|_{H_\tau^1}^2$ and thus can be absorbed in the left-hand side for $\tau \geq \tau_0$ with τ_0 large enough.

We then notice that, with P satisfying the assumptions of Theorem 43, we have

$$P = \tilde{P} + R_1, \quad \text{with} \quad \tilde{P} = -D_t^2 + \sum_{1 \leq i, j \leq d} D_i g^{ij}(x) D_j, \quad \text{and} \quad R_1 \in \text{Diff}_\tau^1(\Omega),$$

where $g^{ij}(x) = g^{ji}(x)$. See also Example 14. According to the previous discussion, it is sufficient to prove Estimate (67) for P replaced by $\tilde{P}$. Applying Lemma 56 and using that $Q_{\varepsilon, \tau}^\Phi$ exactly commutes with $g^{ij}(x)$, we have the exact formula:

$$\begin{aligned} \tilde{P}_{\Phi, \varepsilon} &= -(D_t + i\tau \partial_t \Phi - \varepsilon \Phi''_{t,t} D_t)^2 \\ &\quad + \sum_{1 \leq i, j \leq d} (D_i + i\tau \partial_i \Phi - \varepsilon \Phi''_{t,x_i} D_t) g^{ij}(x) (D_j + i\tau \partial_j \Phi - \varepsilon \Phi''_{t,x_j} D_t). \end{aligned}$$

We now collect all terms with a factor τ to obtain, for some $M \in \text{Diff}_\tau^1(\Omega)$,

$$\tilde{P}_{\Phi, \varepsilon} = -(D_t - \varepsilon \Phi''_{t,t} D_t)^2 + \sum_{1 \leq i, j \leq d} (D_i - \varepsilon \Phi''_{t,x_i} D_t) g^{ij}(x) (D_j - \varepsilon \Phi''_{t,x_j} D_t) + \tau M,$$

and remark that $\tilde{P}_{\Phi, \varepsilon} - \tau M$ is a formally selfadjoint operator. As a consequence, when defining

$$P_{R, \varepsilon} = \frac{\tilde{P}_{\Phi, \varepsilon} + \tilde{P}_{\Phi, \varepsilon}^*}{2}, \quad P_{I, \varepsilon} = \frac{\tilde{P}_{\Phi, \varepsilon} - \tilde{P}_{\Phi, \varepsilon}^*}{2i},$$

we notice that we have, as in the proof of Theorem 25, $P_{I, \varepsilon} = \frac{\tau M - \tau M^*}{2i} =: \tau \widetilde{P_{I, \varepsilon}}$ (that is, τ can be factorized in the skew-adjoint part of $P_{I, \varepsilon}$). With this decomposition, we have $P_{\Phi, \varepsilon} = P_{R, \varepsilon} + i P_{I, \varepsilon} = P_{R, \varepsilon} + i \tau \widetilde{P_{I, \varepsilon}}$, and may now proceed to the key computation. Following the proof and computations of Theorem 25, we obtain

$$\|P_{\Phi, \varepsilon} v\|_{L^2}^2 = \|P_{R, \varepsilon} v\|_{L^2}^2 + \|P_{I, \varepsilon} v\|_{L^2}^2 + \tau \left(i [P_{R, \varepsilon}, \widetilde{P_{I, \varepsilon}}] v, v \right),$$

leading to

$$\frac{1}{\tau} \|P_{\Phi, \varepsilon} v\|_{L^2}^2 \geq (Lv, v),$$

$$\text{with } L = C_1 P_{R, \varepsilon} (-\Delta + \tau^2)^{-1} P_{R, \varepsilon} + C_1 P_{I, \varepsilon} (-\Delta + \tau^2)^{-1} P_{I, \varepsilon} + i [P_{R, \varepsilon}, \widetilde{P_{I, \varepsilon}}],$$

for $\tau \geq \tau_0$, τ_0 large enough, and with C_1 as given by Lemma 60. We thus obtain

$$\frac{1}{\tau} \|P_{\Phi, \varepsilon} v\|_{L^2}^2 + C_1 \|D_t v\|_{L^2}^2 \geq ((L + C_1 D_t^2) v, v).$$

The principal symbol of $L + C_1 D_t^2$ is

$$\frac{1}{i\tau} \{ \overline{p_{\Phi, \varepsilon}}, p_{\Phi, \varepsilon} \}(\mathbf{x}, \xi, \tau) + C_1 \left[\frac{|p_{\Phi, \varepsilon}(\mathbf{x}, \xi, \tau)|^2}{|\xi|^2 + \tau^2} + |\xi_t|^2 \right].$$

We conclude as in the proof of Theorem 25 using the symbolic estimate of Lemma 60 at the point $\mathbf{x}_0$, together with the Gårding inequality of Proposition 15. $\square$

3.3.3. *End of the proof of the Carleman estimate.* – Equipped with the subelliptic estimate (67), we may now proceed with the proof of Theorem 46. Setting $v = Q_{\varepsilon, \tau}^\Phi w = e^{-\frac{\varepsilon}{2\tau}|D_t|^2}(e^{\tau\Phi}w)$, the estimate reads

$$\tau \|v\|_{H_\tau^1}^2 \leq C \|P_{\Phi, \varepsilon} v\|_{L^2}^2 + C e^{-d\tau} \|e^{\tau\Phi} w\|_{H_\tau^1}^2.$$

The latter is very close to (67), except for the last term, and it is very tempting to apply (67) to $v = Q_{\varepsilon, \tau}^\Phi w$. One hope that the term $\tau \|D_t v\|_{L^2}^2$ can be estimated exploiting that the multiplier $e^{-\varepsilon \frac{|D_t|^2}{2\tau}}$ “localizes” where D_t is small. This will be done at the end of the proof. However, the first problem we have to face is that, even if w is compactly supported, the function $v = Q_{\varepsilon, \tau}^\Phi w$ is not compactly supported in the variable t . In fact, the operator $e^{-\frac{\varepsilon}{2\tau}|D_t|^2}$ is not local. We thus need to introduce an additional cutoff in time, and estimate the remainder this produces.

Proof of Theorem 46. – Without any loss of generality, we assume for simplicity that $\mathbf{x}_0 = (0, x_0)$, i.e., $t_0 = 0$. Consider $\varepsilon > 0$ and $r > 0$ as given by Proposition 61. We choose $r_0 > 0$ with $2r_0 = r$, and, all along the proof, we consider functions $u \in C_c^\infty(B(\mathbf{x}_0, r_0/4))$. Let $\chi \in C_c^\infty([-r_0, r_0])$ be such that $\chi = 1$ on $]-r_0/2, r_0/2[$. Since v is not compactly supported in the variable t , we set $f(\mathbf{x}) = \chi(t)v(\mathbf{x})$. Then $\text{supp}(f) \subset [-r_0, r_0] \times B(x_0, r_0/4) \subset B(\mathbf{x}_0, 2r_0) = B(\mathbf{x}_0, r)$. Proposition 61 applies to f . One has

$$v - f = (1 - \chi)Q_{\varepsilon, \tau}^\Phi u = (1 - \chi)e^{-\frac{\varepsilon}{2\tau}|D_t|^2}(\check{\chi}e^{\tau\Phi}u),$$

for $\check{\chi} \in C_c^\infty([-r_0/3, r_0/3])$ with $\check{\chi} = 1$ in a neighborhood of $[-r_0/4, r_0/4]$ so that $\check{\chi}u = u$. We use the following lemma to estimate $v - f$.

