On Critical Points of Eigenvalues of the Montgomery Family of Quartic Oscillators

BERNARD HELFFER & MATTHIEU LÉAUTAUD

ABSTRACT. We discuss spectral properties of the family of quartic oscillators $\mathfrak{h}_{\mathcal{M}}(\alpha) = -\mathrm{d}^2/\mathrm{d}t^2 + (\frac{1}{2}t^2 - \alpha)^2$ on the real line, where $\alpha \in \mathbb{R}$ is a parameter. This operator appears in a variety of applications coming from quantum mechanics to harmonic analysis on Lie groups, Riemannian geometry, and superconductivity. We study the variations of the eigenvalues $\lambda_j(\alpha)$ of $\mathfrak{h}_{\mathcal{M}}(\alpha)$ as functions of the parameter α. We prove that for j sufficiently large, $\alpha \mapsto \lambda_j(\alpha)$ has a unique critical point, which is a nondegenerate minimum. We also prove that the first eigenvalue $\lambda_1(\alpha)$ enjoys the same property, and give a numerically assisted proof that the same holds for the second eigenvalue $\lambda_2(\alpha)$. The proof for excited states relies on a semiclassical reformulation of the problem. In particular, we develop a method that allows us to differentiate with respect to the semiclassical parameter, which may be of independent interest.

CONTENTS

1. INTRODUCTION AND MAIN RESULTS

We consider the spectral properties of the family (indexed by $\alpha \in \mathbb{R}$) of self-adjoint realizations of the second-order differential operator

$$\mathfrak{h}_{\mathcal{M}}(\alpha) := -\frac{\mathrm{d}^2}{\mathrm{d}t^2} + \left(\frac{1}{2}t^2 - \alpha\right)^2,$$

on the real line. This operator appears in many different settings, namely, the following:

- in quantum mechanics (see Simon [42]; see also [41]);
- in the study of irreducible representations of certain nilpotent Lie groups of rank 3 (see [16, 17, 27, 40] and the references therein for considerations on analytic hypoellipticity[1] of hypoelliptic operators; see also [6] and [3] for a recent harmonic analysis on the so-called Engel group, and [8] for the analysis of a related sublaplacian);
- in Riemannian Geometry and the study of Schrödinger operators with magnetic fields on compact manifolds and in superconductivity (see [36]—hence the name of Montgomery attributed to this family—and [1, 14, 21–23, 25, 26, 38, 39]).

Our results concern the eigenvalues of the operator $\mathfrak{h}_{\mathcal{M}}(\alpha)$, acting on

$$D(\mathfrak{h}_{\mathcal{M}}(\alpha)) := \left\{u \in L^2(\mathbb{R}) \mid -u'' + \left(\frac{1}{2}t^2 - \alpha\right)^2 u \in L^2(\mathbb{R})\right\} \subset L^2(\mathbb{R}).$$

We first recall a list of elementary spectral properties of the operator $\mathfrak{h}_{\mathcal{M}}(\alpha)$ (see e.g., [3] for a detailed proof of all items but Item (1); the latter is proved in [42, Lemma II.1]).

Proposition 1.1. *For any $\alpha \in \mathbb{R}$, the operator $(\mathfrak{h}_{\mathcal{M}}(\alpha), D(\mathfrak{h}_{\mathcal{M}}(\alpha)))$ is self-adjoint on $L^2(\mathbb{R})$, with compact resolvent. Its spectrum consists of countably many real eigenvalues with finite multiplicities, accumulating only at $+\infty$. Moreover, we have the following properties:*

(1) *The domain*

$$D(\mathfrak{h}_{\mathcal{M}}(\alpha)) = D(\mathfrak{h}_{\mathcal{M}}(0)) = \{u \in H^2(\mathbb{R}) \mid t^4 u \in L^2(\mathbb{R})\}$$

does not depend on α.

(2) *All eigenvalues are simple and positive, and we may thus write*

$$\mathrm{Sp}(\mathfrak{h}_{\mathcal{M}}(\alpha)) = \{\lambda_j(\alpha) \mid j \in \mathbb{N}^*\}$$

with

$$0 < \lambda_1(\alpha) < \cdots < \lambda_j(\alpha) < \lambda_{j+1}(\alpha) \to +\infty,$$

[1] The existence of $\alpha \in \mathbb{C}$ such that $\mathfrak{h}_{\mathcal{M}}(\alpha)$ is not injective [40] leads to the proof of the non hypoanalyticity of the sublaplacian on the Engel group [16, 17].

$$\dim \ker(\mathfrak{h}_{\mathcal{M}}(\alpha) - \lambda_j(\alpha)) = 1.$$

(3) *All eigenfunctions are real analytic and decay exponentially fast at infinity* (*as well as all their derivatives*).
(4) *For all* $j \in \mathbb{N}^*$, *functions in* $\ker(\mathfrak{h}_{\mathcal{M}}(\alpha) - \lambda_j(\alpha))$ *have the parity of* $j + 1$.

For all $j \in \mathbb{N}^*$, we shall denote by $u_j(\alpha)$ any $L^2(\mathbb{R})$ normalized real-valued function in $\ker(\mathfrak{h}_{\mathcal{M}}(\alpha) - \lambda_j(\alpha))$. Any such family $(u_j(\alpha))_{j\in\mathbb{N}}$ forms a Hilbert basis of $L^2(\mathbb{R})$, according to Item (2).

Some properties of the first eigenvalue of $\mathfrak{h}_{\mathcal{M}}(\alpha)$ have been studied in [19].

The present paper investigates the dependence in α of the eigenvalues $\lambda_j(\alpha)$, and in particular the nature of its critical points. This study is motivated by a question of the second author with H. Bahouri, I. Gallagher, and D. Barilari (see [3]) related to the Schrödinger evolution equation on the Engel group. In that context, the study of dispersive properties involves that of oscillatory integrals whose phases contain $\lambda_j(\alpha)$. It is thus likely that dispersive estimates depend on the critical points of the maps $\alpha \mapsto \lambda_j(\alpha)$ for $j \in \mathbb{N}^*$.

As far as the variations of λ_j are concerned, classical arguments show that $\lambda_j(\alpha) \to +\infty$ as $\alpha \to \pm\infty$ and $\lambda'(\alpha) < 0$ for $\alpha \leq 0$ (see Section 2). Henceforth, λ_j has at least one critical point, namely, a global minimum. The graph of the first six eigenvalues $\lambda_j(\alpha)$ is plotted on Figure 1.1 (see also Section 5.6 for comments on this figure and additional numerical results).

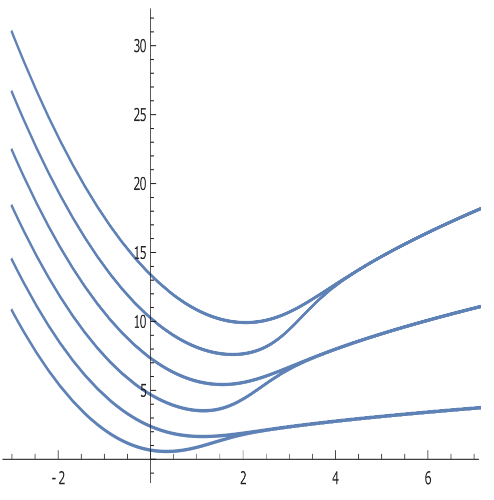

FIGURE 1.1. Graph of $\alpha \mapsto \lambda_j(\alpha)$ for $j = 1, \ldots, 6$.

Our first main result concerns the first eigenvalue λ_1.

Theorem 1.2. *The first eigenvalue* $\alpha \mapsto \lambda_1(\alpha)$ *of* $\mathfrak{h}_{\mathcal{M}}(\alpha)$ *has a unique critical point* α_c. *Moreover,* $\alpha_c \in (0, 1)$, λ_1 *reaches its minimum at* α_c, *that is,* $\lambda_1(\alpha_c) = \min_{\alpha\in\mathbb{R}} \lambda_1(\alpha)$, *and this minimum is nondegenerate, that is,* $\lambda_1''(\alpha_c) > 0$.

Theorem 1.2 was conjectured by Montgomery when replacing "critical point" by "minimum" [36], partially analyzed in [25], stated in Pan-Kwek [39], and verified with a computer assisted[2] but not completely mathematically rigorous proof in [22, 23]. One can also find in [14] a presentation of some of these results. In the case of the minimum, a complete proof was finally given in [19]. Improvements in the proof and a generalization to a larger class of operators appeared in [28].

Our second main result is a numerically assisted proof of the same property for λ_2.

Statement 1.3 (Numerically assisted). *λ_2 has a unique critical point $\alpha_{2,c}$. Moreover, this critical point is positive, corresponds to a minimum, and is nondegenerate.*

Next, we investigate the case of "large eigenvalues," that is, of $\lambda_j(\alpha)$ for sufficiently large values of j. We first provide with a rough asymptotic localization of critical points of λ_j for large j. We set

$$A_{j,c} = \{\alpha \in \mathbb{R} \mid \lambda_j'(\alpha) = 0\}, \quad \text{for } j \in \mathbb{N}^*.$$

Theorem 1.4. *Setting*

$$V(x) := \left(\frac{x^2}{2} - 1\right)^2 \quad \text{and} \quad x_+(E) := \sqrt{2 + 2\sqrt{E}}, \tag{1.1}$$

the function

$$(1, +\infty) \ni E \mapsto F(E) = \int_0^{x_+(E)} \frac{2 - x^2}{\sqrt{E - V(x)}}\, \mathrm{d}x \tag{1.2}$$

is of class C^1 and admits a unique zero E_c in the interval $(1, +\infty)$. For any choice of $\alpha_{j,c}$ in $A_{j,c}$, we have $\lim_{j\to+\infty} \alpha_{j,c} = +\infty$ and

$$\lim_{j\to+\infty} \frac{\lambda_j(\alpha_{j,c})}{\alpha_{j,c}^2} = E_c.$$

A numerical investigation shows that $E_c \approx 2.35$, with a relatively fast convergence with respect to j (see Section 5.6 for more on numerics).

Finally, our last main result proves the analogue of Theorem 1.2 for large eigenvalues.

Theorem 1.5. *There exists j_0 such that for $j \geq j_0$, λ_j admits a unique critical point $\alpha_{j,c}$ corresponding to a nondegenerate minimum. Moreover, we have*

$$\lambda_j''(\alpha_{j,c}) \to -3F'(E_c)G(E_c) > 0, \quad \text{as } j \to +\infty, \tag{1.3}$$

[2] This was done by numerical computations of V. Bonnaillie-Noël.

where $F'(E_c) < 0$ *and, (using the notation* (1.1)*),*

$$G(E) = \left(\int_0^{x_+(E)} (E - V(x))^{1/2}\,\mathrm{d}x \right) \times \left(\int_0^{x_+(E)} (E - V(x))^{-1/2}\,\mathrm{d}x \right)^{-2}. \tag{1.4}$$

Theorems 1.2 and 1.5 together with Statement 1.3 and Figure 1.1 naturally lead us to the following conjecture.

Conjecture 1.6. For any $j \geq 1$, λ_j has a unique critical point $\alpha_{j,c}$. Moreover, this critical point is positive, corresponds to a minimum, and is nondegenerate.

Note that $A_{j,c} \neq \emptyset$ because we will prove in (2.4) that for any j, $\lambda_j(\alpha)$ has at least one minimum. Conjecture 1.6 says in particular that $A_{j,c}$ has precisely one element for all $j \in \mathbb{N}^*$. Note also that, on account of Theorem 1.5, there is only a finite number of eigenvalues which remain uncovered.

As in [3, Section 2], there are different regions of the space of $(j, \alpha) \in \mathbb{N}^* \times \mathbb{R}$ to be considered separately. Here, it is rather straightforward to see that λ_j has no critical point in $\{\alpha \leq 0\}$ (see Corollary 2.2), so there are only two relevant regimes to consider.

The first regime is $\lambda_j(\alpha) \gg \alpha^2$, in which the operator $\mathfrak{h}_{\mathcal{M}}(\alpha)$ may be rescaled to (a small perturbation of) the operator $P^\varepsilon = -\varepsilon^2 \mathrm{d}^2/\mathrm{d}s^2 + s^4/4$. In this regime, we prove in Section 2.6 that there is no critical point asymptotically. The proof relies on a description of semiclassical measures associated with eigenfunctions of the operator P^ε following [33].

The second regime is $\lambda_j(\alpha) \lesssim \alpha^2$, in which the operator $\mathfrak{h}_{\mathcal{M}}(\alpha)$ may be rescaled to the double-well semiclassical Schrödinger operator $P_h = -h^2\mathrm{d}^2/\mathrm{d}x^2 + (x^2/2-1)^2$. It appears that all critical points belong asymptotically to this regime, and a thorough semiclassical analysis is needed to first localize critical points, and then prove their non-degeneracy. This is the heart of the proofs of Theorems 1.4 and 1.5 achieved, respectively, in Sections 3 and 4. The proof of Theorem 1.4 relies on the description of semiclassical measures associated with eigenfunctions of the operator P_h at all possible energy levels $E \in \mathbb{R}_+$ (and the semiclassical reformulation of the first variation formula for eigenvalues). In particular, this analysis yields $\lambda_j'(\alpha) \sim \Phi(E)\alpha$ when $\lambda_j(\alpha)/\alpha^2 \to E$ for an explicit function Φ (see Corollary 3.7 and Proposition 3.13). We then prove that the quantity $\Phi(E)$ vanishes if and only if $E = E_c$.

The description of semiclassical measures is unfortunately not sufficient for the analysis of the second derivative $\lambda_j''(\alpha)$, and thus for the proof of Theorem 1.5. Luckily enough, the energy E_c is a regular energy for the potential $(x^2/2-1)^2$ since $E_c > 1$, and we can push the semiclassical analysis beyond semiclassical measures at the energy E_c. To this end, we use an approximation of eigenfunctions by semiclassical Lagrangian distributions and prove that derivatives of the approximate and the true eigenfunctions are close to each other in the semiclassical limit.

We refer to the beginning of Section 4 for a detailed description of the proof of Theorem 1.5.

Finally, the proof of Theorem 1.2 is completed in Section 5. It relies on several refinements of the proof in [19]. In particular, we first obtain a refined estimate for the localization of possible critical points of λ_1. Then, we prove an upper bound for $\lambda_1(\alpha)$ and a lower bound for $\lambda_3(\alpha)$ which yield a positive lower bound for λ_1'' in this interval. The estimates on λ_1 and λ_3 follow by comparing the operator $\mathfrak{h}_{\mathcal{M}}(\alpha)$ with various harmonic oscillators. Numerical experiments performed by M. Persson-Sundqvist are presented in Subsection 5.6, and allow us to both check the accuracy of the results we obtain and prove Statement 1.3.

2. PRELIMINARIES

2.1. Previous results. We first collect various known basic results (see [3, 22, 25, 36, 39]). First, we have the following:

$$\text{For all } j \in \mathbb{N}^*, \text{ the map } \alpha \mapsto \lambda_j(\alpha) \text{ is continuous and satisfies } \lim_{|\alpha|\to+\infty} \lambda_j(\alpha) = +\infty. \tag{2.1}$$

When $\alpha \to -\infty$, this is a consequence of the direct bound $\lambda_1(\alpha) \geq \alpha^2$ if $\alpha < 0$. When $\alpha \to +\infty$, it results from standard semiclassical analysis (see Section 3 below) of the spectrum of the operator P_h, yielding, for all $k \in \mathbb{N}$,

$$\lambda_{2k+1}(\alpha) \sim \sqrt{2}(2k+1)\sqrt{\alpha}, \qquad \text{as } \alpha \to +\infty, \tag{2.2}$$

$$\lambda_{2k+2}(\alpha) - \lambda_{2k+1}(\alpha) = \mathcal{O}(\alpha^{-\infty}), \quad \text{as } \alpha \to +\infty. \tag{2.3}$$

The $\mathcal{O}(\alpha^{-\infty})$ decay (actually, this is an exponential decay, but this will not be used here) of $\lambda_{2k+2}(\alpha) - \lambda_{2k+1}(\alpha)$ is explained by a double well analysis of the problem (see [30], [18] or [11]). As a consequence, for any $j \in \mathbb{N}^*$,

$$\alpha \mapsto \lambda_j(\alpha) \quad \text{admits at least one minimum.} \tag{2.4}$$

We recall the first variation formula for the eigenvalues λ_j.

Proposition 2.1 (Feynman-Hellmann formula). *The map $\alpha \mapsto \lambda_j(\alpha)$ is C^∞, the map $\alpha \mapsto u_j(\cdot, \alpha)$, $\mathbb{R} \to D(\mathfrak{h}_{\mathcal{M}}(0))$ is of class C^1, and*

$$\begin{aligned} (\partial_\alpha u_j(\cdot,\alpha), u_j(\cdot,\alpha))_{L^2(\mathbb{R})} &= 0, \\ \lambda_j'(\alpha) = -2\int_{\mathbb{R}} \left(\frac{1}{2}t^2 - \alpha\right) u_j(t,\alpha)^2 \,\mathrm{d}t& \\ = -4\int_0^{+\infty} \left(\frac{1}{2}t^2 - \alpha\right) u_j(t,\alpha)^2 \,\mathrm{d}t&. \end{aligned} \tag{2.5}$$

Note that the second equality in (2.5) simply follows from the fact, stated in Proposition 1.1, that eigenfunctions are either odd or even, which implies that $t \mapsto (\frac{1}{2}t^2 - \alpha)u_j(t,\alpha)^2$ is even.

In particular, this implies the following result.

Corollary 2.2. *For all $j \in \mathbb{N}^*$, the function $\alpha \mapsto \lambda_j(\alpha)$ is strictly decreasing for $\alpha \le 0$ and all critical points of λ_j are in $]0, +\infty[$. In addition, λ_j admits a strictly positive minimum which can only be attained for positive α.*

Proof. Recalling that $u_j(\cdot, \alpha)$ is normalized in $L^2(\mathbb{R})$, equation (2.5) can be rewritten as

$$\lambda_j'(\alpha) = -||tu_j(\cdot, \alpha)||^2_{L^2(\mathbb{R})} + 2\alpha.$$

As a consequence, $\lambda_j'(\alpha) < 0$ for all $\alpha < 0$ and $\lambda_j'(0) = -\|tu_j(\cdot, 0)\|^2_{L^2(\mathbb{R})} < 0$ since $u_j(\cdot, \alpha)$ is normalized in $L^2(\mathbb{R})$. ◻

We also record here the following classical lemma, essentially following from Proposition 1.1 Item (4) (see, e.g., [39] for a proof).

Lemma 2.3. *For all $k \in \mathbb{N}^*$, $\alpha \in \mathbb{R}$, we have*

$$\lambda_{2k-1}(\alpha) = \lambda_k^N(\alpha), \quad \lambda_{2k}(\alpha) = \lambda_k^D(\alpha),$$

where $\lambda_k^N(\alpha)$ (respectively, $\lambda_k^D(\alpha)$) denotes the $k-th$ eigenvalue of the Neumann (respectively, Dirichlet) realization of the differential operator $-\mathrm{d}^2/\mathrm{d}t^2 + (\frac{1}{2}t^2 - \alpha)^2$ in $\mathbb{R}^+$.

In particular, we shall use

$$\lambda_1(\alpha) = \lambda_1^N(\alpha), \quad \lambda_3(\alpha) = \lambda_2^N(\alpha).$$

2.2. Dilation operators. In the present article, we shall make an intensive use of dilation operators. We define for $\eta > 0$ the following unitary (dilation) operator

$$\begin{aligned} T_\eta : L^2(\mathbb{R}) &\to L^2(\mathbb{R}), \\ u(x) &\mapsto \eta^{1/2} u(\eta x), \end{aligned} \tag{2.6}$$

having adjoint/inverse $T_\eta^* = T_\eta^{-1} = T_{\eta^{-1}}$.

2.3. Additional properties at a critical point. Of course, multiplying the eigenvalue equation

$$\mathfrak{h}_{\mathcal{M}}(\alpha)u(\cdot, \alpha) = \lambda(\alpha)u(\cdot, \alpha), \quad \|u(\cdot, \alpha)\|_{L^2(\mathbb{R})} = 1, \tag{2.7}$$

by $u(\cdot, \alpha)$ and integrating by parts, we obtain

$$\lambda(\alpha) = ||\partial_t u(\cdot, \alpha)||^2_{L^2(\mathbb{R})} + \left|\left|\left(\frac{t^2}{2} - \alpha\right) u(\cdot, \alpha)\right|\right|^2_{L^2(\mathbb{R})}. \tag{2.8}$$

In particular, we always have

$$\left|\left|\left(\frac{1}{2}t^2 - \alpha\right) u(\cdot, \alpha)\right|\right|^2_{L^2(\mathbb{R})} \le \lambda(\alpha). \tag{2.9}$$

Lemma 2.4. *Assume that* $(\lambda(\alpha), u(\cdot,\alpha))$ *satisfy the eigenvalue equation* (2.7). *Then, we have*

$$\alpha\lambda'(\alpha) + \lambda(\alpha) = 3\left\|\left(\frac{1}{2}t^2 - \alpha\right)u(\cdot,\alpha)\right\|^2_{L^2(\mathbb{R})}. \tag{2.10}$$

As a consequence of this lemma, one can improve a little the estimate (2.9) when $\alpha = \alpha_c$ is a critical point of λ (this is already stated in [39], Proposition 3.5 and (3.14)), namely,

$$3\left\|\left(\frac{1}{2}t^2 - \alpha_c\right)u(\cdot,\alpha_c)\right\|^2 = \lambda(\alpha_c). \tag{2.11}$$

The proof uses invariance by dilation T_η, defined in (2.6), of the spectrum.

Proof. Since T_η defined in (2.6) is unitary, we deduce from (2.7) that, for all $\eta > 0$ and $\alpha \in \mathbb{R}$,

$$T_\eta h_{\mathcal{M}}(\alpha)T_\eta^{-1}T_\eta u(\cdot,\alpha) = \lambda(\alpha)T_\eta u(\cdot,\alpha), \quad \|T_\eta u(\cdot,\alpha)\|_{L^2(\mathbb{R})} = 1,$$

where the conjugated operator is given by

$$T_\eta h_{\mathcal{M}}(\alpha)T_\eta^{-1} = -\frac{1}{\eta^2}\frac{\mathrm{d}^2}{\mathrm{d}t^2} + \left(\frac{1}{2}\eta^2t^2 - \alpha\right)^2.$$

That is to say, the spectrum is invariant and the eigenfunctions are only dilated. Setting $u(t,\eta,\alpha) := T_\eta u(t,\alpha)$, the above equation can be rewritten as

$$\begin{aligned}&-\frac{1}{\eta^2}\partial_t^2 u(t,\eta,\alpha) + \left(\frac{1}{2}\eta^2t^2 - \alpha\right)^2 u(t,\eta,\alpha) = \lambda(\alpha)u(t,\eta,\alpha),\\ &\|u(\cdot,\eta,\alpha)\|_{L^2(\mathbb{R})} = 1.\end{aligned}$$

Differentiating this expression with respect to η, we obtain

$$2\eta^{-3}\partial_t^2 u + 2\eta t^2\left(\frac{1}{2}\eta^2t^2 - \alpha\right)u + \mathfrak{h}_{\mathcal{M}}(\alpha)\,\partial_\eta u = \lambda\,\partial_\eta u,$$

together with

$$(u(\cdot,\eta,\alpha), \partial_\eta u(\cdot,\eta,\alpha))_{L^2(\mathbb{R})} = 0. \tag{2.12}$$

Taking now the scalar product of the previous identity with $u(\cdot,\eta,\alpha)$ and using (2.12), we deduce

$$\begin{aligned}-2\eta^{-3}\|\partial_t u\|^2 + 2\eta\int_{\mathbb{R}} t^2\left(\frac{1}{2}\eta^2t^2 - \alpha\right)u^2(t,\eta,\alpha)\,\mathrm{d}t&\\ + (\mathfrak{h}_{\mathcal{M}}(\alpha)\,\partial_\eta u, u)_{L^2(\mathbb{R})} = 0.&\end{aligned}$$

Selfadjointness of $\mathfrak{h}_{\mathcal{M}}(\alpha)$ together with (2.12) imply

$$(\mathfrak{h}_{\mathcal{M}}(\alpha)\,\partial_\eta u, u)_{L^2(\mathbb{R})} = (\partial_\eta u, \mathfrak{h}_{\mathcal{M}}(\alpha) u)_{L^2(\mathbb{R})} = \lambda(\alpha)(\partial_\eta u, u)_{L^2(\mathbb{R})} = 0,$$

and we finally deduce

$$-\eta^{-3}\|\partial_t u\|^2 + \eta \int_{\mathbb{R}} t^2 \left(\frac{1}{2}\eta^2 t^2 - \alpha\right) u^2(t,\eta,\alpha)\,\mathrm{d}t = 0.$$

Fixing now $\eta = 1$, this can be rewritten as

$$-\|\partial_t u\|^2 + 2\int_{\mathbb{R}} \left(\frac{1}{2}t^2 - \alpha + \alpha\right)\left(\frac{1}{2}t^2 - \alpha\right) u^2(t,\eta,\alpha)\,\mathrm{d}t = 0,$$

and, recalling the first variation formula (2.5), this implies

$$0 = -\|\partial_t u\|^2 + 2\left\|\left(\frac{1}{2}t^2 - \alpha\right) u(\cdot,\alpha)\right\|^2_{L^2(\mathbb{R})} - \alpha\lambda'(\alpha).$$

Combined with the energy identity (2.8), this implies (2.10) and concludes the proof of the lemma. □

2.4. An integration by parts formula. The following lemma will be useful in the proof of Theorem 1.2 and (as a technical device) Theorem 1.5. Although initially due to Pan-Kwek in [39] for λ_1 (see also [10] and [19]), the most elegant proof is given in [28]. We recall it quickly.

Lemma 2.5. *Assume that*

$$(\lambda(\alpha), u(\cdot,\alpha))$$

satisfy the eigenvalue equation (2.7). *Then, we have*

$$\begin{aligned}2\int_0^{+\infty} (t - \sqrt{2\alpha})\left(\frac{t^2}{2} - \alpha\right) u^2(t,\alpha)\,\mathrm{d}t - \frac{\sqrt{\alpha}}{\sqrt{2}}\lambda'(\alpha) \\ = (\lambda(\alpha) - \alpha^2)u(0,\alpha)^2 + u'(0,\alpha)^2.\end{aligned} \tag{2.13}$$

Proof. Using integration by parts together with $u(\cdot,\alpha) \in S(\mathbb{R})$, according to Proposition 1.1, we have

$$\begin{aligned}&\int_0^{+\infty} \frac{\mathrm{d}}{\mathrm{d}t}\left(\left(\frac{1}{2}t^2 - \alpha\right)^2\right) u^2(t,\alpha)\,\mathrm{d}t \\ &\quad = \alpha^2 u^2(0,\alpha) - 2\int_0^{+\infty}\left(\frac{1}{2}t^2 - \alpha\right)^2 u(t,\alpha)u'(t,\alpha)^2\,\mathrm{d}t.\end{aligned}$$

The eigenvalue equation (2.7) then implies

$$\begin{aligned}(2.14)\qquad \int_0^{+\infty} \frac{\mathrm{d}}{\mathrm{d}t}\left(\left(\frac{1}{2}t^2-\alpha\right)^2\right)u^2(t,\alpha)\,\mathrm{d}t \\ = -\alpha^2u^2(0,\alpha) - 2\int_0^{+\infty}\lambda(\alpha)u(t,\alpha)u'(t,\alpha)\,\mathrm{d}t \\ - 2\int_0^{+\infty}u''(t,\alpha)u'(t,\alpha)\,\mathrm{d}t \\ = (\lambda(\alpha)-\alpha^2)u(0,\alpha)^2 + u'(0,\alpha)^2.\end{aligned}$$

We now compute the lefthand side as

$$\begin{aligned}\int_0^{+\infty} \frac{\mathrm{d}}{\mathrm{d}t}\left(\left(\frac{1}{2}t^2-\alpha\right)^2\right)u^2(t,\alpha)\,\mathrm{d}t \\ = \int_0^{+\infty} 2t\left(\frac{1}{2}t^2-\alpha\right)u^2(t,\alpha)\,\mathrm{d}t \\ = \int_0^{+\infty} 2(t-\sqrt{2\alpha})\left(\frac{1}{2}t^2-\alpha\right)u^2(t,\alpha)\,\mathrm{d}t \\ + 2\sqrt{2\alpha}\int_0^{+\infty}\left(\frac{1}{2}t^2-\alpha\right)u^2(t,\alpha)\,\mathrm{d}t.\end{aligned}$$

Using the Feynman-Hellmann formula (2.5), this is rewritten as

$$\begin{aligned}\int_0^{+\infty} \frac{\mathrm{d}}{\mathrm{d}t}\left(\left(\frac{1}{2}t^2-\alpha\right)^2\right)u^2(t,\alpha)\,\mathrm{d}t \\ = 2\int_0^{+\infty}(t-\sqrt{2\alpha})\left(\frac{1}{2}t^2-\alpha\right)u^2(t,\alpha)\,\mathrm{d}t - \frac{\sqrt{\alpha}}{\sqrt{2}}\lambda'(\alpha).\end{aligned}$$

Combined with (2.14), this proves (2.13) and concludes the proof of the lemma. □

As an immediate first application we have the following proposition.

Proposition 2.6. *Assume that j is odd and that $\alpha_{j,c}$ is a critical point of $\lambda_j(\alpha)$. Then,*

$$(2.15)\qquad \alpha_{j,c}^2 < \lambda_j(\alpha_{j,c}).$$

Proof. When j is odd, we recall from Proposition 1.1 that $u_j(\cdot,\alpha)$ is even, so that

$$u_j'(0,\alpha) = 0.$$

We deduce from the universal formula (2.13) that if in addition

$$\lambda_j'(\alpha_{j,c}) = 0,$$

then

$$\begin{aligned}
&(\lambda_j(\alpha_{j,c}) - \alpha_{j,c}^2)u_j(0,\alpha_{j,c})^2 \\
&\quad = 2\int_0^{+\infty} (t\sqrt{2\alpha_{j,c}})\left(\frac{1}{2}t^2 - \alpha_{j,c}\right) u_j^2(t,\alpha_{j,c})\,\mathrm{d}t, \quad j \text{ odd} \\
&\quad = \int_0^{+\infty} (t + \sqrt{2\alpha_{j,c}})(t - \sqrt{2\alpha_{j,c}})^2 u_j^2(t,\alpha_{j,c})\,\mathrm{d}t.
\end{aligned}$$

Observing that the righthand side is positive (u_j does not vanish identically on an open set; otherwise it would vanish identically on $\mathbb{R}$) concludes the proof of the proposition. ◻

2.5. The second derivative. For the analysis of the second derivative, we also need the following formula established in [22] and probably long before. As in the proof of the Feynman-Hellmann formula, we start from

$$\left(\mathfrak{h}_{\mathcal{M}}(\alpha) - \lambda_j(\alpha)\right)\frac{\partial u_j(\cdot,\alpha)}{\partial\alpha} = \left[2\left(\frac{t^2}{2} - \alpha\right) + \lambda_j'(\alpha)\right] u_j(\cdot,\alpha). \tag{2.16}$$

Differentiating once more and taking the scalar product with u_j we obtain

$$\lambda_j''(\alpha) = 2 - 4\int \left(\frac{1}{2}t^2 - \alpha\right) u_j(t,\alpha)\,\partial_\alpha u_j(t,\alpha)\,\mathrm{d}t, \tag{2.17}$$

where we have used that u_j is normalized. This implies

$$\begin{aligned}
\lambda_j''(\alpha) &= 2 - 2\int t^2 u_j(t,\alpha)\,\partial_\alpha u_j(t,\alpha)\,\mathrm{d}t \\
&= 2 - \frac{\mathrm{d}}{\mathrm{d}\alpha}\left(\int t^2 u_j(t,\alpha)^2\,\mathrm{d}t\right).
\end{aligned} \tag{2.18}$$

Note finally that we can rewrite the last equality in a form where the normalization condition is not necessarily assumed:

$$\begin{aligned}
\lambda_j''(\alpha) &= 2 - 2\int t^2 u_j(t,\alpha)\,\partial_\alpha u_j(t,\alpha)\,\mathrm{d}t \\
&= 2 - \frac{\mathrm{d}}{\mathrm{d}\alpha}\left(\left(\int t^2 u_j(t,\alpha)^2\,\mathrm{d}t\right)\left(\int u_j(t,\alpha)^2\,\mathrm{d}t\right)^{-1}\right).
\end{aligned}$$

2.6. No critical point in the regime $\lambda_j(\alpha_j) \gg \alpha_j^2$. We prove in this preliminary section that there is no critical point in the regime $\alpha_j^2/\lambda_j(\alpha_j) \to 0$. The latter are thus necessarily located in the regime $\lambda_j(\alpha_j) \lesssim \alpha_j^2$, which we investigate in the next sections.

Proposition 2.7. *Let $(\alpha_j)_{j\in\mathbb{N}}$ a real sequence, and assume $\lambda_j(\alpha_j) \to +\infty$ and $\alpha_j^2/\lambda_j(\alpha_j) \to 0$, as $j \to +\infty$. Then, we have*

$$\frac{\lambda_j'(\alpha_j)}{\alpha_j\lambda_j(\alpha_j)^{1/2}} \to -K_1 \int_{-\sqrt{2}}^{\sqrt{2}} \frac{s^2}{\sqrt{1-s^4/4}}\,\mathrm{d}s, \qquad \text{where } K_1 = \left(\int_{-\sqrt{2}}^{\sqrt{2}} \frac{\mathrm{d}s}{\sqrt{1-s^4/4}}\right)^{-1}, \tag{2.19}$$

as $j \to +\infty$.

Note that in this regime, α does not necessarily converge to $+\infty$. Note also that $\min_{\alpha\in\mathbb{R}} \lambda_j(\alpha) \to +\infty$ as $j \to +\infty$, so that the assumption $\lambda_j(\alpha_j) \to +\infty$ is implied by $j \to +\infty$.

For the next result, recall that $A_{j,c}$ is the set of critical points of $\alpha \mapsto \lambda_j(\alpha)$.

Corollary 2.8. *For any choice of $\alpha_{j,c}$ in $A_{j,c}$ we have*

$$\limsup_{j\to+\infty} \frac{\lambda_j(\alpha_{j,c})}{\alpha_{j,c}^2} < +\infty.$$

Proof of Corollary 2.8. Suppose that there exists a subsequence j_k such that $\lambda_{j_k}(\alpha_{j_k,c})/\alpha_{j_k,c}^2$ tends to $+\infty$ and $\alpha_{j_k,c} \in A_{j_k,c}$. Then, by applying Proposition 2.7, the lefthand side of (2.19) equals 0 since $\alpha_{j_k,c}$ is critical, whereas the righthand side is negative. This yields a contradiction. □

The proof of Proposition 2.7 relies on a semiclassical rescaling and the explicit description of associated semiclassical measures. The latter point is rather classical in the present 1D setting (see, e.g., [15, pp. 139–143; 24, Theorem 4.1; 33, Theorem 1.6] or [35, Proposition 5.1; 37, Section 6 pp. 190–198] or [31]).

Proof of Proposition 2.7. We drop the index j for readability. We start from the eigenvalue equation (3.3) with $\lambda = \lambda(\alpha)$, in which we set $w = T_{\lambda^{1/4}}u$, where the dilation operator T_η is defined in (2.6). We obtain

$$\left(-\frac{1}{\lambda^{1/2}}\frac{\mathrm{d}^2}{\mathrm{d}s^2} + \left(\lambda^{1/2}\frac{s^2}{2} - \alpha\right)^2\right)w = \lambda w.$$

Dividing by λ, we obtain a semiclassical problem with semiclassical parameter $\varepsilon := \lambda^{-3/4} \to 0$ (by assumption), namely,

$$\left(-\varepsilon^2\frac{\mathrm{d}^2}{\mathrm{d}s^2} + \left(\frac{s^2}{2} - \delta\right)^2\right)w = w, \quad \text{with } \delta = \frac{\alpha}{\lambda^{1/2}} \to 0 \text{ (by assumtion).}$$

Let μ be a semiclassical measure at scale ε_k, associated with a sequence $(w_k, \varepsilon_k, \delta_k)$ of solutions to this equation (see, e.g., [33] for a definition). The energy estimate

$$||\varepsilon_k w_k'||^2_{L^2(\mathbb{R})} + \left|\left|\left(\frac{s^2}{2} - \delta_k\right) w_k\right|\right|^2_{L^2(\mathbb{R})} = ||w_k||^2_{L^2(\mathbb{R})} = 1$$

shows on the one hand that $\|\varepsilon_k w_k'\|_{L^2(\mathbb{R})}$ is bounded, whence

$$\limsup_{k\to+\infty} \|\mathcal{F}(w_k)(\xi)\|_{L^2(\varepsilon_k|\xi|\geq R)} \to 0 \quad \text{as } R \to +\infty$$

(that is to say; the sequence w_k is ε_k-oscillating), and on the other hand that for k sufficiently large so that $\delta_k \leq 1$,

$$\frac{1}{2}\|s^2 w_k\|_{L^2(\mathbb{R})} \leq \left|\left|\left(\frac{s^2}{2} - \delta_k\right) w_k\right|\right|_{L^2(\mathbb{R})} + \|w_k\|_{L^2(\mathbb{R})} \leq 2,$$

whence $\limsup_{k\to+\infty} \|w_k\|_{L^2(|x|\geq R)} \to 0$ as $R \to +\infty$ (that is to say, w_k is compact at infinity). As a consequence (see, e.g., [33]), the semiclassical measure μ is a probability measure on $\mathbb{R}^2$, and its projection on the x-space, $\pi_*\mu$ is a probability measure on $\mathbb{R}$. Then, the semiclassical pseudodifferential calculus shows (see, e.g., [33, 43]) that $\mathrm{supp}(\mu) \subset \{(s,\xi) \in \mathbb{R}^2 \mid p(s,\xi) = 1\}$ with $p(s,\xi) = \xi^2 + s^4/4$, and that μ is H_p-invariant, namely, $H_p\mu = 0$. As a consequence, μ is the unique invariant probability measure supported by $\{(s,\xi) \in \mathbb{R}^2 \mid p(x,\xi) = 1\}$, given explicitly as follows (see [33, Theorem 1.6]): for all $a \in C_c^\infty(\mathbb{R}^2)$,

$$\langle \mu, a\rangle = \frac{K_1}{2}\sum_{\pm}\int_{-\sqrt{2}}^{\sqrt{2}} a(s, \pm\sqrt{1 - s^4/4})\frac{\mathrm{d}s}{\sqrt{1 - s^4/4}},$$
$$K_1 = \left(\int_{-\sqrt{2}}^{\sqrt{2}} \frac{\mathrm{d}s}{\sqrt{1 - s^4/4}}\right)^{-1}.$$

Combined with Lemma 3.4 below (stating in a similar context that eigenfunctions decay at infinity faster than any polynomial), this implies that for all $a \in C^\infty(\mathbb{R})$ having at most polynomial growth at infinity,

$$\int_{\mathbb{R}} a(s)|w_k(s)|^2\,\mathrm{d}s \to K_1\int_{-\sqrt{2}}^{\sqrt{2}} \frac{a(s)}{\sqrt{1 - s^4/4}}\,\mathrm{d}s.$$

Since this convergence holds for any sequence $(w_k, \varepsilon_k, \delta_k)$ of solutions, it always holds for the full sequence (w, ε, δ).

Recalling that $u = T_{\lambda^{-1/4}} w$, we now deduce from the first variation formula (2.5) (and using that eigenfunctions are assumed normalized) that

$$\lambda'(\alpha) = 2\alpha - \int_{\mathbb{R}} t^2|u_k|^2(t)\,\mathrm{d}t = \alpha\left(2 - \lambda^{1/2}\int_{\mathbb{R}} s^2|w_k|^2(s)\,\mathrm{d}s\right).$$

Recalling that $\lambda = \lambda(\alpha) \to +\infty$ by assumption, we finally deduce that

$$\lim \frac{\lambda'(\alpha)}{\alpha\lambda(\alpha)^{1/2}} = \langle \pi_* \mu, s^2 \rangle = -K_1 \int_{-\sqrt{2}}^{\sqrt{2}} \frac{s^2}{\sqrt{1 - s^4/4}} \, ds.$$

As this holds for any subsequence $(u_k, \lambda_k, \alpha_k)$, it holds for any triple (u, λ, α). ◻

3. SEMICLASSICAL MEASURES IN $\lambda_j(\alpha_j) \lesssim \alpha_j^2$ AND PROOF OF THEOREM 1.4

3.1. Semiclassical rescaling. Recalling the definition of the dilation operator T_η in (2.6), and choosing $\eta = \sqrt{\alpha}$ for the scaling parameter, we obtain

$$\begin{aligned} T_{\sqrt{\alpha}} \mathfrak{h}_{\mathcal{M}}(\alpha) T_{\sqrt{\alpha}}^{-1} &= -\frac{1}{\alpha} \frac{d^2}{ds^2} + \alpha^2 \Big(\frac{s^2}{2} - 1 \Big)^2 \\ &= \alpha^2 \Big(-h^2 \frac{d^2}{ds^2} + \Big(\frac{s^2}{2} - 1 \Big)^2 \Big) = \alpha^2 P_h, \end{aligned} \tag{3.1}$$

with

$$h = \alpha^{-3/2}, \quad P_h := -h^2 \frac{d^2}{ds^2} + \Big(\frac{s^2}{2} - 1 \Big)^2. \tag{3.2}$$

Hence, the analysis as $\alpha \to +\infty$ becomes an analysis as $h \to 0$. Recall that we consider the real, L^2 normalized eigenfunction u (respectively, u_k) associated with the eigenvalue λ (respectively, λ_k), that is to say, the solution to

$$\mathfrak{H}_{\mathcal{M}}(\alpha) u = \lambda u, \quad u \in D(\mathfrak{H}_{\mathcal{M}}(\alpha)), \ \|u\|_{L^2(\mathbb{R})} = 1. \tag{3.3}$$

After the semiclassical rescaling (3.1), we set

$$v(\cdot, h) := T_{\sqrt{\alpha}} u(\cdot, \alpha), \quad \text{resp. } v_k(\cdot, h) := T_{\sqrt{\alpha}} u_k(\cdot, \alpha), \tag{3.4}$$

and (recalling that $\alpha = h^{-2/3}$),

$$E(h) := \frac{\lambda(\alpha)}{\alpha^2} = h^{4/3} \lambda(\alpha), \quad \text{resp. } E_k(h) := \frac{\lambda_k(\alpha)}{\alpha^2} = h^{4/3} \lambda_k(\alpha), \tag{3.5}$$

which according to (3.3) now solve the semiclassical eigenvalue equation

$$P_h v = E(h) v, \quad v \in D(P_h), \ \|v\|_{L^2(\mathbb{R})} = 1, \tag{3.6}$$

where we recall that the operator P_h is defined in (3.2).