Lemma 62. – *Suppose $\chi_1 \in C^\infty(\mathbb{R}^n)$, $\chi_2 \in C^\infty(\mathbb{R}^n)$ have all derivatives bounded and be such that $\text{dist}(\text{supp}(\chi_1), \text{supp}(\chi_2)) > 0$. Then, there exist $C, c > 0$ such that for all $u \in \mathcal{S}(\mathbb{R}^n)$ and all $\lambda > 0$, we have*

$$\left\| \chi_1 e^{-\frac{|D_t|^2}{\lambda}} (\chi_2 u) \right\|_{L^2} \leq C e^{-c\lambda} \|u\|_{L^2} \quad \text{and} \quad \left\| \chi_1 e^{-\frac{|D_t|^2}{\lambda}} (\chi_2 u) \right\|_{H_\tau^1} \leq C e^{-c\lambda} \|u\|_{H_\tau^1}.$$

As a consequence of Lemma 62, we obtain, for $\tau > 0$

$$(68) \quad \|v\|_{H_\tau^1} \leq \|f\|_{H_\tau^1} + \|v - f\|_{H_\tau^1} \leq \|f\|_{H_\tau^1} + C e^{-c\frac{\tau}{\varepsilon}} \|e^{\tau\Phi} u\|_{H_\tau^1}.$$

The subelliptic estimate (67) applied to f gives

$$(69) \quad \begin{aligned} \tau \|v\|_{H_\tau^1}^2 &\leq 2\tau \|f\|_{H_\tau^1}^2 + 2C^2 \tau e^{-2c\frac{\tau}{\varepsilon}} \|e^{\tau\Phi} u\|_{H_\tau^1}^2 \\ &\leq 2C \|P_{\Phi, \varepsilon} f\|_{L^2}^2 + 2C\tau \|D_t f\|_{L^2}^2 + 2C^2 \tau e^{-2c\frac{\tau}{\varepsilon}} \|e^{\tau\Phi} u\|_{H_\tau^1}^2, \end{aligned}$$

and we need to estimate the first two terms on the right-hand side in terms of v . Firstly, $\|P_{\Phi, \varepsilon} f\|_{L^2} = \|P_{\Phi, \varepsilon} \chi v\|_{L^2} \leq \|\chi P_{\Phi, \varepsilon} v\|_{L^2} + \|[P_{\Phi, \varepsilon}, \chi]v\|_{L^2}$. For the commutator, we write $[P_{\Phi, \varepsilon}, \chi]v = [P_{\Phi, \varepsilon}, \chi]e^{-\frac{\varepsilon}{2\tau}|D_t|^2}\check{\chi}e^{\tau\Phi}u$. We notice that $[P_{\Phi, \varepsilon}, \chi] \in \text{Diff}_\tau^1$

with coefficients supported in $\text{supp}(\chi'_t)$, that is, away from $\text{supp}(\check{\chi})$. Lemma 62 implies $\|[P_{\Phi,\varepsilon}, \chi]v\|_{L^2} \leq Ce^{-c\frac{\tau}{\varepsilon}} \|e^{\tau\Phi}u\|_{H^1_\tau}$. This yields

$$(70) \quad \|P_{\Phi,\varepsilon}f\|_{L^2} \leq \|P_{\Phi,\varepsilon}v\|_{L^2} + Ce^{-c\frac{\tau}{\varepsilon}} \|e^{\tau\Phi}u\|_{H^1_\tau}.$$

Secondly, in a similar way, by Lemma 62 we obtain

$$(71) \quad \begin{aligned} \|D_t f\|_{L^2} &= \|D_t(\chi v)\|_{L^2} \leq \|\chi D_t v\|_{L^2} + \|\chi'(t)e^{-\frac{\varepsilon}{2\tau}|D_t|^2}\check{\chi}e^{\tau\Phi}u\|_{L^2} \\ &\leq \|D_t v\|_{L^2} + Ce^{-c\frac{\tau}{\varepsilon}} \|e^{\tau\Phi}u\|_{L^2}. \end{aligned}$$

Let ς be a small constant to be fixed in what follows. We separate frequencies of size smaller and bigger than $\varsigma\tau$. We obtain

$$\begin{aligned} \|D_t v\|_{L^2} &= \|D_t e^{-\frac{\varepsilon}{2\tau}|D_t|^2} e^{\tau\Phi}u\|_{L^2} \leq \|D_t 1_{|D_t| \leq \varsigma\tau} v\|_{L^2} + \|D_t 1_{|D_t| \geq \varsigma\tau} e^{-\frac{\varepsilon}{2\tau}|D_t|^2} e^{\tau\Phi}u\|_{L^2} \\ &\leq \|D_t 1_{|D_t| \leq \varsigma\tau} v\|_{L^2} + \max_{\xi_t \in [\varsigma\tau, +\infty)} (\xi_t e^{-\frac{\varepsilon}{2\tau}|\xi_t|^2}) \|e^{\tau\Phi}u\|_{L^2}. \end{aligned}$$

On $\mathbb{R}^+$, the function $s \mapsto se^{-\frac{\varepsilon}{2\tau}s^2}$ reaches its maximum at $s = \sqrt{\frac{\tau}{\varepsilon}}$, and is decreasing on $[\sqrt{\frac{\tau}{\varepsilon}}, +\infty)$. Hence, if $\tau \geq \frac{1}{\varsigma^2\varepsilon}$, then $\sqrt{\frac{\tau}{\varepsilon}} \leq \varsigma\tau$, the function $s \mapsto se^{-\frac{\varepsilon}{2\tau}s^2}$ is decreasing on the interval $[\varsigma\tau, +\infty)$, and thus bounded by its value at $\varsigma\tau$. This yields, for all $\tau \geq \max(\tau_0, \frac{1}{\varsigma^2\varepsilon})$, the estimate

$$(72) \quad \|D_t v\|_{L^2} \leq \varsigma\tau \|v\|_{L^2} + \varsigma\tau e^{-\frac{\tau\varsigma^2\varepsilon}{2}} \|e^{\tau\Phi}u\|_{L^2}.$$

Combining estimates (68)-(69)-(70)-(71)-(72), we have proved that there exists some constants $c > 0$ (depending on ε) and $C > 0$ so that for any $\varsigma > 0$, if $\tau \geq \max(\tau_0, \frac{1}{\varsigma^2\varepsilon})$, then

$$\tau \|v\|_{H^1_\tau}^2 \leq C \|P_{\Phi,\varepsilon}v\|_{L^2}^2 + C\varsigma^2\tau^3 \|v\|_{L^2}^2 + C \left(e^{-c\tau} + \varsigma^2\tau^3 e^{-\tau\varsigma^2\varepsilon} \right) \|e^{\tau\Phi}u\|_{H^1_\tau}^2.$$

We now fix the constant ς small enough so that the term $C\varsigma^2\tau^3 \|v\|_{L^2}^2 \leq C\varsigma^2\tau \|v\|_{H^1_\tau}^2$ can be absorbed in the left-hand side of the estimate. This yields the sought estimate for $\tau \geq \max(\tau_0, \frac{1}{\varsigma^2\varepsilon})$, and concludes the proof of the theorem. $\square$

It remains to prove Lemma 62. Roughly speaking, it expresses the fact that when λ is large, $e^{-\frac{\xi_t^2}{\lambda}}$ spreads out and converges to the function 1: $e^{-\frac{|D_t|^2}{\lambda}}$ converges to the identity and the localization in ξ_t is less effective. In particular, Lemma 62 states that $\chi_1 e^{-\frac{|D_t|^2}{\lambda}} (\chi_2 u)$ converges to $\chi_1 \chi_2 u = 0$ at a rate $e^{-c\lambda}$.

Proof of Lemma 62. – Recalling the convolution form of the action of $e^{-\frac{|D_t|^2}{\lambda}}$ given in (57), we have

$$\begin{aligned} \chi_1 e^{-\frac{|D_t|^2}{\lambda}} (\chi_2 u)(t, x) &= \left(\frac{\lambda}{4\pi} \right)^{\frac{1}{2}} \int_{\mathbb{R}} \chi_1(t, x) e^{-\frac{\lambda}{4}|s-t|^2} (\chi_2 u)(s, x) ds \\ &= \left(\frac{\lambda}{4\pi} \right)^{\frac{1}{2}} \chi_1(t, x) \int_{\mathbb{R}} 1_{|t-s| \geq d} e^{-\frac{\lambda}{4}|s-t|^2} (\chi_2 u)(s, x) ds, \end{aligned}$$

where $d = \text{dist}(\text{supp}(\chi_1), \text{supp}(\chi_2)) > 0$ and we have used the support properties in the second equality. This yields

$$\begin{aligned} |\chi_1 e^{-\frac{|D_t|^2}{\lambda}}(\chi_2 u)|(t, x) &\leq \|\chi_1\|_{L^\infty} \left(\frac{\lambda}{4\pi}\right)^{\frac{1}{2}} \int_{\mathbb{R}} 1_{|t-s| \geq d} e^{-\frac{\lambda}{4}|s-t|^2} |\chi_2 u|(s, x) ds \\ &\leq \|\chi_1\|_{L^\infty} \left(\frac{\lambda}{4\pi}\right)^{\frac{1}{2}} \left(1_{|\cdot| \geq d} e^{-\frac{\lambda}{4}|\cdot|^2} * |\chi_2 u|(\cdot, x)\right)(t). \end{aligned}$$

As a consequence, using the Young inequality, we have

$$(73) \quad \|\chi_1 e^{-\frac{|D_t|^2}{\lambda}}(\chi_2 u)\|_{L^2(\mathbb{R}^n)} \leq \|\chi_1\|_{L^\infty} \left(\frac{\lambda}{4\pi}\right)^{\frac{1}{2}} \|1_{|\cdot| \geq d} e^{-\frac{\lambda}{4}|\cdot|^2}\|_{L^1(\mathbb{R})} \|\chi_2 u\|_{L^2(\mathbb{R}^n)}.$$

Next, we notice that

$$\begin{aligned} \frac{1}{2} \|1_{|\cdot| \geq d} e^{-\frac{\lambda}{4}|\cdot|^2}\|_{L^1(\mathbb{R})} &= \int_d^{+\infty} e^{-\frac{\lambda}{4}s^2} ds = \frac{2}{\sqrt{\lambda}} \int_{d\sqrt{\lambda}/2}^{+\infty} e^{-y^2} dy \\ &= \frac{2}{\sqrt{\lambda}} \int_{d\sqrt{\lambda}/2}^{+\infty} e^{-\frac{y^2}{2}} e^{-\frac{y^2}{2}} dy \leq \frac{2}{\sqrt{\lambda}} e^{-\frac{d^2}{8}\lambda} \int_{d\sqrt{\lambda}/2}^{+\infty} e^{-\frac{y^2}{2}} dy \\ &\leq \frac{2}{\sqrt{\lambda}} e^{-\frac{d^2}{8}\lambda} \int_0^{+\infty} e^{-\frac{y^2}{2}} dy = \frac{2}{\sqrt{\lambda}} e^{-\frac{d^2}{8}\lambda} \sqrt{\frac{\pi}{2}}. \end{aligned}$$

Coming back to (73), we have obtained the existence of a constant $C > 0$ such that for all $\lambda > 0$,

$$\|\chi_1 e^{-\frac{|D_t|^2}{\lambda}}(\chi_2 u)\|_{L^2(\mathbb{R}^n)} \leq C \|\chi_1\|_{L^\infty} \|\chi_2\|_{L^\infty} e^{-\frac{d^2}{8}\lambda} \|u\|_{L^2(\mathbb{R}^n)},$$

which implies the result in L^2 . The proof in H_τ^1 is a consequence of that in L^2 . $\square$

3.4. Semiglobal statements and noncharacteristic hypersurfaces. – In this section, we first describe a geometric setting that encompasses those described in Section 1.1. We then prove a semiglobal unique continuation statement relying on the construction of a family of noncharacteristic hypersurface. This semiglobal statement will be the cornerstone in all the applications discussed in Section 1.1.

3.4.1. Distance, metric, Laplace-Beltrami operator. – We consider a connected d -dimensional Riemannian manifold $(\mathcal{M}, g)$ with or without boundary $\partial\mathcal{M}$. In the case $\partial\mathcal{M} \neq \emptyset$, we denote by $\text{Int}(\mathcal{M})$ the interior of $\mathcal{M}$, so that $\mathcal{M} = \partial\mathcal{M} \sqcup \text{Int}(\mathcal{M})$ (see, e.g., [62, Chapter 1]). The metric g (a bilinear form on $T\mathcal{M}$) is in local charts a smooth family of symmetric elliptic matrices $(g_{ij}(x))$, i.e., $g_{ij}(x)X^iX^j \geq c|X|^2$ for all $X \in T_x\mathcal{M}$. We consider the Riemannian volume density $d\text{Vol}_g$, given in local charts by $d\text{Vol}_g(x) = \sqrt{\det g(x)}dx$. We also define the co-metric g^* to g , defined on $T^*\mathcal{M}$, given in local charts by the smooth family of symmetric elliptic (i.e., satisfying (55)) matrices $g^*(x) = (g_{ij}(x)) = (g^{ij}(x))^{-1}$. Associated with the metric g and the volume

density $d\text{Vol}_g$, the natural (negative) elliptic operator is the Laplace-Beltrami operator on $\mathcal{M}$, given in local charts by

$$\Delta_g f = \sum_{i,j=1}^n \frac{1}{\sqrt{\det g}} \partial_i \left(\sqrt{\det g} g^{ij} \partial_j f \right), \quad f \in C^\infty(\mathcal{M}),$$

which is formally selfadjoint on $L^2(\mathcal{M}) := L^2(\mathcal{M}, d\text{Vol}_g)$.

Given a path $\gamma \in C^1([0, 1]; \text{Int}(\mathcal{M}))$ (or even $\gamma \in W^{1,1}([0, 1]; \text{Int}(\mathcal{M}))$), its length (in the sense of the Riemannian metric g) is given by

$$\text{length}(\gamma) = \int_0^1 |\dot{\gamma}(t)|_{\gamma(t)} dt,$$

where we write $|X|_x^2 := g_x(X, X)$ for $x \in \mathcal{M}$ and $X \in T_x \mathcal{M}$ (i.e., $|X|_x^2 = g_{ij}(x) X^i X^j$ in local charts). This allows one to define the Riemannian distance associated with g as

$$\begin{aligned} \text{dist}(x_1, x_2) &= \inf \{ \text{length}(\gamma), \gamma \in W^{1,1}([0, 1]; \text{Int}(\mathcal{M})), \gamma(0) = x_1; \gamma(1) = x_2 \} \\ &= \inf \{ \text{length}(\gamma), \gamma \in C^\infty([0, 1]; \text{Int}(\mathcal{M})), \gamma(0) = x_1; \gamma(1) = x_2 \}, \end{aligned}$$

where equality in the second line follows from a classical regularization argument.