The first variation formula (2.5) for the eigenvalue equation becomes

$$\lambda'(\alpha) = 2\alpha - ||xu(\cdot,\alpha)||_{L^2}^2 = h^{-2/3}(2 - ||xv(\cdot,h)||_{L^2}^2). \tag{3.7}$$

Localizing critical points of λ is thus equivalent to solving $2 - \|xv(\cdot,h)\|_{L^2}^2 = 0$, where $v(\cdot,h)$ solves (3.6). We finally reformulate the second variation formula (2.18) in the semiclassical setting.

Lemma 3.1. *With $\alpha = h^{-2/3}$ and $v(\cdot,h)$ as in (3.4), we have*

$$\lambda''(\alpha) = \left(1 - \frac{3}{2}h\,\partial_h\right)\left(\int_{\mathbb{R}}(2-s^2)|v(s,h)|^2\,\mathrm{d}s\right). \tag{3.8}$$

Proof. We recall from (3.4) with $h = \alpha^{-3/2}$ that

$$u(t,\alpha) = (T_{\alpha^{-1/2}}v)(t,h) = \alpha^{-1/4}v(\alpha^{-1/2}t,\alpha^{-3/2}). \tag{3.9}$$

From (2.18), we deduce

$$\begin{aligned}
\lambda''(\alpha) &= 2 - \partial_\alpha\left(\int_{\mathbb{R}} x^2\alpha^{-1/2}|v(\alpha^{-1/2}x,\alpha^{-3/2})|^2\,\mathrm{d}x\right)\\
&= 2 - \partial_\alpha\left(\alpha\int_{\mathbb{R}} s^2|v(s,\alpha^{-3/2})|^2\,\mathrm{d}s\right)\\
&= 2 - \left(\int_{\mathbb{R}} s^2|v(s,\alpha^{-3/2})|^2\,\mathrm{d}s\right) - \alpha\,\partial_\alpha\left(\int_{\mathbb{R}} s^2|v(s,\alpha^{-3/2})|^2\,\mathrm{d}s\right)\\
&= 2 - \int_{\mathbb{R}} s^2|v(s,\alpha^{-3/2})|^2\,\mathrm{d}s + \frac{3}{2}\alpha^{-3/2}\,\partial_h\left(\int_{\mathbb{R}} s^2|v(s,\alpha^{-3/2})|^2\,\mathrm{d}s\right)\\
&= 2 - \int_{\mathbb{R}} s^2|v(s,h)|^2\,\mathrm{d}s + \frac{3}{2}h\,\partial_h\left(\int_{\mathbb{R}} s^2|v(s,h)|^2\,\mathrm{d}s\right),
\end{aligned}$$

where we have used that $\partial_\alpha(f(\alpha^{-3/2})) = -\frac{3}{2}\alpha^{-5/2}f'(\alpha^{-3/2})$ in the penultimate equality with $f(h) = \int_{\mathbb{R}} s^2|v(s,h)|^2\,\mathrm{d}s$. Finally, (3.8) follows from using the normalization of $v(\cdot,h)$. ◻

Remark 3.2. Since we have $\lambda(\alpha) = \alpha^2E(\alpha^{-3/2})$, we also have the following direct relations between the variations of the eigenvalues of $\mathfrak{h}_{\mathcal{M}}(\alpha)$ and those of the semiclassical operator P_h (in the present regime):

$$\begin{aligned}
\frac{\lambda(\alpha)}{\alpha^2} &= E(h), \quad h = \alpha^{-3/2},\\
\frac{\lambda'(\alpha)}{\alpha} &= 2E(h) - \frac{3}{2}hE'(h), \quad h = \alpha^{-3/2},\\
\lambda''(\alpha) &= 2E(h) - \frac{9}{4}hE'(h) + \frac{1}{4}h^2E''(h), \quad h = \alpha^{-3/2}.
\end{aligned}$$

3.2. Double well analysis. The operator P_h defined in (3.2) has the form $P_h := -h^2\mathrm{d}^2/\mathrm{d}s^2 + V(s)$ with

$$V(s) = \left(\frac{s^2}{2} - 1\right)^2$$

and the standard multidimensional semiclassical analysis (see, e.g., Helffer-Robert [29], Helffer-Sjöstrand [30], and references therein) can be used. The potential is a symmetric double well potential and the two minima occur at $\pm\sqrt{2}$ with energy 0. The quadratic approximation at the bottom leads to the following result [30].

Lemma 3.3 (Bottom of the spectrum). *For all $\beta \in \mathbb{R}_+ \setminus \sqrt{2}(2\mathbb{N}+1)$, there are $N_\beta \in \mathbb{N}$, $h_0 > 0$ such that*

$$\mathrm{Sp}(P_h) \cap (-\infty, \beta h) = \{E_n(h) \mid n \in \{1, \dots, N_\beta\}\},$$

uniformly for $h \in (0, h_0)$, with $0 < E_n(h) < E_{n+1}(h) < \beta h$ for all $n \in \{1, \dots, N_\beta - 1\}$ and $h \in (0, h_0)$. Moreover, as $h \to 0^+$, we have

$$E_{2j+1}(h) \sim \sqrt{2}(2j+1)h, \tag{3.10}$$

$$E_{2j+2}(h) - E_{2j+1}(h) = \mathcal{O}(h^{+\infty}), \tag{3.11}$$

uniformly for all $j \in \mathbb{N}$ such that $2j + 2 \le N_\beta$.

Note that (3.11) is a weak form of the tunneling analysis. The asymptotics (2.2) and (2.3) are straightforward consequences of (3.10)–(3.11) together with (3.5).

In addition to the two minima, the potential V admits a critical point at $s = 0$ corresponding to a local maximum with $V(0) = 1$. For energy levels E away from $E = 1$, one can describe the whole spectrum by using Bohr-Sommerfeld formulas which are specific to dimension one.

We now recall the localization properties of the eigenfunctions in the so-called classical region determined by their corresponding eigenvalue. For a given energy E, we define the classically allowed region by

$$K_E := V^{-1}((-\infty, E]).$$

Proposition 3.4. *Let $E \in \mathbb{R}$. For any sequence $(E_{j(h)}(h), v_{j(h)}(h))$ of eigenpairs of P_h such that*

$$\limsup_{h\to 0} E_{j(h)}(h) \le E$$

and $\|v_{j(h)}(h)\|_{L^2(\mathbb{R})} = 1$, and any closed interval I such that $I \cap K_E = \varnothing$, we have for any k

$$\|v_{j(h)}(h)\|_{B^k(I)} = \mathcal{O}(h^\infty),$$

where

$$B^k(I) = \{u \in L^2(I) \mid s^\ell u^{(m)}(s) \in L^2(I),\ \forall\,(\ell, m) \text{ s.t. } \ell + m \le k\},$$
$$\|u\|_{B^k(I)} = \sup_{\ell+m\le k} \|s^\ell u^{(m)}\|_{L^2(I)}.$$

This is a consequence of classical Agmon estimates (see [30] or [18]). The latter actually yield an exponential decay of eigenfunctions away from the classically allowed region K_E, but this is not needed in our analysis.

3.3. Semiclassical measures. We now only consider the semiclassical regime $h \to 0^+$ and consider the limit measures for the densities $v(x,h)^2\,\mathrm{d}x$. Taking advantage of the parity of the potential $V = (x^2/2 - 1)^2$, we define for any given energy $E \ge 0$

$$x_+(E) = \sqrt{2 + 2\sqrt{E}} \quad \text{if } E \in [0, +\infty), \tag{3.12a}$$

$$x_-(E) = \begin{cases} \sqrt{2 - 2\sqrt{E}} & \text{if } E \in [0, 1], \\ 0 & \text{if } E \ge 1. \end{cases} \tag{3.12b}$$

Notice that $x_\pm$ is defined so that for all $E \ge 0$,

$$[x_-(E), x_+(E)] = \{x \ge 0 \mid V(x) \le E\} = K_E \cap \mathbb{R}_+$$

is the classically allowed region intersected with $\mathbb{R}^+$. That is,

$$\begin{aligned} K_E &= [-x_+(E), x_+(E)] && \text{if } E \ge 1, \\ K_E &= [-x_+(E), -x_-(E)] \cup [x_-(E), x_+(E)] && \text{if } E \le 1. \end{aligned}$$

The densities $v(x,h)^2\,\mathrm{d}x$ satisfy the following asymptotic repartition.

Proposition 3.5. *Assume v solves (3.6) with $E(h) \to E$ and $h \to 0^+$; then, $v(x,h)^2\,\mathrm{d}x \rightharpoonup \mathfrak{m}_E$ in the sense of measures in $\mathbb{R}$ where $\mathfrak{m}_E$ is the nonnegative Radon measure given by*

$$\mathfrak{m}_E = \frac{1}{2}(\delta_{-\sqrt{2}} + \delta_{\sqrt{2}}) \qquad \text{if } E = 0, \tag{3.13}$$

$$\mathfrak{m}_E = \frac{C(E)}{2}\,\frac{\mathbf{1}_{[-x_+(E),-x_-(E)]}(x)\,\mathrm{d}x}{\sqrt{E - V(x)}} + \frac{C(E)}{2}\,\frac{\mathbf{1}_{[x_-(E),x_+(E)]}(x)\,\mathrm{d}x}{\sqrt{E - V(x)}} \qquad \text{if } E \in (0,1), \tag{3.14}$$

$$\mathfrak{m}_E = \delta_0 \qquad \text{if } E = 1, \tag{3.15}$$

$$\mathfrak{m}_E = \frac{C(E)}{2}\,\frac{\mathbf{1}_{[-x_+(E),x_+(E)]}(x)\,\mathrm{d}x}{\sqrt{E - V(x)}} \qquad \text{if } E > 1, \tag{3.16}$$

where

$$C(E) = \left(\int_{x_-(E)}^{x_+(E)} (E - V(x))^{-1/2}\,\mathrm{d}x \right)^{-1}, \quad E \in (0,1) \cup (1,\infty). \tag{3.17}$$

Note that the expression for $E \in (0,1)$ is also valid for $E > 1$. The continuity of the map $E \mapsto \mathfrak{m}_E$ at $E = 1$ for the weak-* topology of measures is investigated in Proposition 3.9 below.

Proof of Proposition 3.5. Note first that all since eigenfunctions are either odd or even, the measures $v(x,h)^2\,\mathrm{d}x$ are all even and so are their limits. The case $E > 1$ is proved in, for example, [24, Theorem 4.1] (before the introduction of the theory of semiclassical measures) or more recently in [33, Theorem 1.6] (see also pages 139–143 in [15] or [37, Section 6 pp. 190–198] for different approaches on a similar problem and [35, Proposition 5.1] and [31] for related statements). Next, the case $E < 1$ follows the same once restricted to the half real line (thus, a solution to a Dirichlet or a Neumann problem on $\mathbb{R}^+$), and by using parity of the limit measures.

Finally, the case $E = 1$ follows from [9, Section 5]. For the sake of completeness, we give a short proof inspired from the proof of Theorem 16 in Section 2.2 of [33] in the case $E = 1$. In case $E(h) \to 1$, any semiclassical measure μ (see, e.g., [33] for a definition and a proof of these properties) is a probability measure on $\mathbb{R}^2$, supported by $p^{-1}(1) := \{(x,\xi) \in \mathbb{R}^2 \mid p(x,\xi) = 1\}$ where $p(x,\xi) = \xi^2 + (x^2/2 - 1)^2$, and moreover invariant by the Hamilton flow of p inside $p^{-1}(1)$, that is, $0 = H_p\mu = (2\xi\,\partial_x - 2x(x^2/2-1)\,\partial_\xi)\mu$. Now, the set $p^{-1}(1)$ is partitioned into three H_p-invariant disjoint sets:

$$p^{-1}(1) = \{(0,0)\} \sqcup C_+ \sqcup C_-,$$

with

$$C_\pm := \{(x,\xi) \in \mathbb{R}^2 \mid \pm x > 0,\ p(x,\xi) = 1\}.$$

As a consequence, $\mu = \mu_- + \alpha\delta_{(0,0)} + \mu_-$ where $\alpha \in [0,1]$ and $\mu_\pm$ is an invariant finite measure supported by $C_\pm$. But since $C_\pm$ is not compact, the only invariant finite measure it carries is $\mu_\pm = 0$. Since μ is a probability measure, we deduce $\mu = \delta_{(0,0)}$ and $\mathfrak{m}_1 = \pi_*\mu = \delta_0$ (see [33] for a proof of the first equality). Uniqueness of the limit measure shows that the full sequence converges. ◻

Remark 3.6. Note that in the single well problem [33], it is enough that $v(x,h)$ solves (3.6) approximately, that is to say, with a $o(h)$ remainder, in order to characterize the limit measure μ and $\mathfrak{m}_E$. In the double well regime, we actually use here in an essential way that $v(x,h)$ is a genuine eigenfunction; that is, it solves (3.6) exactly (it would actually work if $v(x,h)$ would solve (3.6) approximately with a $O(e^{-S_1/h})$ remainder, with S_1 sufficiently large) because of the tunnelling effect between the two wells (see [30]).

Corollary 3.7. *Let $E \geq 0$, and assume that $\alpha \to +\infty$ and $\lambda(\alpha)/\alpha^2 \to E$; then,*

$$\alpha^{-1}\lambda'(\alpha) \to \Phi(E), \tag{3.18}$$

with

$$\Phi(E) := \begin{cases} 0, & \text{if } E = 0, \\ C(E)F(E), & \text{if } E \in (0,1) \cup (1,\infty), \\ 2, & \text{if } E = 1, \end{cases} \tag{3.19}$$

where, for $E \in (0,1) \cup (1,\infty)$, $C(E)$ is defined in (3.17) *and*

$$\begin{aligned} F(E) &:= \int_{x_-(E)}^{x_+(E)} \frac{2-x^2}{\sqrt{E-V(x)}}\,\mathrm{d}x \\ &= \int_{\sqrt{(2-2\sqrt{E})_+}}^{\sqrt{2+2\sqrt{E}}} \frac{2-x^2}{\sqrt{E-(x^2/2-1)^2}}\,\mathrm{d}x, \quad E \in (0,1) \cup (1,\infty). \end{aligned} \tag{3.20}$$

where $(z)_+ = \max\{z, 0\}$.

Note that the function F in (1.2) in Theorem 1.4 is the restriction to $(1, +\infty)$ of the function defined in (3.20). The properties of the limit object in (3.18), and in particular of the function $F(E)$, are studied in the next section. Note that this corollary is not sufficient for proving there is no critical point near the energy $E = 0$.

Proof. Recalling (3.7), as $h \to 0$ and $E(h) = h^{4/3}\lambda(\alpha) \to E$, we have, using Proposition 3.5,

$$h^{2/3}\lambda'(\alpha) = 2 - \|xv(\cdot,h)\|_{L^2}^2 \to 2 - \int_{\mathbb{R}} x^2\,\mathrm{d}\mathfrak{m}_E(x) = \int_{\mathbb{R}} (2-x^2)\,\mathrm{d}\mathfrak{m}_E(x).$$

That is, according to (3.13)–(3.16), respectively,

$$\begin{aligned} h^{2/3}\lambda'(\alpha) &\to 0 \quad \text{if } E = 0, \\ h^{2/3}\lambda'(\alpha) &\to 2 - C(E)\int_{x_-(E)}^{x_+(E)} \frac{x^2\,\mathrm{d}x}{\sqrt{E-V(x)}} \\ &\qquad = C(E)\left(\int_{x_-(E)}^{x_+(E)} \frac{2-x^2}{\sqrt{E-V(x)}}\,\mathrm{d}x\right), \quad \text{if } E \in (0,1), \\ h^{2/3}\lambda'(\alpha) &\to 2 \quad \text{if } E = 1, \\ h^{2/3}\lambda'(\alpha) &\to 2 - C(E)\int_0^{x_+(E)} \frac{x^2\,\mathrm{d}x}{\sqrt{E-V(x)}} \\ &\qquad = C(E)\left(\int_0^{x_+(E)} \frac{2-x^2}{\sqrt{E-V(x)}}\,\mathrm{d}x\right), \quad \text{if } E > 1. \end{aligned}$$

$\square$

3.4. Study of $\mathfrak{m}_E$ ***and*** $F(E)$ ***near*** $E = 1$**.** Our goal in this section is to prove continuity of the family $\mathfrak{m}_E$ and the function $C(E)F(E)$ through $E = 1$. As a preliminary, we study properties as $E \to 1^{\pm}$ of the integral

$$\mathsf{F}_{\varphi}(E) := \int_{x_-(E)}^{x_+(E)} \frac{\varphi(x)}{\sqrt{E - V(x)}}\,\mathrm{d}x = \int_{\sqrt{(2-2\sqrt{E})_+}}^{\sqrt{2+2\sqrt{E}}} \frac{\varphi(x)}{\sqrt{E-(x^2/2-1)^2}}\,\mathrm{d}x,$$

defined for $\varphi \in C^0(\mathbb{R})$ and $E \in (0,1) \cup (1,+\infty)$.

Lemma 3.8. *There is* $C > 0$ *such that for all* $\delta \in (0,1)$*, there is* $C_\delta > 0$ *such that for all* $\varphi \in C^1(\mathbb{R})$*,*

$$|\mathsf{F}_{\varphi}(E) - \varphi(0)\mathsf{F}_1^{\delta,+}(E)| \le C\delta\|\varphi'\|_{L^\infty(0,1)} + C_\delta\|\varphi\|_{L^\infty(0,3)}, \quad \forall\, E \in (1, 9/4),$$

where

$$\mathsf{F}_1^{\delta,+}(E) = \int_0^{\delta} \frac{1}{\sqrt{E-(x^2/2-1)^2}}\,\mathrm{d}x, \quad E \in (1, 9/4)$$

satisfies, for $\sqrt{E} = 1 + \varepsilon$, $\varepsilon \in (0, 1/2)$*,*

$$0 \le \mathsf{F}_1^{\delta,+}(E) - \int_0^{\delta/\sqrt{2\varepsilon}} \frac{\mathrm{d}y}{\sqrt{y^2+1}} \le C\delta^2 \int_0^{\delta/\sqrt{2\varepsilon}} \frac{\mathrm{d}y}{\sqrt{y^2+1}}, \tag{3.21}$$

and

$$|\mathsf{F}_{\varphi}(E) - \varphi(0)\mathsf{F}_1^{\delta,-}(E)| \le C\delta\|\varphi'\|_{L^\infty(0,1)} + C_\delta\|\varphi\|_{L^\infty(0,3)}, \quad \forall\, E \in (1/4, 1),$$

where

$$\mathsf{F}_1^{\delta,-}(E) = \int_{\sqrt{2-2\sqrt{E}}}^{\delta} \frac{1}{\sqrt{E-(x^2/2-1)^2}}\,\mathrm{d}x, \quad E \in (1/4, 1)$$

satisfies, for $\sqrt{E} = 1 - \varepsilon$, $\varepsilon \in (0, 1/2)$*,*

$$0 \le \mathsf{F}_1^{\delta,-}(E) - \frac{\sqrt{2}}{\sqrt{2-\varepsilon}} \int_1^{\delta/\sqrt{2\varepsilon}} \frac{\mathrm{d}y}{\sqrt{y^2-1}} \le C\delta^2 \int_1^{\delta/\sqrt{2\varepsilon}} \frac{\mathrm{d}y}{\sqrt{y^2-1}}. \tag{3.22}$$

Note that Lemma 3.8 implies that if $\varphi(0) \neq 0$, then

$$\begin{aligned} \mathsf{F}_{\varphi}(E) &\sim \varphi(0)\frac{1}{2}\log\left(\frac{1}{|\sqrt{E}-1|}\right) \\ &\sim \varphi(0)\frac{1}{2}\log\left(\frac{1}{|E-1|}\right) \quad \text{as } E \to 1^{\pm}. \end{aligned} \tag{3.23}$$

As a consequence, we get the following proposition. We denote by $\mathcal{M}_c(\mathbb{R})$ the set of compactly supported Radon measures on $\mathbb{R}$.