Note that in case $\mathcal{M} \subset \mathbb{R}^d$, any (locally uniformly) elliptic operator on $\mathcal{M}$ can be written under the form Δ_g modulo smooth lower-order terms.

3.4.2. The semiglobal theorem. – The key semiglobal result in all applications discussed in Section 1.1 is the following.

Theorem 63 (Semiglobal unique continuation for waves). – *Let $q \in L_{\text{loc}}^\infty(\text{Int}(\mathcal{M}))$ and consider on $\mathbb{R} \times \mathcal{M}$ the operator $P := \partial_t^2 - \Delta_g + q$, with $(\mathcal{M}, g)$ satisfying the assumptions in Section 3.4.1. Let $x_0, x_1 \in \text{Int}(\mathcal{M})$, let ω_0 be neighborhood of x_0 in $\text{Int}(\mathcal{M})$. Then, for any $T > \text{dist}(x_0, x_1)$, there exist $\varepsilon > 0$ and V_{x_1} a neighborhood of x_1 such that*

$$(74) \quad \begin{cases} u \in H_{\text{loc}}^1((-T, T) \times \text{Int}(\mathcal{M})) \\ Pu = 0 \text{ in } (-T, T) \times \text{Int}(\mathcal{M}) \\ u = 0 \text{ in } (-T, T) \times \omega_0 \end{cases} \implies u = 0 \text{ in } (-\varepsilon, \varepsilon) \times V_{x_1}.$$

We refer to Figure 6 for a representation of the geometry of the zone of dependence.

The proof relies first on fixing suitable coordinates along a path joining x_0 and x_1 and having length less than T , and second on constructing in these coordinates appropriate noncharacteristic hypersurfaces in which to apply Theorem 43. Note that the result remains true if one adds to P first-order differential operators with time-independent L_{loc}^∞ coefficients.

The construction of noncharacteristic hypersurfaces in the proof of Theorem 63 below is inspired by [59] (where associated stability estimates in an analytic context are proved, see also [49]). Another geometric construction relying on timelike paths is proposed in [70].

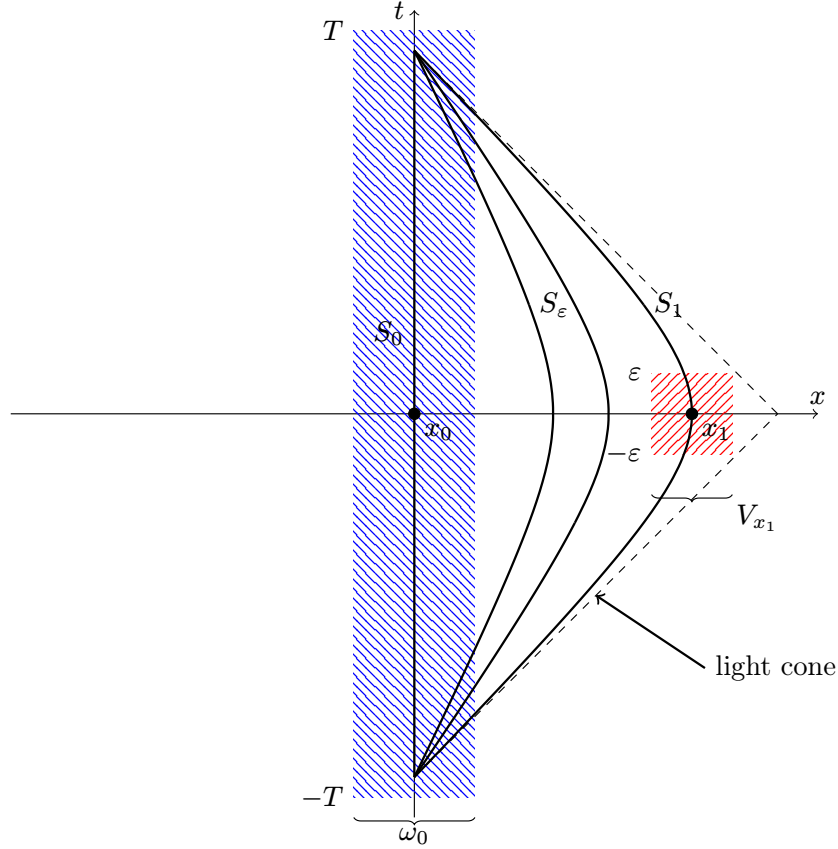


FIGURE 6. Picture of the geometric context of Theorem 63 with the foliation of hypersurfaces $S_\varepsilon = \{\Psi_\varepsilon = 0\}$, $\varepsilon \in [0, 1]$, that are used in the proof. The blue zone $(-T, T) \times \omega_0$ is where the solution is known to be zero while the red zone $(-\varepsilon, \varepsilon) \times V_{x_1}$ is where we propagate the nullity.

Proof of Theorem 63. – According to the definition of dist , there is a smooth injective path $\gamma : [0, 1] \rightarrow \text{Int}(\mathcal{M})$ such that $\gamma(0) = x_0$, $\gamma(1) = x_1$ and $\text{length}(\gamma) = \ell_0$ with $\text{dist}(x_0, x_1) < \ell_0 < T$. According to Lemma 64 below, we can find local coordinates (w, l) near γ in which the path γ by $\gamma(s) = (0, s\ell_0)$ for $s \in [0, 1]$ and the cometric g^* (defined on $T^*\mathcal{M}$) is given by the matrix $m(w, l) \in M_d(\mathbb{R})$ with

$$(75) \quad m(w, l) = \begin{pmatrix} m'(l) & 0 \\ 0 & 1 \end{pmatrix} + \mathcal{O}_{M_d(\mathbb{R})}(|w|), \quad \text{for } w \in B_{\mathbb{R}^{d-1}}(0, \delta), \delta > 0,$$

with $m'(l) \in M_{d-1}(\mathbb{R})$ (uniformly) definite symmetric. With these coordinates in the space variable, and still using the straight time variable, the principal symbol of the wave operator is given by

$$(76) \quad p_2(t, w, l, \xi_t, \xi_w, \xi_l) = p_2(w, l, \xi_t, \xi_w, \xi_l) = -\xi_t^2 + m(w, l)\xi \cdot \xi, \quad \xi = (\xi_w, \xi_l),$$

where we have used ξ_t for the cotangent variable to the time variable and ξ_w, ξ_l for the dual variables to $w \in B_{\mathbb{R}^{d-1}}(0, \delta)$ and $l \in [0, \ell_0]$ respectively. We now aim to apply