Proposition 3.9. *With* $\mathfrak{m}_E$ *defined by* (3.13)–(3.16)*, the map*

$$[0,+\infty) \to \mathcal{M}_c(\mathbb{R}), \quad E \mapsto \mathfrak{m}_E$$

is weakly- continuous: that is, for all* $\varphi \in C^0(\mathbb{R})$ *and all* $E_0 \in [0,+\infty)$*, we have*

$$\langle \mathfrak{m}_E, \varphi\rangle \to \langle \mathfrak{m}_{E_0}, \varphi\rangle, \quad \text{as } E \to E_0 \text{ (resp. } E \to 0^+ \text{ if } E_0 = 0).$$

Note that this property is not strictly needed for our analysis but completes the picture nicely. Its slightly weaker Corollary 3.10 below is, however, needed.

Proof of Proposition 3.9 from Lemma 3.8. First, for $E \in (0,1) \cup (1,+\infty)$, we notice that $C(E) = 1/\mathsf{F}_1(E)$. The measure $\mathfrak{m}_E$ in (3.13)–(3.16) is even and absolutely continuous, so that for any $\varphi \in C^0(\mathbb{R})$,

$$\langle \mathfrak{m}_E, \varphi\rangle = \int_{\mathbb{R}^+} \varphi \, \mathrm{d}\mathfrak{m}_E + \int_{\mathbb{R}^+} \check{\varphi} \, \mathrm{d}\mathfrak{m}_E = \frac{1}{2\mathsf{F}_1(E)}(\mathsf{F}_\varphi(E) + \mathsf{F}_{\check{\varphi}}(E)),$$

where $\check{\varphi}(x) = \varphi(-x)$. This therefore implies the continuity statement for $E \in (0,1) \cup (1,+\infty)$.

Second, we prove the continuity statement at $E = 1$. Given this expression, it suffices to show that $\mathsf{F}_\varphi(E)/\mathsf{F}_1(E) \to \varphi(0)$ as $E \to 1^\pm$. To this end, we assume first that $\varphi \in C^1(\mathbb{R})$. According to Lemma 3.8 (taken for any fixed $\delta \in (0,1)$), we have

$$\frac{\mathsf{F}_\varphi(E)}{\mathsf{F}_1(E)} = \frac{\varphi(0)\mathsf{F}_1(E) + O(1)\|\varphi\|_{C^1(0,3)}}{\mathsf{F}_1(E)} \to \varphi(0) \quad \text{as } E \to 1^\pm,$$

where we have used that $\mathsf{F}_1(E) \to +\infty$ as a consequence of (3.21)–(3.22). This implies that $\langle \mathfrak{m}_E, \varphi\rangle \to \varphi(0)$ for all $\varphi \in C^1(\mathbb{R})$. Finally, if $\varphi \in C^0(\mathbb{R})$, there is $\varphi_n \in C^1(\mathbb{R})$ such that $\|\varphi_n - \varphi\|_{L^\infty(-3,3)} \to 0$, and we write

$$\begin{aligned} |\langle \mathfrak{m}_E, \varphi\rangle - \varphi(0)| &\le |\langle \mathfrak{m}_E, \varphi - \varphi_n\rangle| + |\langle \mathfrak{m}_E, \varphi_n\rangle - \varphi_n(0)| + |\varphi_n(0) - \varphi(0)| \\ &\le 2\|\varphi_n - \varphi\|_{L^\infty(-3,3)} + |\langle \mathfrak{m}_E, \varphi_n\rangle - \varphi_n(0)|, \end{aligned}$$

where we have used that $\mathfrak{m}_E$ is a probability measure.

Given $\varepsilon > 0$, we fix $n \in \mathbb{N}$ such that the first term is $\le \varepsilon$, and then take E close enough to 1 so that the second term is $\le \varepsilon$, and for such E we have $|\langle \mathfrak{m}_E, \varphi\rangle - \varphi(0)| \le 2\varepsilon$. We have thus proved the continuity statement at $E = 1$ for any $\varphi \in C^0(\mathbb{R})$.

Third, we prove the continuity statement at $E = 0^+$. To this end, it suffices to comment that, given any sequence $E_n \to 0^+$, we may extract from the sequence $\nu_n := 2\mathbf{1}_{\mathbb{R}^+}\mathfrak{m}_{E_n} \in \mathcal{M}_c(\mathbb{R}^+)$ of probability measure a converging subsequence in

$\mathcal{M}_c(\mathbb{R}^+)$. Since $\operatorname{supp} \nu_n = [x_-(E_n), x_+(E_n)]$, the limit measure is a probability measure supported in

$$\bigcap_{n\in\mathbb{N}} [x_-(E_n), x_+(E_n)] = \{\sqrt{2}\}.$$

Therefore, the unique possible limit is $\delta_{\sqrt{2}}$, and the whole sequence ν_n converges to $\delta_{\sqrt{2}}$. This implies that

$$2\mathbf{1}_{\mathbb{R}^+}\mathfrak{m}_E \rightharpoonup \delta_{\sqrt{2}} \quad \text{as } E \to 0^+,$$

and, since $\mathfrak{m}_E$ is even, that $\mathfrak{m}_E \rightharpoonup \frac{1}{2}(\delta_{-\sqrt{2}} + \delta_{\sqrt{2}})$.

This concludes the proof of the proposition. □

Corollary 3.10. *With the functions C defined in (3.17) and F in (3.20), the function Φ defined in (3.19) is continuous on $[0, +\infty)$.*

Proof. This follows directly from the fact that $C(E) = 1/\mathsf{F}_1(E)$ and $F(E) = \mathsf{F}_{2-x^2}(E)$, so that

$$\Phi(E) = C(E)F(E) = \frac{\mathsf{F}_{2-x^2}(E)}{\mathsf{F}_1(E)} = \langle \mathfrak{m}_E, 2 - x^2 \rangle,$$

to which Proposition 3.9 applies. □

We finally prove Lemma 3.8.

Proof of Lemma 3.8. First, for $E > 1$, we set $\sqrt{E} = 1 + \varepsilon > 0$ and have

$$\begin{aligned}\mathsf{F}_\varphi(E) &= \int_0^{\sqrt{2+2\sqrt{E}}} \frac{\varphi(x)}{\sqrt{E - (x^2/2 - 1)^2}}\,\mathrm{d}x \\ &= \int_0^{\sqrt{4+2\varepsilon}} \frac{\varphi(x)}{\sqrt{(1+\varepsilon)^2 - (x^2/2 - 1)^2}}\,\mathrm{d}x.\end{aligned}$$

For $\delta \in (0, 1)$ independent of ε, we write $\mathsf{F}_\varphi(E) = \mathsf{G}_\varphi^{\delta,+}(E) + \mathsf{F}_\varphi^{\delta,+}(E)$ with

$$\begin{aligned}\mathsf{G}_\varphi^{\delta,+}(E) &:= \int_\delta^{\sqrt{4+2\varepsilon}} \frac{\varphi(x)}{\sqrt{(1+\varepsilon)^2 - (x^2/2 - 1)^2}}\,\mathrm{d}x \\ &= \int_\delta^{\sqrt{4+2\varepsilon}} \frac{\varphi(x)}{\sqrt{2\varepsilon + \varepsilon^2 - x^4/4 + x^2}}\,\mathrm{d}x.\end{aligned}$$

In particular,

$$|\mathsf{G}_\varphi^{\delta,+}(E)| \leq \|\varphi\|_{L^\infty(0,3)}\mathsf{G}_1^{\delta,+}(E)$$

where

$$\mathsf{G}_1^{\delta,+}(E) \to \int_\delta^2 \frac{1}{\sqrt{x^2 - x^4/4}}\,\mathrm{d}x = \int_\delta^2 \frac{2}{x\sqrt{(2+x)(2-x)}}\,\mathrm{d}x < +\infty,$$

as $\varepsilon \to 0^+$. As a consequence, for any $\delta \in (0,1)$, there is $C_\delta > 0$ such that for all $\varepsilon \in (0, \frac{1}{2})$, with all $\varphi \in C^0(\mathbb{R})$,

$$|\mathsf{G}_\varphi^{\delta,+}(E)| \le C_\delta \|\varphi\|_{L^\infty(0,3)}.$$

It thus remains to consider

$$\begin{aligned}
\mathsf{F}_\varphi^{\delta,+}(E) &:= \int_0^\delta \frac{\varphi(x)}{\sqrt{(1+\varepsilon)^2 - (x^2/2-1)^2}}\,\mathrm{d}x \\
&= \int_0^\delta \frac{\varphi(x)}{\sqrt{(2+\varepsilon - x^2/2)(x^2/2+\varepsilon)}}\,\mathrm{d}x \\
&= \sqrt{2}\int_0^{\delta/\sqrt{2\varepsilon}} \frac{\varphi(\sqrt{2\varepsilon}y)}{\sqrt{2+\varepsilon-\varepsilon y^2}\sqrt{y^2+1}}\,\mathrm{d}y,
\end{aligned}$$

where we have set $x = \sqrt{2\varepsilon}y$. We also notice that

$$y \in [0, \delta/\sqrt{2\varepsilon}] \Longrightarrow \frac{1}{\sqrt{2+\varepsilon}} \le \frac{1}{\sqrt{2+\varepsilon-\varepsilon y^2}} \le \frac{1}{\sqrt{2+\varepsilon-\delta^2/2}}.$$

As a consequence, we have

$$\begin{aligned}
&|\mathsf{F}_\varphi^{\delta,+}(E) - \varphi(0)\mathsf{F}_1^{\delta,+}(E)| \\
&\quad = \sqrt{2}\left|\int_0^{\delta/\sqrt{2\varepsilon}} \frac{\varphi(\sqrt{2\varepsilon}y) - \varphi(0)}{\sqrt{2+\varepsilon-\varepsilon y^2}\sqrt{y^2+1}}\,\mathrm{d}y\right| \\
&\quad \le \sqrt{2}\sqrt{2\varepsilon}\|\varphi'\|_{L^\infty(0,1)} \int_1^{\delta/\sqrt{2\varepsilon}} \frac{\mathrm{d}y}{\sqrt{2+\varepsilon-\varepsilon y^2}\sqrt{y^2+1}} \\
&\quad \le C\delta\|\varphi'\|_{L^\infty(0,1)},
\end{aligned}$$

uniformly for $\varepsilon \in (0, \frac{1}{2})$. Finally, we have

$$\frac{\sqrt{2}}{\sqrt{2+\varepsilon}}\int_0^{\delta/\sqrt{2\varepsilon}} \frac{\mathrm{d}y}{\sqrt{y^2+1}} \le \mathsf{F}_1^{\delta,+}(E) \le \frac{\sqrt{2}}{\sqrt{2+\varepsilon-\delta^2/2}}\int_1^{\delta/\sqrt{2\varepsilon}} \frac{\mathrm{d}y}{\sqrt{y^2+1}},$$

from which (3.21) follows.

We now consider the case $E < 1$. Setting $\sqrt{E} = 1 - \varepsilon$ with $\varepsilon > 0$, we have

$$\begin{aligned}\mathsf{F}_\varphi(E) &= \int_{\sqrt{(2-2\sqrt{E})}}^{\sqrt{2+2\sqrt{E}}} \frac{\varphi(x)}{\sqrt{E-(x^2/2-1)^2}}\,\mathrm{d}x \\ &= \int_{\sqrt{2\varepsilon}}^{\sqrt{4-2\varepsilon}} \frac{\varphi(x)}{\sqrt{(1-\varepsilon)^2-(x^2/2-1)^2}}\,\mathrm{d}x.\end{aligned}$$

For $\delta \in (0,1)$ independent of ε, we have $\mathsf{F}_\varphi(E) = \mathsf{G}_\varphi^{\delta,-}(E) + \mathsf{F}_\varphi^{\delta,-}(E)$ with

$$\begin{aligned}\mathsf{G}_\varphi^{\delta,-}(E) &:= \int_{\delta}^{\sqrt{4-2\varepsilon}} \frac{\varphi(x)}{\sqrt{(1-\varepsilon)^2-(x^2/2-1)^2}}\,\mathrm{d}x \\ &= \int_{\delta}^{\sqrt{4-2\varepsilon}} \frac{\varphi(x)}{\sqrt{-2\varepsilon+\varepsilon^2-x^4/4+x^2}}\,\mathrm{d}x.\end{aligned}$$

In particular,

$$|\mathsf{G}_\varphi^{\delta,-}(E)| \le \|\varphi\|_{L^\infty(0,3)} \mathsf{G}_1^{\delta,-}(E)$$

where

$$\mathsf{G}_1^{\delta,-}(E) \to \int_\delta^2 \frac{1}{\sqrt{x^2-x^4/4}}\,\mathrm{d}x = \int_\delta^2 \frac{2}{x\sqrt{(2+x)(2-x)}}\,\mathrm{d}x < +\infty,$$

as $\varepsilon \to 0^+$. As a consequence, for any $\delta \in (0,1)$, there is $C_\delta > 0$ such that for all $\varepsilon \in (0, \frac{1}{2})$, with all $\varphi \in C^0(\mathbb{R})$,

$$|\mathsf{G}_\varphi^{\delta,-}(E)| \le C_\delta \|\varphi\|_{L^\infty(0,3)}.$$

It thus remains to consider

$$\begin{aligned}\mathsf{F}_\varphi^{\delta,-}(E) &:= \int_{\sqrt{2\varepsilon}}^{\delta} \frac{\varphi(x)}{\sqrt{(1-\varepsilon)^2-(x^2/2-1)^2}}\,\mathrm{d}x \\ &= \int_{\sqrt{2\varepsilon}}^{\delta} \frac{\varphi(x)}{\sqrt{(2-\varepsilon-x^2/2)(x^2/2-\varepsilon)}}\,\mathrm{d}x \\ &= \sqrt{2}\int_1^{\delta/\sqrt{2\varepsilon}} \frac{\varphi(\sqrt{2\varepsilon}y)}{\sqrt{2-\varepsilon-\varepsilon y^2}\sqrt{y^2-1}}\,\mathrm{d}y,\end{aligned}$$

where we have set $x = \sqrt{2\varepsilon}y$. We also notice that

$$y \in [1, \delta/\sqrt{2\varepsilon}] \Longrightarrow \frac{1}{\sqrt{2-\varepsilon}} \le \frac{1}{\sqrt{2-\varepsilon-\varepsilon y^2}} \le \frac{1}{\sqrt{2-\varepsilon-\delta^2/2}}.$$

As a consequence, we have

$$
\begin{aligned}
&|\mathsf{F}_{\varphi}^{\delta,-}(E) - \varphi(0)\mathsf{F}_1^{\delta,-}(E)| \\
&\quad = \sqrt{2}\left| \int_1^{\delta/\sqrt{2\varepsilon}} \frac{\varphi(\sqrt{2\varepsilon}y) - \varphi(0)}{\sqrt{2-\varepsilon-\varepsilon y^2}\sqrt{y^2-1}}\,\mathrm{d}y \right| \\
&\quad \le \sqrt{2}\sqrt{2\varepsilon}\|\varphi'\|_{L^\infty(0,1)} \int_1^{\delta/\sqrt{2\varepsilon}} \frac{\mathrm{d}y}{\sqrt{2-\varepsilon-\varepsilon y^2}\sqrt{y^2-1}} \\
&\quad \le C\delta\|\varphi'\|_{L^\infty(0,1)},
\end{aligned}
$$

uniformly for $\varepsilon \in (0, \frac{1}{2})$. Finally, we have

$$
\frac{\sqrt{2}}{\sqrt{2-\varepsilon}} \int_1^{\delta/\sqrt{2\varepsilon}} \frac{\mathrm{d}y}{\sqrt{y^2-1}} \le \mathsf{F}_1^{\delta,-}(E) \le \frac{\sqrt{2}}{\sqrt{2-\varepsilon-\delta^2/2}} \int_1^{\delta/\sqrt{2\varepsilon}} \frac{\mathrm{d}y}{\sqrt{y^2-1}},
$$

from which (3.22) follows. $\square$

3.5. Analysis of $F(E)$**.** To prove Theorem 1.4, we now study properties of the function F defined in (3.20). We first rewrite this quantity in a more customary form.

Lemma 3.11. *We have*

$$
\begin{aligned}
F(E) &= -2^{1/2}E^{1/4}\theta_+(E^{-1/2}) \\
&= -2^{1/2}\eta^{-1/2}\theta_+(\eta), \qquad \text{for } \eta = E^{-1/2} \in (0,1), \\
F(E) &= -2^{1/2}E^{1/4}\theta_-(E^{-1/2}) \\
&= -2^{1/2}\eta^{-1/2}\theta_-(\eta), \qquad \text{for } \eta = E^{-1/2} \in (1,+\infty),
\end{aligned}
$$

with

$$
\theta_+(\eta) := \int_{-\eta}^{1} \tau(1-\tau^2)^{-1/2}(\tau+\eta)^{-1/2}\,\mathrm{d}\tau, \quad \text{for } \eta = E^{-1/2} \in (0,1), \tag{3.24}
$$

$$
\theta_-(\eta) := \int_{-1}^{1} \tau(1-\tau^2)^{-1/2}(\tau+\eta)^{-1/2}\,\mathrm{d}\tau, \quad \text{for } \eta = E^{-1/2} \in (1,+\infty). \tag{3.25}
$$

This reduces the study of F to that of θ_+ and θ_-.

Proof. After a change of variable $\sigma = x^2/2 - 1$ (i.e., $x = \sqrt{2(\sigma+1)}$, $\mathrm{d}x = (2(\sigma+1))^{-1/2}\,\mathrm{d}\sigma$), we obtain from (3.20) that

$$
F(E) = -2\int_{(1-E^{1/2})_+-1}^{E^{1/2}} \sigma(E-\sigma^2)^{-1/2}(2(\sigma+1))^{-1/2}\,\mathrm{d}\sigma.
$$

The change of variable $\sigma = E^{1/2}\tau$ then leads to

$$F(E) = -2E^{1/2}\int_{(E^{-1/2}-1)_+-E^{-1/2}}^{1} \tau(1-\tau^2)^{-1/2}(2(E^{1/2}\tau+1))^{-1/2}\,\mathrm{d}\tau,$$

that is,

$$F(E) = -2^{1/2}E^{1/2}\int_{-E^{-1/2}}^{1} \tau(1-\tau^2)^{-1/2}(E^{1/2}\tau+1)^{-1/2}\,\mathrm{d}\tau, \qquad \text{for } E \in (1,+\infty),$$

$$F(E) = -2^{1/2}E^{1/2}\int_{-1}^{1} \tau(1-\tau^2)^{-1/2}(E^{1/2}\tau+1)^{-1/2}\,\mathrm{d}\tau, \qquad \text{for } E \in (0,1).$$

Setting $\eta = E^{-1/2}$, this can be rewritten as

$$F(E) = -2^{1/2}\eta^{-1/2}\int_{-\eta}^{1} \tau(1-\tau^2)^{-1/2}(\tau+\eta)^{-1/2}\,\mathrm{d}\tau, \qquad \text{for } \eta = E^{-1/2} \in (0,1),$$

$$F(E) = -2^{1/2}\eta^{-1/2}\int_{-1}^{1} \tau(1-\tau^2)^{-1/2}(\tau+\eta)^{-1/2}\,\mathrm{d}\tau, \qquad \text{for } \eta = E^{-1/2} \in (1,+\infty),$$

which is the sought result. □

We may now analyze properties of the function F in (3.20). The goal of this section is to prove the existence and uniqueness of E_c where F vanishes, which is part of the statement of Theorem 1.4.

Proposition 3.12. *The following statements hold:*

(1) *The function F is of class C^1 and we have $F' < 0$ on $(1,+\infty)$.*
(2) $\lim_{E\to+\infty} F(E) = -\infty$.
(3) $\lim_{E\to1^+} F(E) = +\infty$ *and* $\lim_{E\to1^-} F(E) = +\infty$.
(4) *F admits on $(1,\infty)$ a unique zero $E_c \in (1,\infty)$ and $F'(E_c) < 0$. Numerically $E_c \approx 2.35$.*
(5) *$F(E) > 0$ for all $E \in (0,1)$.*

Note that in Item (3), the divergence is logarithmic according to (3.23) (taken for $F(E) = \mathsf{F}_{2-x^2}(E)$).

Proof. We use Lemma 3.11 to reduce the analysis of the variation of F on $(1,\infty)$ to that of θ_+ defined in (3.24), on $(0,1)$. We decompose θ_+ as $\theta_+ = \theta_1 + \theta_2$ with

$$\theta_1(\eta) := \int_0^1 \tau(1-\tau^2)^{-1/2}(\tau+\eta)^{-1/2}\,\mathrm{d}\tau, \qquad \eta \in (0,1),$$

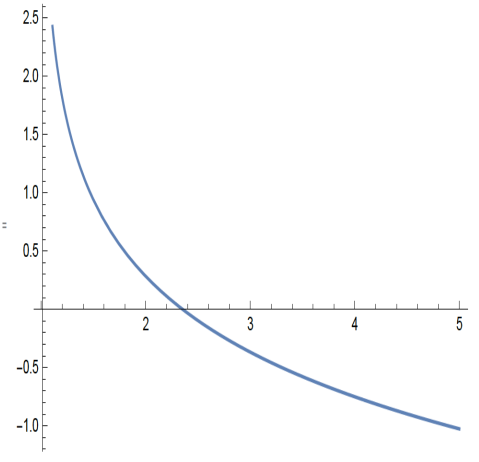

FIGURE 3.1. Graph of $E \mapsto F(E)$ computed with Matematica for $E \in (1.01, 5)$. The change of sign occurs for $E_c \approx 2.35$.

and

$$\theta_2(\eta) := \int_{-\eta}^{0} \tau(1-\tau^2)^{-1/2}(\tau+\eta)^{-1/2}\,\mathrm{d}\tau, \quad \eta \in (0,1).$$

It is clear that θ_1 is of class C^1 and decreasing. For the analysis of θ_2, we start by an integration by part, which yields

$$\begin{aligned}\theta_2(\eta) &= -2\int_{-\eta}^{0} (\tau(1-\tau^2)^{-1/2})'(\tau+\eta)^{1/2}\,\mathrm{d}\tau \\ &= -2\int_{-\eta}^{0} (1-\tau^2)^{-3/2}(\tau+\eta)^{1/2}\,\mathrm{d}\tau.\end{aligned}$$

In this form we see that θ_2 is of class C^1 and that its derivative is given by

$$\theta_2'(\eta) = -\int_{-\eta}^{0} (1-\tau^2)^{-3/2}(\tau+\eta)^{-1/2}\,\mathrm{d}\tau.$$

This yields $\theta_2'(\eta) < 0$ for $\eta \in (0,1)$ so that θ_2 is decreasing, and thus so is $\theta_+ = \theta_1 + \theta_2$. Recalling from Lemma 3.11 that $F(E) = -2^{1/2}E^{1/4}\theta_+(E^{-1/2})$ for $E \in (1, +\infty)$, we deduce that $F' < 0$ on $(1, +\infty)$, which is Item (1).

Concerning Item (2), we notice that θ_+ is continuous at 0^+ with

$$\theta_+(0) = \int_0^1 \tau^{1/2}(1-\tau^2)^{-1/2}\,\mathrm{d}\tau > 0.$$

As a consequence, we deduce that

$$F(E) \sim -2^{1/2}E^{1/4}\theta_+(0) \to -\infty \quad \text{as } E \to +\infty.$$

Item (3) is a direct consequence of Lemma 3.8 since $F(E) = \mathsf{F}_{2-x^2}(E)$ (see, e.g., (3.23) with $\varphi(x) = 2 - x^2$).

Finally, Item (4) is a direct consequence of Items (1), (2), and (3) (numerically $E_c \sim 2.35$).

To examine the function F on the interval $(0,1)$, we rather study according to Lemma 3.11 the function

$$\theta_-(\eta) = \int_{-1}^{1} \tau(1-\tau^2)^{-1/2}(\tau+\eta)^{-1/2}\,\mathrm{d}\tau, \quad \text{for } \eta \in (1,+\infty).$$

Splitting the integral and changing variables, we rewrite it as

$$\theta_-(\eta) = \int_0^1 \tau(1-\tau^2)^{-1/2}((\eta+\tau)^{-1/2} - (\eta-\tau)^{-1/2})\,\mathrm{d}\tau.$$

Noticing that $(\eta+\tau)^{-1/2} < (\eta-\tau)^{-1/2}$, we see this implies that $\theta_-(\eta) < 0$ for $\eta \in (1,\infty)$, and from Lemma 3.11 we deduce that $F > 0$ on $(0,1)$, whence we get Item (5). ◻

Note that so far, we have proved in Corollary 3.7 that

$$\begin{aligned} \alpha \to +\infty, \ \frac{\lambda(\alpha)}{\alpha^2} &\to E \in [0,+\infty) \\ \Longrightarrow \frac{\lambda'(\alpha)}{\alpha} &\to \Phi(E) = C(E)F(E) = \langle \mathfrak{m}_E, 2-x^2\rangle. \end{aligned}$$

This is a useful piece of information as long as $\Phi(E) \neq 0$. We have already seen that $\Phi(E_c) = 0$. We have seen that $\Phi(0) = 0$ so that the description of the measure $\mathfrak{m}_E$ is not enough to prove that λ' does not vanish in this regime. To prove Theorem 1.4, an additional analysis is hence required near $E = 0$, which is performed in the next section.

***3.6. Analysis in the case* $E < 1$.** The main goal of this subsection is to prove that $\lambda'(\alpha)$ does not vanish in the regime $\lambda(\alpha)/\alpha^2 \to 0$. We prove a slightly more precise result.

Proposition 3.13. *Assume that $\alpha \to +\infty$ and $\lambda(\alpha)/\alpha^2 \to E \in [0,1)$. Then, for all $\delta > 0$, there is $\alpha_0 > 0$ such that for all $\alpha \geq \alpha_0$, we have*

$$\lambda'(\alpha) \geq \left(\frac{4}{3\sqrt{2} + 2\sqrt{E} + \sqrt{2}} - \delta\right)\alpha^{-1/2}.$$

In particular, there is no critical point in this regime. Note that the case $\lambda(\alpha)/\alpha^2 \to 0$ was not covered yet, and Proposition 3.13 yields

$$\alpha \to +\infty, \ \frac{\lambda(\alpha)}{\alpha^2} \to 0 \Longrightarrow \lambda'(\alpha) \geq \left(\frac{1}{\sqrt{2}} - \delta\right) \alpha^{-1/2}.$$

The vanishing rate of the derivative as $(2\alpha)^{-1/2}$ is consistent (and optimal) since it corresponds to that of the formal derivation of the low-lying eigenvalue (2.2)–(3.10). This asymptotic behavior is worse than in the case $\lambda(\alpha)/\alpha^2 \to E \in (0,1)$ (in which case $\lambda'(E) \sim \Phi(E)\alpha$ as $\alpha \to +\infty$ according to Corollary 3.7).

Proof. We start from equation (2.13), which we reformulate in the semiclassical setting. Recalling that $v(\cdot, h)$ is defined in (3.9), we have

$$u(0,\alpha) = \alpha^{-1/4} v(0, \alpha^{-3/2}) \quad \text{and} \quad \partial_t u(0,\alpha) = \alpha^{-3/4}\, \partial_x v(0, \alpha^{-3/2}).$$

Equation (2.13) thus can be rewritten as

$$\begin{aligned} 2\int_0^{+\infty} (t - \sqrt{2\alpha}) \left(\frac{t^2}{2} - \alpha\right) \alpha^{-1/2} v(\alpha^{-1/2} t, \alpha^{-2/3})^2 \, \mathrm{d}t - \frac{\sqrt{\alpha}}{\sqrt{2}} \lambda'(\alpha) \\ = (\lambda(\alpha) - \alpha^2)\alpha^{-1/2} v(0, \alpha^{-3/2})^2 + \alpha^{-3/2}\, \partial_x v(0, \alpha^{-3/2})^2. \end{aligned}$$

Changing variable $x = \alpha^{-1/2} t$ in the integral and recalling that $h = \alpha^{-3/2}$, we have obtained

$$\begin{aligned} 2h^{-1} \int_0^{\infty} (x - \sqrt{2}) \left(\frac{x^2}{2} - 1\right) v(x,h)^2 \, \mathrm{d}x - \frac{h^{-1/3}}{\sqrt{2}} \lambda'(\alpha) \\ = (\lambda(\alpha) - h^{-4/3}) h^{1/3} v(0,h)^2 + h\, \partial_x v(0,h)^2. \end{aligned}$$

We now recall $v(\cdot, h)$ solves the semiclassical eigenvalue equation (3.6), associated with a family of eigenvalues $E(h) := \lambda(\alpha)/\alpha^2 = h^{4/3}\lambda(\alpha) \to E < 1$. Hence, the point $x = 0$ stands in the classically forbidden region $\mathbb{R}\backslash K_E$ at energy E. According to Proposition 3.4 combined with rough Sobolev embeddings, we thus have

$$h\, \partial_x v(0,h)^2 = O(h^\infty), \quad (\lambda(\alpha) - h^{-4/3}) h^{1/3} v(0,h)^2 = O(h^\infty).$$

Hence, we have obtained, as $h = \alpha^{-3/2} \to 0^+$,

$$\begin{aligned} \lambda'(\alpha) &= \sqrt{2} h^{-2/3} \int_0^{\infty} (x - \sqrt{2})(x^2 - 2) v(x,h)^2 \, \mathrm{d}x + O(h^\infty) \\ &= \sqrt{2} h^{-2/3} \int_0^{\infty} \frac{(x + \sqrt{2})}{(x + \sqrt{2})} (x - \sqrt{2})(x^2 - 2) v(x,h)^2 \, \mathrm{d}x + O(h^\infty) \\ &= \sqrt{2} h^{-2/3} \int_0^{\infty} \frac{1}{(x + \sqrt{2})} (x^2 - 2)^2 v(x,h)^2 \, \mathrm{d}x + O(h^\infty). \end{aligned}$$

Using again twice the localization in K_E given by Proposition 3.4, we obtain, for any $\varepsilon > 0$,

$$
\begin{aligned}
(3.26)\quad & \lambda'(\alpha) \\
&= 4\sqrt{2}h^{-2/3}\int_0^{x_+(E)+\varepsilon} \frac{1}{(x+\sqrt{2})}\left(\frac{x^2}{2}-1\right)^2 v(x,h)^2\,\mathrm{d}x + O_\varepsilon(h^\infty) \\
&\geq 4\sqrt{2}h^{-2/3}\frac{1}{(x_+(E)+\varepsilon+\sqrt{2})}\int_0^{x_+(E)+\varepsilon}\left(\frac{x^2}{2}-1\right)^2 v(x,h)^2\,\mathrm{d}x + O_\varepsilon(h^\infty) \\
&\geq \frac{4\sqrt{2}}{(x_+(E)+\varepsilon+\sqrt{2})}h^{-2/3}\int_0^{\infty}\left(\frac{x^2}{2}-1\right)^2 v(x,h)^2\,\mathrm{d}x + O_\varepsilon(h^\infty),
\end{aligned}
$$

where we have written $\mathcal{Q} = O_\varepsilon(h^\infty)$ if, for all $N \in \mathbb{N}$, $\varepsilon > 0$, there is $C_{N,\varepsilon} > 0$ such that $|\mathcal{Q}| \leq C_{N,\varepsilon}h^N$. We now rewrite (2.10) in the semiclassical setting using (3.9) and $x = \alpha^{-1/2}t$ to obtain

$$h^{-2/3}\lambda'(\alpha) + \lambda(\alpha) = 3\int_{\mathbb{R}}\left(\frac{1}{2}t^2-\alpha\right)^2 u(\cdot,\alpha)^2\,\mathrm{d}t.$$

Using the parity of $v(\cdot,h)^2$ for any eigenfunction, we deduce

$$\lambda'(\alpha) + h^{2/3}\lambda(\alpha) = 6h^{-2/3}\int_0^\infty\left(\frac{x^2}{2}-1\right)^2 v(x,h)^2\,\mathrm{d}x.$$

Combining this together with (3.26) yields

$$\lambda'(\alpha) \geq \frac{2\sqrt{2}}{3(x_+(E)+\varepsilon+\sqrt{2})}(\lambda'(\alpha)+h^{2/3}\lambda(\alpha)) + O_\varepsilon(h^\infty).$$

We then notice that $2\sqrt{2}/(3(x_+(E)+\varepsilon+\sqrt{2})) < 1$ since $x_+(E) \geq \sqrt{2} \geq 0$. We deduce that

$$
\begin{aligned}
\lambda'(\alpha) &\geq \frac{2\sqrt{2}}{3(x_+(E)+\varepsilon)+\sqrt{2}}h^{2/3}\lambda(\alpha) + O_\varepsilon(h^\infty) \\
&= \frac{2\sqrt{2}}{3(x_+(E)+\varepsilon)+\sqrt{2}}h^{-2/3}E(h) + O_\varepsilon(h^\infty),
\end{aligned}
$$

after having recalled (3.5). Finally, we always have $E(h) \geq E_1(h)$ where, according to the bottom of the well asymptotics (3.10), $E_1(h) \sim \sqrt{2}h$ as $h \to 0$. As a consequence, for all $\delta > 0$, there exists $h_0 > 0$ such that

$$\lambda'(\alpha) \geq \left(\frac{4}{3x_+(E)+\sqrt{2}} - \delta\right)h^{1/3} \quad \text{for all } h \in (0,h_0).$$

This concludes the proof of the lemma by using that $h^{1/3} = \alpha^{-1/2}$ and by recalling the explicit value of $x_+(E)$ in (3.12). ◻

3.7. End of the proof of Theorem 1.4 Existence and uniqueness of E_c were proven in Proposition 3.12. Next, according to Corollary 2.8, there exists $E_m > 0$ such that $\alpha_{j,c} \to +\infty$ and $\lambda_j(\alpha_{j,c}) \leq E_m \alpha_{j,c}^2$ for any $\alpha_{j,c} \in A_{j,c}$. Up to extracting a subsequence, we may assume that $\lambda_j(\alpha_{j,c})/\alpha_{j,c}^2 \to E \in [0, E_m]$ as $j \to +\infty$. We have $E \notin [0,1)$ since Proposition 3.13 would then imply $\lambda_j'(\alpha_{j,c}) > 0$. Now for $E \in [1, E_m]$, Corollary 3.7 implies $\alpha_{j,c}^{-1}\lambda_j'(\alpha_{j,c}) \to \Phi(E)$. If $E \neq E_c$, then $\Phi(E) > 0$ and thus $\lambda_j'(\alpha_{j,c}) \sim \Phi(E)\alpha_{j,c}$ which contradicts $\lambda_j'(\alpha_{j,c}) = 0$. As a consequence, we necessarily have $\lambda_j(\alpha_{j,c})/\alpha_{j,c}^2 \to E_c$ (since this holds for any subsequence, this holds for the full sequence), which concludes the proof of Theorem 1.4.

4. Analysis Beyond Semiclassical Measures and Proof of Theorem 1.5

4.1. Introduction According to the results of Section 3, we have now identified the location of critical points of λ_j. In the semiclassical rescaling of the problem, the latter only arise (in the semiclassical limit) near the energy $E_c > 1$. The goal of the present section is to analyze the second derivative of λ_j around these critical points, that is to say, the semiclassical rescaling of the problem, near the energy E_c. To this end, we need to analyze precisely the righthand side in (the semiclassical version of) the second variation formula (3.8). The latter, however, involves the derivative with respect to the semiclassical parameter h of quantities like $(av_h^j, v_h^j)_{L^2}$ (or of the eigenfunctions themselves), where a is a fixed polynomial and v_h^j is an eigenfunction associated with the eigenvalue $E_j(h) \to E_c$. This piece of information is not encoded in the semiclassical measures studied in Section 3.

The main goal of this section is thus essentially to give an explicit approximation $v_h^{j,\mathrm{app}}$ of the eigenfunction v_h^j (associated with the eigenvalue $E_j(h) \to E_c$), that is differentiable with respect to h, and such that $\partial_h v_h^{j,\mathrm{app}}$ is close to $\partial_h v_h^j$ in the semiclassical limit. Luckily, the energy E_c is a regular level set of the potential $(x^2/2-1)^2$, and we use this fact all throughout the proof.

In Subsection 4.2, we therefore introduce a slightly more general setting of a 1D Schrödinger operator at a regular energy level. This allows us to focus on the sole relevant properties of the potential $(x^2/2-1)^2$ at the energy E_c. We also introduce additional classical quantities which arise in the semiclassical limit.

Our analysis of eigenvalues near E_c will rely on the Bohr-Sommerfeld quantization rules at this energy level, which we review in Subsection 4.3, with an emphasis on the labelling of the eigenvalues.

In Subsection 4.4, we develop a rather systematic approach to compare approximate and exact eigenvalues/eigenfunctions. We assume there that approximate eigenvalues/eigenfunctions are constructed, and give sufficient conditions on the approximate eigenfunctions $v_h^{j,\mathrm{app}}$ so that $\partial_h v_h^{j,\mathrm{app}}$ is close to $\partial_h v_h^j$.

In Subsection 4.5, we shall recall a classical way for constructing approximate eigenvalues/eigenfunctions (which eventually leads in particular to the Bohr-Sommerfeld quantization conditions discussed in Section 4.3) relying on WKB expansions or semiclassical Lagrangian distributions. This approach was initiated by V.P. Maslov in a rather formal way but a complete mathematical proof can be found in the article by J. Duistermaat [12] and in a more formalized way in the article by B. Candelpergher and J.C. Nosmas [5]. For the analysis modulo $\mathcal{O}(h^\infty)$, we also refer to the arguments given in the appendix of Helffer-Robert [24] (itself relying on [12] and [5]). Here, we need to revisit this construction and control that we can differentiate with respect to the parameters.[3]

4.2. Setting and classical quantities. Our approach in this section involves refined semiclassical analysis at a noncritical energy (namely E_c) for the semiclassical Schrödinger operator

$$P_h = -h^2 \frac{\mathrm{d}^2}{\mathrm{d}x^2} + V(x). \tag{4.1}$$

Throughout this section, we consider this operator and let $E_0 \in \mathbb{R}$ be an energy level satisfying the following assumption.

Assumption 4.1 (Non-degeneracy at energy level E_0***).*** The potential V is C^∞, real valued and even. The classically allowed region at energy E_0, $K_{E_0} = V^{-1}((-\infty, E_0])$ is compact and connected. The energy is noncritical for V in the sense that for $x \in \mathbb{R}$, $V(x) = E_0 \Longrightarrow V'(x) \neq 0$.

The last part of Assumption 4.1 ensures that, if we define the Hamiltonian $p(x,\xi) = \xi^2 + V(x)$, we have $\mathrm{d}p \neq 0$ on the energy layer $p^{-1}(\{E_0\}) \subset \mathbb{R}^2$. Hence, Assumption 4.1 implies that $p^{-1}(\{E_0\})$ is a smooth compact connected 1D submanifold of $\mathbb{R}^2$.

Note that the evenness of the potential is actually not needed for the results presented in this section, but it simplifies notation and proofs slightly. Our goal is to apply the results of the present section to $V(x) = (x^2/2 - 1)^2$ and $E_0 = E_c \approx 2.35 > 1$ which satisfy Assumption 4.1. Note that if (V, E_0) satisfy Assumption 4.1, then there exists $\varepsilon_0 > 0$ small such that (V, E) also satisfy Assumption 4.1 for all $E \in I_{E_0}(\varepsilon_0)$, where we have denoted by

$$I_{E_0}(\varepsilon_0) := (E_0 - \varepsilon_0, E_0 + \varepsilon_0) \tag{4.2}$$

such an interval of non-degenerate energies. Moreover, we have

$$K_E = V^{-1}((-\infty, E]) = [-x_+(E), x_+(E)] \quad \text{for all } E \in I_{E_0}(\varepsilon_0),$$

[3] As mentioned to us by Didier Robert, we could also construct approximate eigenfunctions using coherent states instead of Lagrangian distributions.

where $x_+(E)$ is the unique $x \in \mathbb{R}^+$ such that $V(x_+(E)) = E$. Note also that under Assumption 4.1, P_h is essentially self-adjoint and from now on P_h denotes this realization.

To conclude these preliminaries, let us now introduce some classical quantities (in the sense that they arise from classical mechanics). For $\mu > \inf V$, we first introduce the energy

$$\mathcal{E}(\mu) = \frac{1}{2\pi} \int_{\xi^2+V\le\mu} \mathrm{d}x\,\mathrm{d}\xi.$$

Under Assumption 4.1 and with the notation (4.2), for all $\mu \in I_{E_0}(\varepsilon_0)$, the energy $\mathcal{E}(\mu)$ is finite and can be rewritten as

$$\mathcal{E}(\mu) = \frac{1}{\pi} \int_{-x_+(\mu)}^{x_+(\mu)} \sqrt{\mu - V(x)}\,\mathrm{d}x. \tag{4.3}$$

Still using Assumption 4.1, $\mu \mapsto \mathcal{E}(\mu)$ is smooth on $I_{E_0}(\varepsilon_0)$ and satisfies

$$\mathcal{E}'(\mu) = \frac{1}{2\pi} \int_{-x_+(\mu)}^{x_+(\mu)} \frac{\mathrm{d}x}{\sqrt{\mu - V(x)}}.$$

In particular, $\mathcal{E}$ is a strictly increasing function on $I_{E_0}(\varepsilon_0)$. Note that, in the case where $V(x) = (x^2/2-1)^2$, we have

$$\pi\mathcal{E}'(\mu) = C(\mu)^{-1}, \tag{4.4}$$

where $C(\mu) = (\int_0^{x_+(\mu)} (\mu - V(x))^{-1/2}\,\mathrm{d}x)^{-1}$ was introduced in (3.17).