Theorem 43 and we need to construct appropriate noncharacteristic hypersurfaces. To this aim, we let t_0 with $\ell_0 < t_0 < T$. For $b < \delta$ small, to be fixed in what follows, we define

$$D = \left\{ (t, w) \in [-t_0, t_0] \times B_{\mathbb{R}^{d-1}}(0, \delta) \mid \left(\frac{w}{b} \right)^2 + \left(\frac{t}{t_0} \right)^2 \leq 1 \right\},$$

$$G(t, w, \varepsilon) = \varepsilon \ell_0 \zeta \left(\sqrt{\left(\frac{w}{b} \right)^2 + \left(\frac{t}{t_0} \right)^2} \right), \quad \Psi_\varepsilon(t, w, l) := G(t, w, \varepsilon) - l, \quad \varepsilon \in [0, 1],$$

where ζ is a fixed function such that

$$\zeta : [-1, 1] \rightarrow \mathbb{R} \text{ even, } \zeta(\pm 1) = 0, \quad \zeta(0) = 1, \quad \zeta(s) \geq 0, \quad |\zeta'(s)| \leq \alpha, \text{ for } s \in [-1, 1],$$

with $1 < \alpha < \frac{t_0}{\ell_0}$. This is possible since $\frac{t_0}{\ell_0} > 1$. Note that the fact that ζ is even implies that $G(t, w, \varepsilon)$ is actually smooth. Note also that the point $(t = 0, w = 0, l = \ell_0)$ corresponding in the local coordinates to x_1 belongs to the hypersurface $\{\Psi_1 = 0\}$. We have

$$d\Psi_\varepsilon(t, w, l) = \varepsilon \ell_0 \left(\left(\frac{w}{b} \right)^2 + \left(\frac{t}{t_0} \right)^2 \right)^{-1/2} \zeta' \left(\sqrt{\left(\frac{w}{b} \right)^2 + \left(\frac{t}{t_0} \right)^2} \right) \left(\frac{tdt}{t_0^2} + \frac{wdw}{b^2} \right) - dl.$$

Given the form of the principal symbol of the wave operator in these coordinates (see (75)-(76)), we obtain

$$\begin{aligned} p(w, l, d\Psi_\varepsilon(t, w, l)) &= -\varepsilon^2 \ell_0^2 \frac{t^2}{t_0^4} \left(\left(\frac{w}{b} \right)^2 + \left(\frac{t}{t_0} \right)^2 \right)^{-1} |\zeta'|^2 \\ &\quad + \ell_0^2 \frac{\varepsilon^2}{b^4} \langle m'(l)w, w \rangle \left(\left(\frac{w}{b} \right)^2 + \left(\frac{t}{t_0} \right)^2 \right)^{-1} |\zeta'|^2 + 1 \\ &\quad + O(|w|^2) \left(1 + \frac{\varepsilon^2 \ell_0^2}{b^4} |w|^2 \left(\left(\frac{w}{b} \right)^2 + \left(\frac{t}{t_0} \right)^2 \right)^{-1} |\zeta'|^2 \right), \end{aligned}$$

where $|\zeta'|^2$ is evaluated at the point $\sqrt{\left(\frac{w}{b} \right)^2 + \left(\frac{t}{t_0} \right)^2}$. Now, since $\alpha < \frac{t_0}{\ell_0}$ and $m'(l)$ is uniformly definite positive (for $l \in [0, \ell_0]$), there exists $\eta > 0$ and $b > 0$ small enough so that for $|w| \leq b$, we have

$$1 + O(|w|^2) \geq \alpha^2 \frac{\ell_0^2}{t_0^2} + \eta,$$

$$\langle m'(l)w, w \rangle + O(|w|^2)|w|^2 \geq \frac{1}{2} \langle m'(l)w, w \rangle \geq 0.$$

Hence, there exists a sufficiently small neighborhood (taking again b small enough) of the path (i.e., of $w = 0$), in which we have (for any $\varepsilon \in [0, 1]$), and

any $(t, w, l) \in D \times [0, \ell_0]$,

$$\begin{aligned} p(w, l, d\Psi_\varepsilon(t, w, l)) &\geq -\frac{\varepsilon^2}{t_0^2} \ell_0^2 \left(\frac{t}{t_0}\right)^2 \left(\left(\frac{w}{b}\right)^2 + \left(\frac{t}{t_0}\right)^2\right)^{-1} |\zeta'|^2 + \alpha^2 \frac{\ell_0^2}{t_0^2} + \eta \\ &\geq -\frac{\ell_0^2}{t_0^2} |\zeta'|^2 + \alpha^2 \frac{\ell_0^2}{t_0^2} + \eta \geq \eta. \end{aligned}$$

As a consequence, for any $\varepsilon \in [0, 1]$, the hypersurface $\{\Psi_\varepsilon = 0\}$ is noncharacteristic for P near any of its points. Theorem 43 thus applies. Now, define $K_\varepsilon := \{(t, w, l) \in D \times [0, \ell_0], l \leq G(t, w, \varepsilon)\} \cap \{l \geq 0\}$ for $\varepsilon \in [0, 1]$ and consider $\varepsilon_0 := \sup\{\varepsilon \in [0, 1], u = 0 \text{ in } K_\varepsilon\}$. A continuity argument yields that $u = 0$ on K_{ε_0} . A compactness argument on the compact set K_ε (taking into account the “corners”) and successive applications of Theorem 43 proves that $\varepsilon_0 = 1$. Theorem 43 applied once again across $\{\Psi_1 = 0\}$ then implies that $u = 0$ in a neighborhood of $\{\Psi_1 = 0\}$ which contains the point $(t, w, l) = (0, 0, \ell_0) = x_1$ (in these coordinates), and concludes the proof of the theorem. $\square$

The following result is proved, e.g., in [53].

Lemma 64. – *Let $\gamma : [0, 1] \rightarrow \text{Int}(\mathcal{M})$ be a smooth path without self intersection (i.e., γ is injective) of length ℓ_0 so that $\gamma(0) = x_0$ and $\gamma(1) = x_1$. Then, there are coordinates $(w, l) \in B_{\mathbb{R}^{d-1}}(0, \varepsilon) \times [0, \ell_0]$ in an open neighborhood U of $\gamma([0, 1])$ such that*

- $\gamma([0, 1]) = \{w = 0\} \times [0, \ell_0]$,
- the cometric g^* (defined on $T^*\mathcal{M}$) is of the form

$$m(w, l) = \begin{pmatrix} m'(l) & 0 \\ 0 & 1 \end{pmatrix} + O_{M_d(\mathbb{R})}(|w|),$$

where m' is a smooth family on $[0, 1]$ of positive definite matrices in $M_{d-1}(\mathbb{R})$.

Remark 65. – Note that all hypersurfaces constructed in the proof of Theorem 63 have points where they are *not* strongly pseudoconvex, e.g., for $t = 0$ (actually, one could prove that they are strongly pseudoconvex near none of their points). As a consequence of Theorems 41 and 42, for each of these these hypersurfaces, one can modify the operator by a time dependent zero order term so that unique continuation does not hold.

3.5. Global unique continuation statements: back to applications. – We now come back to the motivating applications presented in Section 1.1.

3.5.1. *Region of dependence.* – The following result might be seen as a counterpart to finite speed of propagation for waves. Recall that we set

$$\text{dist}(x, E) = \inf_{y \in E} \text{dist}(x, y), \quad \text{for } E \subset \mathcal{M}, x \in \mathcal{M}.$$

Theorem 66. – *Let $(\mathcal{M}, g)$ be a connected Riemannian manifold with or without boundary $\partial\mathcal{M}$, $q \in L^\infty_{\text{loc}}(\text{Int}(\mathcal{M}))$, and $\omega \subset \mathcal{M}$ be a nonempty open set. Let $u \in H^1_{\text{loc}}((-T, T) \times \mathcal{M})$ be such that $\partial_t^2 u - \Delta_g u + qu = 0$ in $\mathcal{D}'((-T, T) \times \text{Int}(\mathcal{M}))$. Assume that $u|_{(-T, T) \times \omega} = 0$, then $u|_{\mathcal{U}} = 0$ where*

$$\mathcal{U} = \mathcal{U}(\omega, T) := \{(t, x) \in (-T, T) \times \mathcal{M}, \text{dist}(x, \omega) < T - |t|\}.$$

Similarly, if $\Gamma \subset \partial\mathcal{M}$ is a nonempty open set of $\partial\mathcal{M}$, if in addition $q \in L^\infty$, $u \in H^1$ in a neighborhood of Γ , and if $u|_{(-T, T) \times \Gamma} = \partial_n u|_{(-T, T) \times \Gamma} = 0$, then $u|_{\mathcal{U}'} = 0$ where $\mathcal{U}' = \mathcal{U}(\Gamma, T)$.