4.3. Bohr-Sommerfeld condition and labelling of eigenvalues.

4.3.1. Spectrum and Bohr-Sommerfeld condition. We start by discussing eigenvalues of P_h and their approximations (see, e.g., [29]). Eigenfunctions and their approximations are discussed later on (although these approximations are of course intimately related).

Under Assumption 4.1 and with the notation (4.2), the spectrum of P_h, denoted $\mathrm{Sp}(P_h)$, satisfies

$$\mathrm{Sp}(P_h) \cap (-\infty, E_0 + \varepsilon_0) = \{E_k(h) \mid 1 \le k \le J(h)\}, \tag{4.5}$$

where $J(h) \in \mathbb{N}$. Moreover, a Sturm-Liouville argument (as in Proposition 1.1) shows that $E_k(h)$ has multiplicity one for all $k \in \{1, \dots, J(h)\}$, and we thus order eigenvalues increasingly as

$$E_k(h) < E_{k+1}(h), \quad \text{for all } 1 \le k < J(h).$$

Under Assumption 4.1 and with the notation (4.2), for all $\mu \in I_{E_0}(\varepsilon_0)$, we consider the pairs $(h, j) \in (0, h_0] \times \mathbb{N}$ satisfying the Bohr-Sommerfeld quantization condition

$$\mathcal{E}(\mu) = \left(j + \frac{1}{2}\right) h. \tag{4.6}$$

As proven in, for example, [29] or the appendix of [24] (see also Theorem 6.3 and the following corollaries in [5]), there is a family of functions $f_\ell \in C^\infty(I_{E_0}(\varepsilon_0))$ for $\ell \in \mathbb{N}$, such that the following holds. There is a unique $E(j, h, \mu) \in \mathrm{Sp}(P_h)$ such that

$$E(j, h, \mu) = \mu + \sum_{\ell \geq 1} f_\ell(\mu) h^\ell + \mathcal{O}(h^\infty) \quad \text{uniformly,}$$
$$\text{for } (\mu, j, h) \text{ satisfying (4.6) with } \mu \in I_{E_0}(\varepsilon_0).$$

Conversely, any eigenvalue of P_h in $I_{E_0}(\varepsilon_0)$ can be obtained for h small enough in this way. In other words, if we define

$$E^j(h) := E(j, h, \mu_j(h)) = \mu_j(h) + \sum_{\ell \geq 1} f_\ell(\mu_j(h)) h^\ell + \mathcal{O}(h^\infty), \tag{4.7}$$

with

$$\mu_j(h) = \mathcal{E}^{-1}\left(\left(j + \frac{1}{2}\right) h\right), \tag{4.8}$$

and

$$\mathcal{J}(h) := \{j \in \mathbb{N} \mid E^j(h) \in I_{E_0}(\varepsilon_0)\},$$

there is $h_0 > 0$ such that for all $h \in (0, h_0]$ we have

$$\mathrm{Sp}(P_h) \cap I_{E_0}(\varepsilon_0) = \{E^j(h) \mid j \in \mathcal{J}(h)\}. \tag{4.9}$$

Note that $\mu_j(h) < \mu_{j+1}(h)$ so that $E^j(h) < E^{j+1}(h)$ for all $h \in (0, h_0)$.

Note also that, as a direct consequence of (4.7)–(4.8) and a Taylor expansion of $\mathcal{E}^{-1}$, the spectrum of P_h in the energy window $I_{E_0}(\varepsilon_0)$ satisfies a gap condition: there exist $C, h_0 > 0$ such that for all $h \in (0, h_0]$ and $j \in \mathcal{J}(h)$,

$$d(E^j(h), \mathrm{Sp}(P_h) \setminus \{E^j(h)\}) \geq \frac{1}{C} h. \tag{4.10}$$

4.3.2. Labelling of eigenvalues. So far, the set $\mathrm{Sp}(P_h) \cap I_{E_0}(\varepsilon_0)$ is described in two different ways according to (4.5)–(4.9), by using, respectively, $E_k(h)$ and $E^j(h)$. Since eigenvalues are simple, there is for all $(j, h) \in \mathcal{J}(h) \times (0, 1)$ a unique $\hat{j} = \hat{j}(j, h)$ such that

$$E^j(h) = E_{\hat{j}}(h),$$

where $\hat{\jmath}$ refers to the "global labelling" of the eigenvalue; that is, $E_{\hat{\jmath}}(h)$ is the $\hat{\jmath}$'s eigenvalue of P_h. Note that in the case when V has a unique nondegenerate minimum without any other critical points below E_0 (which is not the case in which we are interested in our application to $V(x) = (x^2/2-1)^2$), then we have $\hat{\jmath} = j$ (see [29]). Without this assumption, by using the asymptotic Weyl formula with remainder (see again [29]) there are $C_0, h_0 > 0$ such that

$$|\hat{\jmath}(j,h) - j| \leq C_0, \quad \text{for all } j \in \mathcal{J}(h),\ h \in (0,h_0),$$

that is, for all those j such that $E^j(h) \in I_{E_0}(\varepsilon_0)$. More recently, Y. Colin de Verdière showed in [7, Theorem 3] the following more precise result.

Proposition 4.2. *Under Assumption 4.1 and with the notation (4.2), there exist $h_0 > 0$ and $\varepsilon_1 \in (0,\varepsilon_0)$ such that $\hat{\jmath} = j$ for any eigenvalue $E^j(h) \in I_{E_0}(\varepsilon_1)$ for all $h \in (0,h_0)$.*

In other words, the proposition says that $E^j(h) = E_j(h)$ when $E^j(h)$ is close to E_0.

Proof. We just sketch the main lines of the proof since no detail is given in [7]. The idea is to construct a deformation of the potential V onto a harmonic potential for which the spectrum is explicit. We introduce, for $t \in [0,1]$,

$$W_t(x) = (1-t)V(x) + tR_0x^2, \quad \text{with } R_0 = E_0 x_+(E_0)^{-2}.$$

With this choice, we have

$$W_0 = V, \qquad W_1 = R_0 x^2, \qquad \text{and } W_t(x_+(E_0)) = E_0$$

for all $t \in [0,1]$. Moreover, W_t satisfies the same Assumption 4.1, in which the nondegeneracy is uniform with respect to $t \in [0,1]$. We finally define

$$a(t) = \frac{1}{2\pi}\int_{\xi^2 + W_t(x) \leq E_0} \mathrm{d}x\,\mathrm{d}\xi = \frac{1}{\pi}\int_{-x_+(E_0)}^{x_+(E_0)} \sqrt{E_0 - W_t(x)}\,\mathrm{d}x,$$

where $x_+(E_0)$ is for all $t \in [0,1]$ the unique $x \geq 0$ such that $W_t(x_+(E_0)) = E_0$, and

$$V_t(x) := W_t\left(\frac{a(t)}{a(0)}x\right).$$

With this choice, we have that

$$V_0(x) = V(x), \qquad V_1(x) = R_0(a(1)^2/a(0)^2)x^2,$$

and

$$\mathcal{E}_t(E_0) := \frac{1}{2\pi}\int_{\xi^2 + V_t(x) \leq E_0} \mathrm{d}x\,\mathrm{d}\xi$$

satisfy

$$
\begin{aligned}
\mathcal{E}_t(E_0) &= \frac{1}{\pi}\int_{-(a(0)/a(t))x_+(E_0)}^{(a(0)/a(t))x_+(E_0)} \sqrt{E_0 - V_t(x)}\,\mathrm{d}x \\
&= \frac{1}{\pi}\int_{-(a(0)/a(t))x_+(E_0)}^{(a(0)/a(t))x_+(E_0)} \sqrt{E_0 - W_t\left(\frac{a(t)}{a(0)}x\right)}\,\mathrm{d}x \\
&= \frac{a(0)}{a(t)}\frac{1}{\pi}\int_{-x_+(E_0)}^{x_+(E_0)} \sqrt{E_0 - W_t(y)}\,\mathrm{d}y = a(0) = \mathcal{E}(E_0).
\end{aligned} \tag{4.11}
$$

We now consider the spectra of the family of operators

$$
P_{h,t} := -h^2\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V_t(x), \quad t \in [0,1],
$$

where

$$
P_{h,0} = P_h, \quad P_{h,1} = -h^2\frac{\mathrm{d}^2}{\mathrm{d}x^2} + R_0\frac{a(1)^2}{a(0)^2}x^2.
$$

In view of (4.5)–(4.9), we may write

$$
\begin{aligned}
\mathrm{Sp}(P_{h,t}) \cap (-\infty, E_0 + \varepsilon_0) &= \{E_k(h,t) \mid 1 \le k \le J(h,t)\}, \\
\mathrm{Sp}(P_{h,t}) \cap I_{E_0}(\varepsilon_0) &= \{E^j(h,t) \mid j \in \mathcal{J}(h,t)\},
\end{aligned}
$$

and since eigenvalues are simple, we may denote by $\hat{\jmath}(j,h,t)$ the unique integer such that

$$
E^j(h,t) = E_{\hat{\jmath}(j,h,t)}(h,t).
$$

The simplicity of the spectrum therefore also implies that the maps $t \mapsto E^j(h,t)$ and $t \mapsto E_k(h,t)$ are all continuous $[0,1] \to \mathbb{R}$ for any fixed j,k,h. We can then apply the above Bohr-Sommerfeld theory (4.7)–(4.8), uniformly with respect to t for the eigenvalues close to E_0, namely,

$$
E^j(h,t) = \mu_j(h,t) + hg_1(\mu_j(h,t),t) + \mathcal{O}(h^2),
$$

with $\mu_j(h,t) = \mathcal{E}_t^{-1}((j+\frac{1}{2})h)$, for all j,h,t such that $\mu_j(h,t) \in I_{E_0}(\varepsilon_0)$. By using (4.11), there are then $\varepsilon_1 < \varepsilon_0$ and $h_0 > 0$ so that, defining $\mathcal{J}^{\varepsilon_1}(h) \subset \mathcal{J}(h,1)$ by

$$
\mathrm{Sp}(P_{h,1}) \cap I_{E_0}(\varepsilon_1) = \{E^j(h,1) \mid j \in \mathcal{J}^{\varepsilon_1}(h)\},
$$

we have

$$
E^j(h,t) \in I_{E_0}(\varepsilon_0) \quad \text{for all } h \in (0,h_0),\ t \in [0,1],\ j \in \mathcal{J}^{\varepsilon_1}(h).
$$

As a consequence, for $h \in (0,h_0)$ and $j \in \mathcal{J}^{\varepsilon_1}(h)$, the map

$$t \mapsto \hat{\jmath}(j,h,t) - j, \quad [0,1] \to \mathbb{N}$$

is continuous. Hence, we have $\hat{\jmath}(j,h,t) - j$ constant for $t \in [0,1]$. But for $t = 1$ we are considering the harmonic oscillator $P_{h,1}$, for which we know that $\hat{\jmath}(j,h,1) = j$. This implies that $\hat{\jmath}(j,h,0) = j$ for the operator $P_{h,0} = P_h$. ◻

4.4. Functional analysis: a comparison between approximate eigenfunctions and eigenfunctions. In this subsection, we compare approximate eigenvalues and eigenfunctions with exact ones, and investigate their derivatives with respect to the semiclassical parameter h (having in mind the formula (3.8) for the second variation). The approach here is "abstract" in the sense that we do not construct the approximate eigenvalues/eigenfunctions but rather assume their existence. A way of constructing the latter approximate eigenvalues/eigenfunctions is reviewed in Subsection 4.5 below.

Proposition 4.3. *We consider the operator P_h in (4.1) under Assumption 4.1 and with the notation (4.2). Assume further that, for $C_0 > 0$, $k > 1$, and $\varepsilon_1 \in (0,\varepsilon_0)$, we have $E^{\mathrm{app}}(h) \in I_{E_0}(\varepsilon_1)$, $v_h^{\mathrm{app}} \in D(P_h)$ (respectively, real valued) and $r_h \in L^2(\mathbb{R})$ satisfying*

$$P_h v_h^{\mathrm{app}} = E^{\mathrm{app}}(h) v_h^{\mathrm{app}} + r_h, \ \|v_h^{\mathrm{app}}\|_{L^2(\mathbb{R})} = 1, \ \|r_h\|_{L^2(\mathbb{R})} \le C_0 h^k. \tag{4.12}$$

Then, there exist C, $h_0 > 0$ and $E(h) \in \mathrm{Sp}(P_h) \cap I_{E_0}(\varepsilon_1)$ such that

$$|E(h) - E^{\mathrm{app}}(h)| \le C_0 h^k, \tag{4.13}$$

and $v_h \in D(P_h)$ (respectively real valued) such that for all $h \in (0,h_0)$,

$$P_h v_h = E(h) v_h, \quad \|v_h\|_{L^2(\mathbb{R})} = 1, \quad \|v_h - v_h^{\mathrm{app}}\|_{L^2(\mathbb{R})} \le C h^{k-1}. \tag{4.14}$$

Notice that, since P_h has real coefficients, we may always assume that v_h^{app} is real valued (taking otherwise its real part or imaginary part and renormalizing). The proof is rather classical but we recall it for the sake of completeness. Note also that, according to the gap condition (4.10) and the assumption $k > 1$, there is at most one eigenvalue of P_h satisfying (4.13) for h small enough.

Proof. According to (4.12), we have $\|(P_h - E^{\mathrm{app}}(h)) v_h^{\mathrm{app}}\|_{L^2(\mathbb{R})} = \|r_h\|_{L^2(\mathbb{R})} \le C_0 h^k \|v_h^{\mathrm{app}}\|_{L^2(\mathbb{R})}$ and thus $\|(P_h - E^{\mathrm{app}}(h))^{-1}\|_{L^2(\mathbb{R}) \to L^2(\mathbb{R})} \ge (C_0 h^k)^{-1}$ (where $\|(P_h - E^{\mathrm{app}}(h))^{-1}\|_{L^2(\mathbb{R}) \to L^2(\mathbb{R})} \in (0,+\infty]$). Since the operator P_h is selfadjoint, the spectral theorem yields that if $z \in \mathbb{C} \setminus \mathrm{Sp}(P_h)$, we have

$$\|(P_h - z)^{-1}\|_{L^2(\mathbb{R}) \to L^2(\mathbb{R})} = \frac{1}{d(z, \mathrm{Sp}(P_h))}.$$

As a consequence, we have obtained that

$$\text{Either } E^{\mathrm{app}}(h) \in \mathrm{Sp}(P_h) \quad \text{or } \frac{1}{d(E^{\mathrm{app}}(h), \mathrm{Sp}(P_h))} \geq (Ch^k)^{-1}.$$

In any case, we deduce $d(E^{\mathrm{app}}(h), \mathrm{Sp}(P_h)) \leq C_0 h^k$. Given $E^{\mathrm{app}}(h) \in I_{E_0}(\varepsilon_1)$, that $C_0 h^k \to 0$, and that $\mathrm{Sp}(P_h) \cap I_{E_0}(\varepsilon_0)$ consists only in eigenvalues (see (4.5) as consequence of Assumption 4.1), there exist $E(h) \in \mathrm{Sp}(P_h) \cap I_{E_0}(\varepsilon_0)$ with (4.13) and $\tilde{v}_h \in D(P_h)$ satisfying $\|\tilde{v}_h\|_{L^2(\mathbb{R})} = 1$ such that

$$P_h \tilde{v}_h = E(h)\tilde{v}_h.$$

We may assume that $\tilde{v}_h$ is real valued since P_h has real coefficients. We denote by $\Pi_{E(h)}$ the orthogonal projection in $L^2(\mathbb{R})$ onto $\ker(P_h - E(h))$, which, on account to the simplicity of the spectrum, may be written $\Pi_{E(h)} w = (w, \tilde{v}_h)_{L^2(\mathbb{R})}\tilde{v}_h$. We have $\Pi_{E(h)} v_h^{\mathrm{app}}$ real valued if $\tilde{v}_h$ and v_h^{app} are, and

$$(P_h - E(h))\Pi_{E(h)} v_h^{\mathrm{app}} = 0.$$

Moreover, we denote by $P_h = \int_{\mathbb{R}} \lambda \, \mathrm{d}\mathsf{P}(\lambda)$ the spectral resolution of the operator P_h, and comment that the projector-valued measure $\mathrm{d}\mathsf{P}$ is a linear combination of Dirac masses on $I_{E_0}(\varepsilon_0)$ (the spectrum is discrete on $I_{E_0}(\varepsilon_0)$). We thus have

$$\begin{aligned} ||(1 - \Pi_{E(h)})v_h^{\mathrm{app}}||^2_{L^2(\mathbb{R})} &= \int_{\mathbb{R}\setminus\{E(h)\}} \mathrm{d}(\mathsf{P}(\lambda)v_h^{\mathrm{app}}, v_h^{\mathrm{app}})_{L^2(\mathbb{R})} \\ &= \int_{\mathbb{R}\setminus\{E(h)\}} \frac{(\lambda - E(h))^2}{(\lambda - E(h))^2} \mathrm{d}(\mathsf{P}(\lambda)v_h^{\mathrm{app}}, v_h^{\mathrm{app}})_{L^2(\mathbb{R})}. \end{aligned}$$

Recalling the gap property (4.10) (consequence of Assumption 4.1), we deduce

$$\begin{aligned} &||(1 - \Pi_{E(h)})v_h^{\mathrm{app}}||^2_{L^2(\mathbb{R})} \\ &\leq \frac{1}{d(E(h), \mathrm{Sp}(P_h) \setminus E(h))^2} \int_{\mathbb{R}\setminus\{E(h)\}} (\lambda - E(h))^2 \, \mathrm{d}(\mathsf{P}(\lambda)v_h^{\mathrm{app}}, v_h^{\mathrm{app}})_{L^2(\mathbb{R})} \\ &\leq \frac{C^2}{h^2} \int_{\mathbb{R}} (\lambda - E(h))^2 \, \mathrm{d}\left(\mathsf{P}(\lambda)v_h^{\mathrm{app}}, v_h^{\mathrm{app}}\right)_{L^2(\mathbb{R})} \\ &\leq \frac{C^2}{h^2} ||(P_h - E(h))v_h^{\mathrm{app}}||^2_{L^2(\mathbb{R})} \\ &\leq \frac{C^2}{h^2} 2||(P_h - E^{\mathrm{app}}(h))v_h^{\mathrm{app}}||^2_{L^2(\mathbb{R})} + \frac{C^2}{h^2} 2|E(h) - E^{\mathrm{app}}(h)|^2 ||v_h^{\mathrm{app}}||^2_{L^2(\mathbb{R})} \\ &\leq Ch^{2k-2}, \end{aligned} \tag{4.15}$$

according to (4.12)–(4.13). As a consequence, we have

$$c_h^2 := ||\Pi_{E(h)} v_h^{\mathrm{app}}||^2_{L^2(\mathbb{R})} \tag{4.16}$$

$$= ||v_h^{\mathrm{app}}||^2_{L^2(\mathbb{R})} - ||(1-\Pi_{E(h)}) v_h^{\mathrm{app}}||^2_{L^2(\mathbb{R})}$$

$$\geq 1 - Ch^{2k-2}. \tag{4.17}$$

We now set $v_h := c_h^{-1}\Pi_{E(h)} v_h^{\mathrm{app}}$, which is a normalized eigenfunction satisfying, according to (4.15)–(4.16),

$$\begin{aligned} ||v_h - v_h^{\mathrm{app}}||^2_{L^2(\mathbb{R})} &= c_h^{-2}||\Pi_{E(h)} v_h^{\mathrm{app}} - c_h v_h^{\mathrm{app}}||^2_{L^2(\mathbb{R})} \\ &\leq 2c_h^{-2}||\Pi_{E(h)} v_h^{\mathrm{app}} - v_h^{\mathrm{app}}||^2_{L^2(\mathbb{R})} \\ &\quad + 2c_h^{-2}(1-c_h)||v_h^{\mathrm{app}}||^2_{L^2(\mathbb{R})} \\ &\leq 4\frac{Ch^{2k-2}}{1-Ch^{2k-2}}. \end{aligned}$$

This concludes the proof of the lemma. ◻

We have now to prove that, under some additional assumptions, derivatives with respect to h of the approximate eigenvalues/eigenfunctions are close to derivatives with respect to h of the associated real eigenvalues/eigenfunctions.

Proposition 4.4. *We consider the operator P_h in (4.1) under Assumption 4.1 and with the notation (4.2). Assume further that there are $m \in \mathbb{N}$, $C > 0$ such that $|V(x)| \leq C(1+|x|)^m$. Let $\varepsilon_1 \in (0, \varepsilon_0)$ and*

$$I_j := \left\{h \in (0,1) \mid \mathcal{E}^{-1}\left(\left(j+\frac{1}{2}\right)h\right) \in I_{E_0}(\varepsilon_1)\right\}, \quad j \in \mathbb{N}^*.$$

Then, for all $j \in \mathbb{N}^*$, the map $I_j \ni h \mapsto E^j(h) \in \mathrm{Sp}(P_h)$ belongs to $C^1(I_j)$, and for any solution to

$$P_h v_h^j = E^j(h) v_h^j, \quad h \in I_j, \tag{4.18}$$

the map $h \mapsto v_h^j$ belongs to $C^1(I_j; S(\mathbb{R}))$.