The first statement in this result is a direct consequence of Theorem 63 together with translation invariance in time. Concerning the second statement, it suffices to notice that for any extension $\widetilde{\mathcal{M}}$ of $\mathcal{M}$ such that $\partial\mathcal{M} \setminus \Gamma \subset \partial\widetilde{\mathcal{M}} \cap \partial\mathcal{M}$, one may extend q as (any) $\tilde{q} \in L^\infty_{\text{loc}}(\widetilde{\mathcal{M}})$ and extend u as $\tilde{u}$ equal to 0 in $\widetilde{\mathcal{M}} \setminus \mathcal{M}$. The first part of the statement then applies to $\tilde{u}$ and yields the second statement.

Theorem 66 provides the largest region $\mathcal{U}(\omega, T)$ of time-space where solutions to wave equations vanishing on the cylinder $(-T, T) \times \omega$ has to vanish. That one cannot improve the size of the region $\mathcal{U}(\omega, T)$ is a consequence of finite speed of propagation, see, e.g., the constructions in [79, 80].

Theorem 66 is also the unique continuation statement used in the Boundary Control method to solve the hyperbolic inverse problem presented in Section 1.1.3. We refer the reader to [71] for a presentation of the method of [12].

3.5.2. *Penetration into shadow and approximate controllability.* – We are now prepared to discuss the motivation to “Penetration into shadow” and to Approximate controllability introduced in Sections 1.1.1 and 1.1.2. Given two subsets $E_0, E_1 \subset \mathcal{M}$, we introduce the largest distance of E_0 to a point of E_1 (note that this quantity is not symmetric with respect to E_0, E_1):

$$(77) \quad \mathcal{L}(E_1, E_0) := \sup_{x \in E_1} \text{dist}(x, E_0), \quad \text{dist}(x, E) = \inf_{y \in E} \text{dist}(x, y).$$

The following result is a direct corollary of Theorem 63 (or Theorem 66) together with a compactness argument.

Theorem 67 (Global unique continuation for waves). – *Let $(\mathcal{M}, g)$ be a connected Riemannian manifold with or without boundary $\partial\mathcal{M}$, $q \in L^\infty_{\text{loc}}(\text{Int}(\mathcal{M}))$. Let $\omega \subset \mathcal{M}$ be a nonempty open set and $K \subset \mathcal{M}$ be a compact set. If $T > \mathcal{L}(K, \omega)$, and if u satisfies*

$$(78) \quad \begin{aligned} u &\in H^1_{\text{loc}}((-T, T) \times \text{Int}(\mathcal{M})), & (\partial_t^2 - \Delta_g + q)u &= 0 \text{ in } (-T, T) \times \text{Int}(\mathcal{M}), \\ & & u &= 0 \text{ on } (-T, T) \times \omega, \end{aligned}$$

then, there is $\varepsilon > 0$ such that $u = 0$ identically in $(-\varepsilon, \varepsilon) \times K$.

This result answers Question (2) by the affirmative if $T > \mathcal{L}(K, \omega)$. Under this condition, all waves supported at time 0 in K are visible from $(-T, T) \times \omega$. As a corollary (together with well-posedness for the Boundary-value problem), we deduce the following global unique continuation result.

Theorem 68. – *Let $(\mathcal{M}, g)$ be a compact connected Riemannian manifold with or without boundary $\partial\mathcal{M}$, $q \in L^\infty(\text{Int}(\mathcal{M}))$ and fix $T > 2\mathcal{L}(\mathcal{M}, \omega)$. Then (5) holds.*

Proof. – Indeed, under the assumption of (5), the solution w to (4) belongs to

$$C^0([0, T]; H_0^1(\mathcal{M})) \cap C^1([0, T]; L^2(\mathcal{M})) \subset H^1((0, T) \times \text{Int}(\mathcal{M}))$$

and we can thus apply Theorem 67 to $K = \mathcal{M}$ and $u(t) = w(t + T/2)$, defined on the time interval $(-T/2, T/2)$. According to time invariance of the wave equation, u satisfies (78) with $T/2$ in place of T and we deduce that $u = 0$ in $\mathcal{D}'((-\varepsilon, \varepsilon) \times \mathcal{M})$. That is to say $w = 0$ in $\mathcal{D}'((T/2 - \varepsilon, T/2 + \varepsilon) \times \mathcal{M})$ but since $C^0([0, T]; H_0^1(\mathcal{M})) \cap C^1([0, T]; L^2(\mathcal{M}))$ this implies $(w, \partial_t w)(T/2) = 0$ and well-posedness of the Boundary-value problem (4) implies $(w_0, w_1) = (0, 0)$. $\square$

Recalling the discussion of Section 1.1.2, we now have, as a last corollary, the following.

Corollary 69. – *Assume $(\mathcal{M}, g)$ is a compact connected Riemannian manifold with or without boundary $\partial\mathcal{M}$, $q \in L^\infty(\text{Int}(\mathcal{M}))$, let $\omega \subset \mathcal{M}$ be a nonempty open set, and fix $T > 2\mathcal{L}(\mathcal{M}, \omega)$. Then, Equation (3) is approximately controllable from (χ_ω, T) .*

Again, on account to finite speed of propagation, the minimal times $\mathcal{L}(K, \omega)$, $\mathcal{L}(\mathcal{M}, \omega)$, $2\mathcal{L}(\mathcal{M}, \omega)$ needed in Theorems 67, 68 and Corollary 69 are all optimal, see, e.g., [79, 80].

4. Notes

To conclude, we briefly discuss the related question of obtaining quantitative estimates associated to unique continuation, historical notes and pointers to the literature.

4.1. Quantitative unique continuation and the cost of approximate controls. – All the results we presented in this course have their *quantitative* counterpart. We discuss a few of them in this section and only provide statements without any proofs (as opposed to the previous sections). The latter are rather technical and would not fit in these introductory lecture notes.

The local quantitative estimates associated to the Hörmander Theorem 17 are of Hölder type. This was first noticed by Isakov [39] (under a slightly different form) and Bahouri [7] and leads to the following type of statement.

Theorem 70. – *Under the assumptions of Theorem 17, there exist a neighborhood V of x_0 , $\alpha \in (0, 1)$ and $C > 0$ so that we have*

$$\|u\|_{H^1(V)} \leq C \|u\|_{H^1(\Omega \cap \{\Psi \geq \Psi(x_0)\})}^\alpha \|u\|_{H^1(\Omega)}^{1-\alpha}$$

for all $u \in H^1(\Omega)$ solution of $Pu = 0$.

It was also understood in [76, 61], for elliptic operators, that this kind of interpolation estimates can be iterated to deduce global results. We also refer to [49] for a global geometric statement for general operators, across a global foliation of strongly pseudoconvex hypersurfaces.