Suppose also we are given $E^{j,\mathrm{app}}(h) \in C^1(I_j)$ and $v_h^{j,\mathrm{app}} \in C^1(I_j; S(\mathbb{R}))$ satisfying the following assumptions:

$$E^{j,\mathrm{app}}(h) - E^j(h) = \mathcal{O}(h^\infty) \qquad \text{for } h \in I_j, \tag{4.19}$$

$$\|v_h^j - v_h^{j,\mathrm{app}}\|_{L^2(\mathbb{R})} = \mathcal{O}(h^\infty) \qquad \text{for } h \in I_j, \tag{4.20}$$

$$P_h v_h^{j,\mathrm{app}} = E^{j,\mathrm{app}}(h) v_h^{j,\mathrm{app}} + r_h^j \quad \text{for } h \in I_j, \tag{4.21}$$

$$\|v_h^j\|_{L^2} = 1, \quad \|v_h^{j,\mathrm{app}}\|_{L^2} = 1, \tag{4.22}$$

$$\|r_h^j\|_{L^2} = \mathcal{O}(h^\infty) \qquad \text{for } h \in I_j, \tag{4.23}$$

$$\|\partial_h r_h^j\|_{L^2} = \mathcal{O}(h^\infty) \qquad \text{for } h \in I_j, \tag{4.24}$$

there exists $k \in \mathbb{N}$ *such that*

$$\|\partial_h v_h^{j,\mathrm{app}}\|_{L^2(\mathbb{R})} = \mathcal{O}(h^{-k}) \quad \text{for } h \in I_j, \tag{4.25}$$

and there exists a compact set $K \subset \mathbb{R}$ *such that*

$$\operatorname{supp} v_h^{j,\mathrm{app}} \subset K \qquad \text{for } h \in I_j,\ j \in \mathbb{N}^*. \tag{4.26}$$

Then, we have

$$(E^{j,\mathrm{app}})'(h) - (E^j)'(h) = \mathcal{O}(h^\infty) \quad \text{uniformly for } h \in I_j,\ j \in \mathbb{N}^*, \tag{4.27}$$

$$\|\partial_h v_h^j - \partial_h v_h^{j,\mathrm{app}}\|_{L^2(\mathbb{R})} = \mathcal{O}(h^\infty) \quad \text{uniformly for } h \in I_j. \tag{4.28}$$

In the statement of the proposition, we keep the subscript j to help us remember that, for eigenvalues $E^j(h)$ in the energy window $I_{E_0}(\varepsilon_1)$, the product jh belongs to a fixed interval and thus $j \to +\infty$ simultaneously as $h \to 0^+$. We will, however, drop the dependence with respect to j in the proof, since this statement only concerns a fixed couple of eigenvalue/eigenfunction and approximate eigenvalue/eigenfunction.

Note that the existence of $E^j(h), v_h^j$ such that (4.19) and (4.20) are satisfied is actually a consequence of Lemma 4.3 once (4.21)–(4.22)–(4.23) are satisfied.

Note that in our application we can choose K in (4.26) to be any compact neighborhood of $K_{E_0+\varepsilon_0} = V^{-1}((-\infty, E_0+\varepsilon_0])$.

Note, finally, that this proposition is written with the assumption of remainder estimates in $\mathcal{O}(h^\infty)$. In applications such $\mathcal{O}(h^\infty)$ are issued from Borel summation providing differentiable remainders, and in which one can always impose differentiability with respect to parameters. One could also write an analogue of this proposition with $\mathcal{O}(h^N)$ remainders, keeping track of the size N—but we choose not to do this for the sake of the presentation.

Proof. First, notice that eigenvalues $E(h)$ are simple from Sturm-Liouville theory and are thus C^∞ with respect to $h \in I_j$. That $h \mapsto v_h$ belongs to $C^1(I_j; S(\mathbb{R}))$ follows from the following computations. In the interval I_j, we differentiate (4.18) and (4.21) with respect to h, and obtain

$$(-h^2\,\partial_x^2 + V - E(h))\,\partial_h v_h = E'(h) v_h + 2h\,\partial_x^2 v_h, \tag{4.29}$$

and

$$\begin{aligned}(-h^2\,\partial_x^2 + V - E^{\mathrm{app}}(h))\,\partial_h v_h^{\mathrm{app}} &= \\ &= (E^{\mathrm{app}})'(h) v_h^{\mathrm{app}} + \partial_h r_h + 2h\,\partial_x^2 v_h^{\mathrm{app}}.\end{aligned} \tag{4.30}$$

We first compare $E'(h)$ and $(E^{\mathrm{app}})'(h)$ and then compare $\partial_h v_h$ and $\partial_h v_h^{\mathrm{app}}$. Differentiating (4.22), we obtain

$$(v_h, \partial_h v_h)_{L^2(\mathbb{R})} = 0, \quad (v_h^{\mathrm{app}}, \partial_h v_h^{\mathrm{app}})_{L^2(\mathbb{R})} = 0. \tag{4.31}$$

Taking the scalar product with v_h in (4.29) and using (4.31), selfadjointness of P_h, and (4.18), we obtain

$$E'(h) = -2h(v_h, \partial_x^2 v_h)_{L^2(\mathbb{R})}, \tag{4.32}$$

and taking the scalar product with v_h^{app} in (4.30) and using (4.31) selfadjointness of P_h and (4.21), we obtain

$$\begin{aligned}(E^{\mathrm{app}})'(h) = &-2h(v_h^{\mathrm{app}}, \partial_x^2 v_h^{\mathrm{app}})_{L^2(\mathbb{R})} \\ &- (\partial_h r_h, v_h^{\mathrm{app}})_{L^2(\mathbb{R})} + (r_h, \partial_h v_h^{\mathrm{app}})_{L^2(\mathbb{R})}.\end{aligned} \tag{4.33}$$

Subtracting (4.32) from (4.33), we deduce

$$\begin{aligned}&|(E^{\mathrm{app}})'(h) - E'(h)| \\ &\quad\le 2h|(v_h, \partial_x^2 v_h)_{L^2(\mathbb{R})} - (v_h^{\mathrm{app}}, \partial_x^2 v_h^{\mathrm{app}})_{L^2(\mathbb{R})}| \\ &\qquad + |(\partial_h r_h, v_h^{\mathrm{app}})_{L^2(\mathbb{R})}| + |(r_h, \partial_h v_h^{\mathrm{app}})_{L^2(\mathbb{R})}|.\end{aligned} \tag{4.34}$$

The last two terms in the righthand side are $\mathcal{O}(h^\infty)$ according to Assumptions (4.22), (4.23), (4.24), and (4.25), respectively, on v_h^{app}, r_h, $\partial_h r_h$, and $\partial_h v_h^{\mathrm{app}}$. For the first term on the righthand side of (4.34), we write

$$\begin{aligned}&(v_h, \partial_x^2 v_h)_{L^2(\mathbb{R})} - (v_h^{\mathrm{app}}, \partial_x^2 v_h^{\mathrm{app}})_{L^2(\mathbb{R})} \\ &\quad= (v_h - v_h^{\mathrm{app}}, \partial_x^2 v_h)_{L^2(\mathbb{R})} + (v_h^{\mathrm{app}}, \partial_x^2 v_h - \partial_x^2 v_h^{\mathrm{app}})_{L^2(\mathbb{R})} \\ &\quad= (v_h - v_h^{\mathrm{app}}, \partial_x^2 v_h + \partial_x^2 v_h^{\mathrm{app}})_{L^2(\mathbb{R})}.\end{aligned} \tag{4.35}$$

Then, we will obtain again that this term is $\mathcal{O}(h^\infty)$ by using (4.20), once we have proved the following semiclassically temperate control of $\partial_x^2 v_h$ and $\partial_x^2 v_h^{\mathrm{app}}$. The first control can be deduced from (4.18) that we rewrite in the form

$$\partial_x^2 v_h = h^{-2} V(x) v_h - h^{-2} E(h) v_h.$$

Letting $\tilde{K}$ be a compact neighborhood of $K_{E_0+\varepsilon_0} = V^{-1}(]-\infty, E_0 + \varepsilon_0])$ and recalling that $E(h) \in I_{E_0}(\varepsilon_0)$, we thus have

$$\begin{aligned}\|\partial_x^2 v_h\|_{L^2(\mathbb{R})} &\le h^{-2}\|V v_h\|_{L^2(\mathbb{R})} + h^{-2}|E(h)|\,\|v_h\|_{L^2(\mathbb{R})} \\ &\le Ch^{-2}\|v_h\|_{L^2(\tilde{K})} + h^{-2}\|V v_h\|_{L^2(\mathbb{R}\setminus\tilde{K})} + Ch^{-2}\|v_h\|_{L^2(\mathbb{R})}.\end{aligned}$$

Using semiclassical control of v_h in the classically forbidden region (see Proposition 3.4) together with L^2 normalization (4.22), we obtain

$$\|\partial_x^2 v_h\|_{L^2(\mathbb{R})} \le Ch^{-2}. \tag{4.36}$$

The estimate

$$\|\partial_x^2 v_h^{\mathrm{app}}\|_{L^2(\mathbb{R})} \le Ch^{-2}, \tag{4.37}$$

follows the exact same arguments from the assumptions (4.21), (4.22), (4.23), and (4.26). The last two inequalities, combined with (4.35) and (4.34), imply (4.27).

We now turn to the proof of (4.28). We first prove that

$$V(x)(v_h - v_h^{\mathrm{app}}) = \mathcal{O}_{L^2(\mathbb{R})}(h^\infty) \quad \text{and} \quad \partial_x^2(v_h - v_h^{\mathrm{app}}) = \mathcal{O}_{L^2(\mathbb{R})}(h^\infty). \tag{4.38}$$

Indeed, we first have (with $\tilde{K}$ a compact neighborhood of $K_{E_0+\varepsilon_0}$)

$$\begin{aligned} &\|V(v_h - v_h^{\mathrm{app}})\|_{L^2(\mathbb{R})} \\ &\quad \le \|V(v_h - v_h^{\mathrm{app}})\|_{L^2(K\cup\tilde{K})} + \|V(v_h - v_h^{\mathrm{app}})\|_{L^2(\mathbb{R}\setminus(K\cup\tilde{K}))} \\ &\quad \le C\|v_h - v_h^{\mathrm{app}}\|_{L^2(K\cup\tilde{K})} + \|Vv_h\|_{L^2(\mathbb{R}\setminus(K\cup\tilde{K}))} \\ &\quad \le C\|v_h - v_h^{\mathrm{app}}\|_{L^2(\mathbb{R})} + \mathcal{O}(h^\infty) = \mathcal{O}(h^\infty), \end{aligned} \tag{4.39}$$

where we have used (4.26) in the second line, Proposition 3.4 together with (4.22) and the assumption $|V(x)| \le C(1+|x|)^m$ in the last inequality, and (4.20) in the last equality. Then, subtracting (4.21) from (4.18), we obtain

$$\begin{aligned} &-h^2\partial_x^2(v_h - v_h^{\mathrm{app}}) \\ &\quad = -V(v_h - v_h^{\mathrm{app}}) + E(h)v_h - E^{\mathrm{app}}(h)v_h^{\mathrm{app}} - r_h \\ &\quad = -V(v_h - v_h^{\mathrm{app}}) + (E(h) - E^{\mathrm{app}}(h))v_h + E^{\mathrm{app}}(h)(v_h - v_h^{\mathrm{app}}) - r_h \\ &\quad = \mathcal{O}_{L^2(\mathbb{R})}(h^\infty), \end{aligned}$$

after having used (4.20), (4.19), and $E(h) \in I_{E_0}(\varepsilon_0)$, together with the last estimate. This concludes the proof of (4.38). Note also that we deduce from (4.33) together with (4.22), (4.23), (4.24), (4.25), and (4.37) the existence of $k \in \mathbb{N}$ such that

$$(E^{j,\mathrm{app}})'(h) = \mathcal{O}(h^{-k}) \quad \text{uniformly for } h \in I_j. \tag{4.40}$$

We now rewrite the difference of (4.30) and (4.29) under the form

$$\begin{aligned} &(P_h - E(h))(\partial_h v_h - \partial_h v_h^{\mathrm{app}}) \\ &\quad = (E(h) - E^{\mathrm{app}}(h))\,\partial_h v_h^{\mathrm{app}} + 2h\,\partial_x^2(v_h - v_h^{\mathrm{app}}) \\ &\qquad + (E'(h) - (E^{\mathrm{app}})'(h))v_h - (E^{\mathrm{app}})'(h)(v_h^{\mathrm{app}} - v_h) - \partial_h r_h. \end{aligned}$$

Using (4.19), (4.22), (4.20), (4.24), (4.38), and (4.40), this implies

$$(P_h - E(h))(\partial_h v_h - \partial_h v_h^{\mathrm{app}}) = \mathcal{O}_{L^2(\mathbb{R})}(h^\infty). \tag{4.41}$$

Now recalling (4.31), we deduce from (4.20) and (4.25) that

$$\begin{aligned}((\partial_h v_h - \partial_h v_h^{\mathrm{app}}), v_h)_{L^2(\mathbb{R})} &= (\partial_h v_h^{\mathrm{app}}, v_h)_{L^2(\mathbb{R})} \\ &= (\partial_h v_h^{\mathrm{app}}, v_h - v_h^{\mathrm{app}})_{L^2(\mathbb{R})} = \mathcal{O}(h^\infty).\end{aligned}$$

We now denote by $\Pi_{E(h)}$ the orthogonal projection in $L^2(\mathbb{R})$ onto $\ker(P_h - E(h))$, which we may write $\Pi_{E(h)} w = (w, v_h)_{L^2(\mathbb{R})} v_h$ according to the simplicity of the spectrum. On the one hand, the last identity implies

$$\|\Pi_{E(h)}(\partial_h v_h - \partial_h v_h^{\mathrm{app}})\|_{L^2(\mathbb{R})} = \mathcal{O}(h^\infty).$$

On the other hand, proceeding exactly as in (4.15) in the proof of Lemma 4.3, using the approximate equation (4.41) together with the gap condition (4.10), we obtain

$$\|(1 - \Pi_{E(h)})(\partial_h v_h - \partial_h v_h^{\mathrm{app}})\|_{L^2(\mathbb{R})} = \mathcal{O}(h^\infty).$$

Combining the last two identities concludes the proof of (4.28), and thus that of the proposition. □

Recalling the expression of $\lambda_j''(\alpha)$ in the semiclassical scaling in Lemma 3.1, given a C^∞ real-valued function a with polynomial growth, we are now interested in the asymptotic expansion of the quantity

$$\mathfrak{M}_a^j(h) := \int a(x)|v_h^j(x)|^2 \,\mathrm{d}x, \qquad \text{for } h \in I_j, \tag{4.42}$$

and its derivative with respect to h:

$$(\mathfrak{M}_a^j)'(h) = 2\,\mathrm{Re} \int a(x)(\partial_h v_h^j)(x)\overline{v_h^j(x)}\,\mathrm{d}x, \quad \text{for } h \in I_j.$$

We also define its approximate analogue

$$\mathfrak{M}_a^{j,\mathrm{app}}(h) := \int a(x)|v_h^{j,\mathrm{app}}(x)|^2 \,\mathrm{d}x, \qquad \text{for } h \in I_j, \tag{4.43}$$

and obtain the following corollary of Proposition 4.3 (applied for all $k \in \mathbb{N}$) and Proposition 4.4.

Corollary 4.5. *Under the assumptions of Proposition 4.4, we have*

$$\mathfrak{M}_a^j(h) - \mathfrak{M}_a^{j,\mathrm{app}}(h) = \mathcal{O}(h^\infty) \tag{4.44}$$

and

$$(\mathfrak{M}_a^j)'(h) - (\mathfrak{M}_a^{j,\mathrm{app}})'(h) = \mathcal{O}(h^\infty), \quad \text{uniformly for } h \in I_j, \tag{4.45}$$

for any $a \in C^\infty(\mathbb{R})$ with polynomial growth.

Proof. The validity of (4.44) for $a \in C^\infty(\mathbb{R})$ with polynomial growth follows from the same decomposition as in (4.39). Concerning (4.45), we decompose

$$\begin{aligned}(\mathfrak{M}_a^j)'(h) - (\mathfrak{M}_a^{j,\mathrm{app}})'(h) &= 2\,\mathrm{Re}\int a(x)(\partial_h v_h^j)(x)\overline{v_h^j(x)}\,\mathrm{d}x \\ &\quad - 2\,\mathrm{Re}\int a(x)(\partial_h v_h^{j,\mathrm{app}})(x)\overline{v_h^{j,\mathrm{app}}(x)}\,\mathrm{d}x \\ &= 2\,\mathrm{Re}(av_h^j, \partial_h v_h^j - \partial_h v_h^{j,\mathrm{app}})_{L^2} \\ &\quad + 2\,\mathrm{Re}(\partial_h v_h^{j,\mathrm{app}}, a(v_h^j - v_h^{j,\mathrm{app}}))_{L^2}\end{aligned}$$

whence

$$\begin{aligned}|(\mathfrak{M}_a^j)'(h) - (\mathfrak{M}_a^{j,\mathrm{app}})'(h)| &\le 2\|av_h^j\|_{L^2}\,\|\partial_h v_h^j - \partial_h v_h^{j,\mathrm{app}}\|_{L^2} \\ &\quad + 2\|\partial_h v_h^{j,\mathrm{app}}\|_{L^2}\,\|a(v_h^j - v_h^{j,\mathrm{app}})\|_{L^2}.\end{aligned}$$

According to (4.25) and the same decomposition as in (4.39), we have

$$\|\partial_h v_h^{j,\mathrm{app}}\|_{L^2}\,\|a(v_h^j - v_h^{j,\mathrm{app}})\|_{L^2} = \mathcal{O}(h^\infty),$$

and according to Proposition 3.4 together with (4.28), we also have

$$\|av_h^j\|_{L^2}\,\|\partial_h v_h^j - \partial_h v_h^{j,\mathrm{app}}\|_{L^2} = \mathcal{O}(h^\infty).$$

The last three statements conclude the proof of (4.45) and of the corollary. ◻

4.5. Semiclassical Lagrangian distributions revisited. Here, we recall a classical way of constructing approximate eigenfunctions for 1D-Schrödinger operators at a regular energy level E_0, through WKB expansions or so-called semiclassical Lagrangian distributions. We briefly review such a construction and check that one can differentiate with respect to parameters (in particular h). We then consider the product of two such objects.

4.5.1. Semiclassical Lagrangian distributions. We only need part of the local theory of Lagrangian distributions. We refer to the first section of [12] (pages 209 to 232), to [5] or [13] (see also [32, Section 25.1] in the non-semiclassical homogeneous setting). Given a smooth family of compact Lagrangian submanifolds $\mu \mapsto \Lambda_\mu \subset T^*\mathbb{R}^n = \mathbb{R}^{2n}$, with μ a parameter in a small interval $\mu \in I_{E_0}(\varepsilon_0) = (E_0 - \varepsilon_0, E_0 + \varepsilon_0)$, a semiclassical Lagrangian distribution (or Lagrangian function, or Lagrangian state) associated with Λ_μ is a linear combination of integrals in the form (cf. [12, (1.2.1)] or [5])

$$I(x,\mu,h) = (2\pi h)^{-k/2}\int_U e^{i\varphi(x,\theta,\mu)/h} a(x,\theta,\mu,h)\,\mathrm{d}\theta, \tag{4.46}$$

where the following hold:

- $k \in \mathbb{N}$, $U \subset \mathbb{R}^k$ is an open set.
- $\varphi \in C^\infty(\mathbb{R}^n \times U \times \mathbb{R})$ is a real-valued phase-function associated with the Lagrangian Λ_μ in the following sense: on the critical set (depending on μ)

$$C_{\varphi,\mu} := \{(x,\theta) \mid d_\theta\varphi(x,\theta,\mu) = 0\},$$

the differentials $d_x(\partial_{\theta_1}\varphi), \dots, d_x(\partial_{\theta_k}\varphi)$ are linearly independent, and

$$\{(x, d_x\varphi(x,\theta,\mu)) \mid (x,\theta) \in C_{\varphi,\mu}\} \subset \Lambda_\mu.$$

- a is in $C^\infty(\mathbb{R}^n \times U \times I_{E_0}(\varepsilon_0) \times (0,h_0))$, uniformly compactly supported in $(x,\theta) \in \mathbb{R}^n \times U$ and admitting (as well as its derivatives) an expansion in the form

$$\partial^\beta_{(x,\theta,\mu)} a(x,\theta,\mu,h) \sim \sum_{r=0}^{+\infty} a_{r,\beta}(x,\theta,\mu)h^r. \tag{4.47}$$

Here, $\mu \in \mathbb{R}$ is an energy parameter and the dependence on μ is C^∞. Note that the asymptotic expansion for derivatives of a is not always explicit in most references (e.g., [12]). Note also that, from the Borel summation process, the expansion (4.47) is also differentiable with respect to h and μ. The case $k = 0$ in (4.46) simply corresponds to $I(x,h,\mu) = e^{i\varphi(x,\mu)/h}a(x,\mu,h)$, whereas, in general, (4.46) has to be understood in the sense of oscillatory integrals (see [12] or [15, Chapter 1]). Note finally that, since Λ_μ is assumed to be compact here, any semiclassical Lagrangian distribution $I(x,\mu,h)$ as defined above is actually a smooth function of $(x,\mu,h) \in \mathbb{R}^n \times I_{\varepsilon_0}(E_0) \times (0,h_0)$. Another important feature is that, if $I(x,\mu,h)$ is a semiclassical Lagrangian distribution associated with Λ_μ and $a \in C^\infty(\mathbb{R}^n)$ (or more generally $a \in S^m(T^*\mathbb{R}^n)$), then $a(x)I(x,\mu,h)$ (or more generally $\mathrm{Op}_h(a)I(\cdot,\mu,h)$ for any semiclassical quantization Op_h; see [43]) is as well a semiclassical Lagrangian distribution associated with Λ_μ.