Concerning the unique continuation result described in Section 3, the quantitative stability estimates are of logarithmic type. The following (global) result is the quantitative version of the global unique continuation result of Theorem 68, proved in [49] (see also [52] for obtaining the observation term in L^2 norm). It followed several earlier results on the subject and we refer to Section 4.2 for some summary of the previous literature.

Theorem 71 (Quantitative unique continuation for waves [49]). –

Let $(\mathcal{M}, g)$ be a compact Riemannian manifold with (or without) boundary. Suppose ω is a nonempty open subset of $\mathcal{M}$ and any $T > 2\mathcal{L}(\mathcal{M}, \omega)$. There exist $C, \kappa, \mu_0 > 0$ such that for any $(u_0, u_1) \in H_0^1(\mathcal{M}) \times L^2(\mathcal{M}) \setminus \{(0, 0)\}$ and the solution u of

$$(79) \quad \begin{cases} \partial_t^2 u - \Delta_g u = 0 & \text{in } (0, T) \times \text{Int}(\mathcal{M}), \\ u|_{\partial \mathcal{M}} = 0 & \text{in } (0, T) \times \partial \mathcal{M}, \\ (u, \partial_t u)|_{t=0} = (u_0, u_1) & \text{in } \text{Int}(\mathcal{M}), \end{cases}$$

satisfies the following three estimates

(80)

$$\|(u_0, u_1)\|_{L^2 \times H^{-1}} \leq C e^{\kappa \mu} \|u\|_{L^2((0,T) \times \omega)} + \frac{1}{\mu} \|(u_0, u_1)\|_{H^1 \times L^2}, \quad \text{for all } \mu \geq \mu_0,$$

(81)

$$\|(u_0, u_1)\|_{L^2 \times H^{-1}} \leq C \frac{\|(u_0, u_1)\|_{H^1 \times L^2}}{\log \left(1 + \frac{\|(u_0, u_1)\|_{H^1 \times L^2}}{\|u\|_{L^2((0,T) \times \omega)}} \right)},$$

$$(82) \quad \|(u_0, u_1)\|_{H^1 \times L^2} \leq C e^{\kappa \Lambda} \|u\|_{L^2((0,T) \times \omega)}, \quad \text{with } \Lambda = \frac{\|(u_0, u_1)\|_{H^1 \times L^2}}{\|(u_0, u_1)\|_{L^2 \times H^{-1}}}.$$

The three inequalities (80)–(81)–(82) are actually equivalent (up to changing the values of the constants). Note that the statement (80) remains valid for all $\mu > 0$ (not only $\mu \geq \mu_0$), the estimate for μ bounded being trivial and useless. In Estimate (81), the function on the right-hand side is to be understood as being $(\log(1 + \frac{1}{x}))^{-1}$ for $x > 0$ and 0 for $x = 0$. In Estimate (82), Λ has to be considered as the typical frequency of the initial data. Estimate (82) provides a cost of observability of the order of an exponential of the typical frequency. As an illustration, taking for initial data $(u_0, u_1) = (\psi_\lambda, 0)$ with ψ_λ a normalized eigenfunction of the Laplace-Dirichlet operator on $\mathcal{M}$, associated to the eigenvalue λ , one has $\Lambda \sim \sqrt{\lambda}$ and (82) recovers the tunneling estimate $\|\psi_\lambda\|_{L^2(\omega)} \geq C^{-1} e^{-C\sqrt{\lambda}}$ (see [19, 61, 56, 57, 51] for a discussion on quantitative unique continuation for eigenfunctions of elliptic operators). A generalization to hypoelliptic operators has been proved in [52].

As stated by Lebeau [59, Section 2, pages 5 and 6] in the analytic context, the exponential (resp. logarithmic) dependence in Estimates (80)–(82) (resp. in (81)) is sharp in general: the form of the estimates of Theorem 71 is optimal if the geometric control condition of [74, 8, 9] is violated.

As a consequence of Theorem 71, we obtain the cost of the approximate controllability of the wave equation (i.e., a quantitative version of Corollary 69).

Theorem 72 (Cost of approximate controls [49]). – Assume $\mathcal{M}$ is compact and connected. Suppose ω is nonempty open set of $\mathcal{M}$ and $T > 2\mathcal{L}(\mathcal{M}, \omega)$. There exist $C, c > 0$ such that for any $\varepsilon > 0$ and any $(u_0, u_1) \in H_0^1(\mathcal{M}) \times L^2(\mathcal{M})$, there exists $f \in L^2((0, T) \times \omega)$ with

$$\|f\|_{L^2((0,T) \times \omega)} \leq C e^{\frac{c}{\varepsilon}} \|(u_0, u_1)\|_{H_0^1(\mathcal{M}) \times L^2(\mathcal{M})},$$

such that the solution of (3) satisfies

$$\|(u, \partial_t u)|_{t=T}\|_{L^2(\mathcal{M}) \times H^{-1}(\mathcal{M})} \leq \varepsilon \|(u_0, u_1)\|_{H_0^1(\mathcal{M}) \times L^2(\mathcal{M})}.$$

An application of Theorem 71 to the exact controllability problem (and a uniform quantitative version of it, with dependence of the observability constants with respect to different parameters) is developed in [48].

4.2. Historical remarks on unique continuation for waves. – The first general unique continuation result of the form (8) is the Holmgren-John Theorem 8 (due to Holmgren [26] in a special case, and to John [41] in the general case). This local unique continuation result enjoys a global version proved by John [41], where uniqueness is propagated through a family of noncharacteristic hypersurfaces (see Section 3.4.2 above).

When focusing on operators with (only) smooth (C^∞) coefficients, the first result is due to Carleman [16] who had the idea to conjugate the operator with an exponential weight to obtain unique continuation. He proved the result in the case of elliptic operators of order 2 in dimension 2. Calderón [15] extended the result to operators with simple characteristics. Namely, to situations where $p_\Psi(x_0, \xi) = 0 \implies \{p_\Psi, \Psi\}(x_0, \xi) \neq 0$. The most general result of Theorem 17 was proved by Hörmander [29, Chapter VIII], [33, Chapter XXVIII], still using Carleman estimates. Uniqueness across a hypersurface holds assuming the strong pseudoconvexity condition (Definition 16). Other works consider the limit case where higher-order of cancelation are considered. We refer, e.g., to [92, 65] for more details on this topic. Finally, Lerner and Robbiano [67] (see also [65, Section 6.4]) assume a slightly weaker form of the strong pseudoconvexity condition (21) (Definition 16) in which $\{p_2, \{p_2, \Psi\}\}(x_0, \xi) > 0$ is replaced by $\{p_2, \{p_2, \Psi\}\}(x_0, \xi) \geq 0$, at the cost of assuming that the support of the solution itself is compact.

Motivation to study the wave operator arises both from sismology [83], control theory [68, 69] and inverse problems [12, 71]. For the wave operator $P = \partial_t^2 - \Delta_g$ on $\Omega = (-T, T) \times \mathcal{M}$, the central question remained for a long time the validity of the unique continuation property (5). If $\mathcal{M}$ is analytic (and connected), the Holmgren-John Theorem 8 applies, which, together with the argument of John [41], allows one to prove unique continuation from $(-T, T) \times \omega$ for any nonempty open set ω as soon as $T > \mathcal{L}(\mathcal{M}, \omega)$. Removing the analyticity condition on $\mathcal{M}$ has led to a considerable difficulty, since Hörmander general uniqueness result does not apply in this setting and the Carleman-Hörmander strategy fails, as showed by the Alinhac-Baouendi counterexamples of Theorems 41 and 42.

This uniqueness problem in the C^∞ setting was first solved by Rauch-Taylor [73] and Lerner [63] in the case $T = \infty$, and $\mathcal{M} = \mathbb{R}^d$ (under different assumptions at infinity). Then, Robbiano [75] managed to prove that unique continuation from $(-T, T) \times \omega$ holds in any domain $\mathcal{M}$ as soon as $\omega \neq \emptyset$ and $T \geq C_0 \mathcal{L}(\mathcal{M}, \omega)$, with C_0 sufficiently large (a result qualified as “striking and surprising” by Hörmander [34]). Hörmander [32] improved this result down to $T > \sqrt{\frac{27}{23}} \mathcal{L}(\mathcal{M}, \omega)$. That these two results fail to hold in time $\mathcal{L}$ translates the fact that the local uniqueness results of these two authors are not valid across any noncharacteristic hypersurface. The proof of local uniqueness results across any noncharacteristic hypersurface for $\partial_t^2 - \Delta_g$ was reached by Tataru in [85] (Theorem 43), leading to the global unique continuation result in optimal time $T > \mathcal{L}(\mathcal{M}, \omega)$ (Theorem 66 and Corollary 67). The result of Tataru is not restricted to the wave operator: it holds for operators

with coefficients that are analytic in part of the variables, interpolating between the Holmgren theorem and the Hörmander theorem. Technical assumptions of this article were successively removed by Robbiano-Zuily [77], Hörmander [34] and Tataru [87], leading to a very general local unique continuation result for operators with partially analytic coefficients (containing as particular cases both Holmgren and Hörmander theorems).

The local, semiglobal and global quantitative versions of all these unique continuation results are proved in [49]. For the particular case of wave equations with time-independent coefficients, a global quantitative statement takes the form of Theorem 71. In the analytic setting, this result is a global quantitative version of the Holmgren-John theorem and can be proved with the theory developed by Lebeau in [59] (related local estimate were already proved by John [41]). In the C^∞ case, Robbiano [76] first proved a similar inequality for T sufficiently large and $Ce^{\kappa\mu}$ replaced by $Ce^{\kappa\mu^2}$, improved by Phung [72] to $C_\varepsilon e^{\kappa_\varepsilon\mu^{1+\varepsilon}}$. In [86], Tataru suggested a strategy to obtain $C_\varepsilon e^{\kappa_\varepsilon\mu^{1+\varepsilon}}$ in optimal time (in domains without boundaries). At the same time we proved the above Theorem 71, Bosi, Kurylev and Lassas [14] obtained a related result with $C_\varepsilon e^{\kappa_\varepsilon\mu^{1+\varepsilon}}$ dependence, on manifolds without boundaries.

4.3. Related references

4.3.1. *Treatises on unique continuation.* – There are many references on unique continuation and the related topics of Carleman estimates. We mention here a nonexhaustive list of books and monographs treating these topics. The classical reference on unique continuation for partial differential operators is the Chapter XXVIII of Lars Hörmander's treatise [33]. The latter gives a more general framework for what is described in Chapter 2. The recent book of Nicolas Lerner [65] also contains several results covered in the present notes. A short version of this book in the form of lecture notes can be found in [64]. We also refer to the book of Claude Zuily [92] and the notes of Daniel Tataru [86] for related questions. Carleman estimates for hyperbolic equations are also studied in [13] with a special emphasis on applications to inverse problems. The presentation of Chapter 3, concerning the wave operator, is inspired by the article [34]. Finally, the survey article by Jérôme Le Rousseau and Gilles Lebeau [56] is a smooth introduction to unique continuation for elliptic operators (which are only alluded here in Remark 18) and some of its applications to unique continuation and control of parabolic equations. See also the lecture notes [18] by Jérémie Dardé and Sylvain Ervedoza or [51, Appendix A] for an elementary route to Carleman estimates for elliptic operators. We finally refer to the recent books of Jérôme Le Rousseau, Gilles Lebeau and Luc Robbiano [57, 58] for an extensive treatment of Carleman estimates and unique continuation for elliptic operators, with plenty of applications.

4.3.2. *Some other recent developments for waves.* – We briefly discuss recent developments in the field of unique continuation and Carleman estimates for waves, that are not alluded in Section 4.2. The list is of course very far from exhaustive. Most

of these developments concern elaborations on global Carleman estimates for waves in a (global) setting close to that of Theorem 17, with applications to control theory, inverse problems or general relativity.

Global Carleman estimates for waves were proved in [24, Chapter 4] and [89] with applications to controllability. Various boundary conditions have been considered in [38]. Global Carleman estimates with limited regularity coefficients were investigated in [36], with applications to controllability and inverse problems (determining lower-order terms in the equation).

Admitting lower-order terms in unique continuation results is often crucial for treating nonlinear problems. For instance, a nonlinearity of the form u^p can be treated as a term qu with potential $q = u^{p-1}$, in general having limited regularity. A global unique continuation statement (close to Proposition 24 above) was proved in [78], with application to energy decay for nonlinear waves. Sometimes having nonlinear problems in mind, it has therefore been a goal to minimize the regularity of the admissible lower-order terms. This led to some “dispersive” Carleman estimates with Strichartz type spaces. The literature is vast, and we refer for instance to [45, 20, 46] and the references therein.

Global unique continuation results for nonlinear waves have been proved in [43, 44, 54], taking advantage of the nonlinear term. The global geometric assumptions, like Geometric Control Condition or weak trapping, allows one to prove the analyticity in time of the nonlinear solution. Results in the spirit of Section 3 can then be used to conclude the unique continuation.

Another problem not covered by these notes is the so called “strong unique continuation” property. Instead of assuming that the solution u is zero on an open set ω , one assumes infinite order of vanishing at a point or along a submanifold. We refer for instance to [60] for the case of unique continuation for waves from $[0, T] \times \{x_0\}$ and to [90, 91] for associated quantitative estimates.

Numerical analysis and algorithms were introduced in [11] based on Carleman estimates. Carleman estimates in a geometric context (extending Proposition 24 above) were proved in [21, 81, 40], with application to observability estimates and null-controllability. Recent developments involving Carleman estimates for waves to solve inverse problems include [10] (for developing recovery algorithms) or [1, 2] (adapting the Boundary Control method to variable Lorentzian metrics and describing geometrical situations in which the Hörmander condition of Theorem 17 holds). In [55], the author keeps track of two large parameters (here τ and λ) in Carleman estimates in the context of the (general version of the) Hörmander Theorem 17.

Unique continuation problems for wave operators also arise from mathematical general relativity. In [37, 66], the authors proved a unique continuation statement through transversally intersecting characteristic hypersurfaces, relying on the Hörmander Theorem 17. See also [3, 27, 28, 17] where unique continuation statements for waves from infinity are proved in various geometric settings, motivated by questions in general relativity.

The quantitative Theorem 71 was generalized in [52] to hypoelliptic waves, that is to say in which the elliptic operator Δ_g is replaced by a Hörmander sum of squares of vector fields. See also the review article [50]. Finally, Theorem 71 was also generalized recently in [23] to the case in which the metric g has a jump discontinuity across a hypersurface (a situation arising, e.g., in sismology).

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