***4.5.2. Semiclassical Lagrangian distributions approximating eigenfunctions in dimension* 1.** In the present 1D context ($n = 1$), under Assumption 4.1 and with the notation (4.2), the Lagrangian manifold is the energy level

$$\Lambda_\mu = \{(x,\xi) \in \mathbb{R}^2 \mid \xi^2 + V(x) = \mu\}, \quad \mu \in I_{E_0}(\varepsilon_0).$$

We denote $\pi_x : \mathbb{R}_x \times \mathbb{R}_\xi \to \mathbb{R}_x$ and $\pi_\xi : \mathbb{R}_x \times \mathbb{R}_\xi \to \mathbb{R}_\xi$ the canonical projections, and notice that

$$\pi_x(\Lambda_\mu) = [-x_+(\mu), x_+(\mu)]$$

(recall that Assumption 4.1 includes that V is even). Notice that away from the two points $(\pm x_+(\mu), 0) \in \Lambda_\mu$, Λ_μ project nicely on the x variable and near these two points, Λ_μ projects nicely on the ξ variable. Therefore (see, e.g., [15, Exercise 12.3]), one can show that all semiclassical Lagrangian distributions/functions associated with Λ_μ can be written as

$$I(x,\mu,h) = \sum_\pm a_\pm(x,\mu,h)e^{\pm(i/h)\varphi(x,\mu)} + (2\pi h)^{-1/2}\int_{\mathbb{R}} b_\pm(x,\xi,\mu,h)e^{(i/h)(x\xi\pm\psi(\xi))}\,\mathrm{d}\xi, \tag{4.48}$$

$$\varphi(x,\mu) = \int_0^x \sqrt{\mu - V(s)}\,\mathrm{d}s, \quad \psi(\xi,\mu) = \int_0^\xi x_+(\mu-\zeta^2)\,\mathrm{d}\zeta, \tag{4.49}$$

with $\mu \in I_{E_0}(\varepsilon_0)$, $\operatorname{supp}_x a_\pm \subset [-x_+(E_0-\varepsilon_0), x_+(E_0-\varepsilon_0)]$ and $\operatorname{supp}_\xi b_\pm$ being a fixed small set containing 0. Note that, in these two sets, respectively,

$$\Lambda_\mu \cap (-x_+(E_0-\varepsilon_0), x_+(E_0-\varepsilon_0)) \cap \{\pm\xi > 0\} = \{(x \pm \varphi'(x))\},$$

or

$$\Lambda_\mu = \{(\pm\psi'(\xi),\xi)\}$$

(where the last equality holds only locally).

The next important feature we use is that, under the quantization relation (4.8), that is, $\mu = \mu(h) = \mu_j(h)$, there exist $E_j^{\mathrm{app}}(h) = \mu_j(h) + \mathcal{O}(h) \to E_0$ and amplitudes $a_\pm(x,\mu_j(h),h)$, $b_\pm(x,\xi,\mu_j(h),h)$ satisfying (4.47) such that, setting

$$u_j^{\mathrm{app}}(x,h) := c_h^{-1} I(x,\mu_j(h),h), \quad c_h := \|I(x,\mu_j(h),h)\|_{L^2(\mathbb{R})} \to c_0 > 0,$$

where $I(x,\mu,h)$ is the Lagrangian function (4.48) associated with these particular $a_\pm, b_\pm$, we have

$$P_h u_j^{\mathrm{app}}(x,h) = E_j^{\mathrm{app}}(h)u_j^{\mathrm{app}}(x,h) + r(x,h), \tag{4.50}$$

with

$$\|u_j^{\mathrm{app}}(\cdot,h)\|_{L^2(\mathbb{R})} = 1,\ r(\cdot,h) = \mathcal{O}_{C^\infty}(h^\infty),\ \partial_h r(\cdot,h) = \mathcal{O}_{C^\infty}(h^\infty). \tag{4.51}$$

Note that this function is not real valued, which is not an issue in the above results.

4.5.3. Product of semiclassical Lagrangian distributions. In this paragraph, we are interested in the L^2 inner product of two general Lagrangian distributions associated with the same Lagrangian submanifold. We recall that from Assumption 4.1, and equation (4.4), we have $\mathcal{E}'(E_0) = (\pi C(E_0))^{-1} > 0$.

Proposition 4.6. *Assume that $I_1(\cdot,\mu,h)$, $I_2(\cdot,\mu,h)$ are two semiclassical Lagrangian distributions associated with the same Lagrangian submanifold Λ_μ, and define the quantity*

$$K(h,\mu) = (I_1(\cdot,\mu,h), I_2(\cdot,\mu,h))_{L^2(\mathbb{R})} = \int_{\mathbb{R}} I_1(x,\mu,h)\overline{I_2(x,\mu,h)}\,\mathrm{d}x.$$

Then, under Assumption 4.1, assuming $\mu(h) \to E_0$ and that it satisfies the quantization condition (4.8), we have

$$h\,\partial_h(K(h,\mu(h))) \to k_0'(E_0)\mathcal{E}(E_0)\mathcal{E}'(E_0)^{-1}, \quad \text{as } h \to 0^+, \tag{4.52}$$

where $k_0(E) := \lim_{h\to 0} K(h,\mu(h))$ when $\mu(h) \to E$ under (4.8) is a smooth function of E near $E = E_0$.

Proof. It is proven in [12, p. 224] that for $\mu = \mu(h)$, $K(h,\mu(h))$ has a complete expansion in the form (for any $L > 0$)

$$K(h,\mu(h)) = \sum_{\ell=0}^{L} k_\ell(\mu(h))h^\ell + r_L(h),$$

where $k_\ell \in C^\infty(I_{E_0}(\varepsilon_1))$ and

$$r_L(h) = \mathcal{O}(h^{L+1}) \quad \text{and} \quad \partial_h r_L(h) = \mathcal{O}(h^L). \tag{4.53}$$

This follows from the inspection of the proof which is simply a stationary phase theorem with μ as a parameter, the oscillatory term disappearing when the quantization relation is satisfied. The possibility to differentiate with respect to the parameter h or to the parameter μ when applying the stationary phase theorem is proved, for example, in [34] (see Part I, Section 1 and particularly, lines 5–11 on page 31).

With this property, we get by a term by term differentiation

$$\partial_h(K(h,\mu(h))) = \sum_{\ell=1}^{L} \ell k_\ell(\mu(h))h^{\ell-1} + \sum_{\ell=0}^{L} k_\ell'(\mu(h))\mu'(h)h^\ell + \partial_h r_L(h).$$

Then, notice that the quantization relation (4.8) implies

$$\mathcal{E}'(\mu(h))\mu'(h) = \frac{\mathcal{E}(\mu(h))}{h}.$$

Recalling that $\mathcal{E}'(\mu) > 0$ for $\mu \in I_{E_0}(\varepsilon_0)$, we deduce that

$$\begin{aligned}\partial_h(K(h,\mu(h))) = &\sum_{\ell=1}^{L} \ell k_\ell(\mu(h))h^{\ell-1}\\ &+ \sum_{\ell=0}^{L} k_\ell'(\mu(h))\mathcal{E}(\mu(h))\mathcal{E}'(\mu(h))^{-1}h^{\ell-1} + \partial_h r_L(h).\end{aligned}$$

According to (4.53) and the fact that $\mu(h) \in I_{E_0}(\varepsilon_0)$, ordering the terms in the two sums, this rewrites

$$\partial_h(K(h,\mu(h))) = h^{-1}k_0'(\mu(h))\mathcal{E}(\mu(h))\mathcal{E}'(\mu(h))^{-1} + \mathcal{O}(1), \quad \text{as } h \to 0^+.$$

In our application $\mu(h) \to E_0$, and we deduce (4.52). ∎

From the above results, we now deduce the following corollary.

Corollary 4.7. *Under Assumption 4.1, we assume $\mu(h) \to E_0$ and that it satisfies the quantization condition (4.8). For all $a \in C^\infty(\mathbb{R})$ with polynomial growth,*

$$\mathfrak{M}_a^j(h) \to \langle \mathfrak{m}_{E_0}, a\rangle, \quad \text{as } h \to 0^+,$$
$$h\,\partial_h \mathfrak{M}_a^j(h) \to \frac{\mathrm{d}}{\mathrm{d}\mu}(\langle \mathfrak{m}_\mu, a\rangle)\Big|_{\mu=E_0} \mathcal{I}(E_0)\mathcal{I}'(E_0)^{-1}, \quad \text{as } h \to 0^+,$$

where $\mathfrak{M}_a^j(h)$ is defined in (4.42) and $\mathfrak{m}_\mu$ in (3.16).

Proof. First, according to (4.50)–(4.51) above, under the quantization condition (4.8) $\mu(h) = \mu_j(h)$, there is an approximate eigenfunction $u_j^{\mathrm{app}}(\cdot, h)$ associated with the eigenvalue $E_j^{\mathrm{app}}(h)$, such that $u_j^{\mathrm{app}}(\cdot, h)$ is a semiclassical Lagrangian distribution associated with $\Lambda_{\mu(h)}$.

Second, as a consequence of Proposition 4.3, the gap property (4.10), and the simplicity of the spectrum (following from the Sturm-Liouville theory), there is a unique eigenvalue $E_j(h)$ in an $\mathcal{O}(h)$ neighborhood of $E_j^{\mathrm{app}}(h)$, and we have $E_j(h) - E_j^{\mathrm{app}}(h) = \mathcal{O}(h^\infty)$. Moreover, still according to Proposition 4.3, there is a normalized eigenfunction $u_j(\cdot, h)$ of P_h associated with the eigenvalue $E_j(h)$ and such that $\|u_j(\cdot, h) - u_j^{\mathrm{app}}(\cdot, h)\|_{L^2(\mathbb{R})} = \mathcal{O}(h^\infty)$.

Third, notice that all assumptions of Proposition 4.4 are satisfied by $u_j(\cdot, h)$, $E_j(h)$, $u_j^{\mathrm{app}}(\cdot, h)$, $E_j^{\mathrm{app}}(h)$ (in particular as a consequence of the fact that semiclassical Lagrangian distributions are differentiable with respect to h and yield a semiclassically temperate function). Proposition 4.4 together with equation (4.45) in Corollary 4.5 then yields

$$(\mathfrak{M}_a^j)'(h) - (\mathfrak{M}_a^{j,\mathrm{app}})'(h) = \mathcal{O}(h^\infty),$$

where these two quantities are defined in (4.42)–(4.43) respectively, with $v_h^j = u_j(\cdot, h)$ and $v_h^{j,\mathrm{app}} = u_j^{\mathrm{app}}(\cdot, h)$.

Fourth and last, it thus suffices to compute $(\mathfrak{M}_a^{j,\mathrm{app}})'(h)$. To this end, recalling that $u_h^{j,\mathrm{app}}$ and $au_h^{j,\mathrm{app}}$ are two semiclassical Lagrangian distributions, we use Proposition 4.6 with $I_1 = au_h^{j,\mathrm{app}}$ and $I_2 = u_h^{j,\mathrm{app}}$. Noticing that in this case, $k_0(\mu) = \langle \mathfrak{m}_\mu, a\rangle$, equation (4.52) thus can be rewritten as

$$\begin{aligned} h(\mathfrak{M}_a^{j,\mathrm{app}})'(h) &= h\,\partial_h(K(h, \mu_j(h))) \\ &\to \frac{\mathrm{d}}{\mathrm{d}\mu}(\langle \mathfrak{m}_\mu, a\rangle)(\mu = E_0)\mathcal{I}(E_0)\mathcal{I}'(E_0)^{-1}, \quad \text{as } h \to 0^+, \end{aligned}$$

which is the sought result. □

4.6. The second derivative: end of proof of Theorem 1.5. We now want to compute the asymptotics of $\lambda_j''(\alpha_{j,c})$. Using (3.8), λ_j'' can be rewritten as

$$\begin{aligned}\lambda_j''(\alpha) &= \left(1 - \frac{3}{2}h\,\partial_h\right)\left(\int_{\mathbb{R}} (2-s^2)|v_j(s,h)|^2\,\mathrm{d}s\right),\\ &= \left(1 - \frac{3}{2}h\,\partial_h\right)\mathfrak{M}^j_{(2-s^2)}(h),\end{aligned}$$

where $h = \alpha^{-3/2}$ and $\mathfrak{M}^j_a(h)$ is defined in (4.42). We now apply Corollary 4.7 with $a(x) = 2-x^2$ and $E_0 = E_c$ (which satisfies Assumption 4.1). We notice that, with this choice of a, we have $\langle \mathfrak{m}_\mu, a\rangle = \Phi(\mu)$ where Φ is defined in (3.19), and satisfies $\Phi(E) = C(E)F(E)$ for E near $E_c > 1$. As a consequence of the fact that $\Phi(E_c) = \langle \mathfrak{m}_{E_c}, 2-x^2\rangle = 0$ (by definition of E_c), combined with Corollary 4.7, we obtain at a critical point $\alpha_{j,c}$

$$\lambda_j''(\alpha_{j,c}) \to -\frac{3}{2}\Phi'(E_c)\mathcal{I}(E_c)\mathcal{I}'(E_c)^{-1}, \qquad \text{as } j \to +\infty. \tag{4.54}$$

Next, we notice that, by definition of E_c, we have

$$\Phi'(E_c) = C'(E_c)F(E_c) + C(E_c)F'(E_c) = C(E_c)F'(E_c).$$

According to Item (1) in Proposition 3.12, we have $F'(E_c) < 0$. Using Assumption 4.1, and equation (4.4), we have $\mathcal{I}'(E_0) = (\pi C(E_0))^{-1} > 0$, so that (4.54) can be rewritten as

$$\lambda_j''(\alpha_{j,c}) \to -\frac{3\pi}{2}F'(E_c)C(E_c)^2\mathcal{I}(E_c) > 0, \quad \text{as } j \to +\infty.$$

The expression in (1.3)–(1.4) then follows from the definitions of C and $\mathcal{I}$ in (3.17) and (4.3), respectively.

Note finally that this property also implies the uniqueness of the critical point of λ_j for j large enough (otherwise λ_j would have a local maximum), and concludes the proof of Theorem 1.5.

5. Analysis of the First Two Eigenvalues

In this section, we give a complete mathematical proof that λ_1 has a unique critical point (improving [19], which was always considering the minimum), and a numerically assisted proof that the same property holds for λ_2. Unfortunately, we do not see how to adapt the strategy to more excited eigenvalues.

5.1. Strategy of the proof. In the case $j = 1$, the proof is a combination of Lemma 5.1 and Proposition 2.6. In the case $j = 2$, we cannot use Proposition 2.6 and we need some numerical help. Of course, it is less efficient than in the analysis of the De Gennes model[4] where we have equality in Proposition 2.6, but this will be enough. In the case $j = 1$, we will start by using Proposition 2.6 to deduce an upper bound for α_c, and then use Lemma 5.1 to conclude. The proof will be a consequence of a good upper bound for $\lambda_1(\alpha)$ and a good lower bound for $\lambda_3(\alpha)$.

5.2. A preliminary estimate. The following lemma has been proved in [22].

Lemma 5.1. *If α_c is a critical point of λ_1 (respectively, λ_2) and if*

$$3\lambda_3(\alpha_c) > 7\lambda_1(\alpha_c) \quad \text{(respectively, } 3\lambda_4(\alpha_c) > 7\lambda_2(\alpha_c)\text{)},$$

then

$$\lambda_1''(\alpha_c) > 0 \qquad \text{(respectively } \lambda_2''(\alpha_c) > 0\text{)}.$$

Let us recall the proof for the sake of completeness, which relies on the following general Lemma 5.2. Note that Lemma 5.1 also holds for λ_2 instead of λ_1, as a direct consequence of Lemma 5.2. We will verify numerically that this condition is indeed satisfied for λ_2.

Lemma 5.2. *For all $j \in \mathbb{N}$ and α_c a critical point of λ_j, we have*

$$\lambda_j''(\alpha_c) \geq \frac{2}{3}\frac{3\lambda_{j+2}(\alpha_c) - 7\lambda_j(\alpha_c)}{\lambda_{j+2}(\alpha_c) - \lambda_j(\alpha_c)}, \quad \text{if } j \in \{1, 2\},$$

$$\lambda_j''(\alpha_c) \geq 2 - \frac{8}{3}\lambda_j(\alpha_c)\max\left\{\frac{1}{\lambda_j(\alpha_c) - \lambda_{j-2}(\alpha_c)}, \frac{1}{\lambda_{j+2}(\alpha_c) - \lambda_j(\alpha_c)}\right\}, \quad \text{if } j \geq 3.$$

Remark 5.3. Unfortunately, the inequalities obtained for λ_j with $j \geq 3$ are less readable/useful. Also, they become inaccurate for large eigenvalues since the gap is asymptotically negligible with respect to the eigenvalue.

Proof. We start from the expression of $\lambda_j''(\alpha)$ in (2.17), in which the Cauchy-Schwarz inequality yields

$$\lambda_j''(\alpha) \geq 2 - 4\left\|\left(\frac{1}{2}t^2 - \alpha\right)u_j(\cdot, \alpha)\right\|_{L^2(\mathbb{R})}\|\partial_\alpha u_j(\cdot, \alpha)\|_{L^2(\mathbb{R})}, \tag{5.1}$$

and the remaining part of the proof consists in estimating the righthand side. To estimate $\|\partial_\alpha u_j(\cdot, \alpha)\|_{L^2(\mathbb{R})}$, we recall the equation satisfied by $\partial_\alpha u_j(\cdot, \alpha)$ in

[4]The De Gennes model corresponds to the family of Harmonic oscillators $D_t^2 + (t + \alpha)^2$ on the half line with Neumann condition with groundstate energy $\mu(\alpha)$. It has been shown in [10] that λ_1 has a unique minimum and that this minimum is non degenerate.

(2.16). We set

$$H_j(\alpha) := \begin{cases} \{v \in L^2(\mathbb{R}) \mid v \text{ even}, \ v \perp u_j(\cdot,\alpha)\} & \text{if } j \text{ is odd}, \\ \{v \in L^2(\mathbb{R}) \mid v \text{ odd}, \ v \perp u_j(\cdot,\alpha)\} & \text{if } j \text{ is even}, \end{cases}$$

and notice that $\partial_\alpha u_j(\cdot,\alpha) \in H_j(\alpha)$ for all $\alpha \in \mathbb{R}$, $j \in \mathbb{N}^*$. Moreover, if $\Pi_{H_j(\alpha)}$ denotes the orthogonal projector onto $H_j(\alpha)$, we have $\mathfrak{h}_{\mathcal{M}}(\alpha)\Pi_{H_j(\alpha)} = \Pi_{H_j(\alpha)}\mathfrak{h}_{\mathcal{M}}(\alpha)$, and the operator

$$P_j(\alpha) := (\mathfrak{h}_{\mathcal{M}}(\alpha) - \lambda_j(\alpha))\Pi_{H_j(\alpha)}$$

is an isomorphism of $H_j(\alpha)$, as $\ker(\mathfrak{h}_{\mathcal{M}}(\alpha) - \lambda_j(\alpha)) = \operatorname{span} u_j(\cdot,\alpha)$ (see Proposition 1.1). Now if α_c is a critical point of λ_j, then (2.16) can be rewritten as

$$(\mathfrak{h}_{\mathcal{M}}(\alpha_c) - \lambda_j(\alpha_c))\,\partial_\alpha u_j(\cdot,\alpha_c) = 2\left(\frac{t^2}{2} - \alpha_c\right) u_j(\cdot,\alpha_c). \tag{5.2}$$

Note also that, according to the first variation formula (2.5),

$$\left(\frac{t^2}{2} - \alpha_c\right) u_j(\cdot,\alpha_c) \in H_j(\alpha_c)$$

as well. Hence, (5.2) is an equation in $H_j(\alpha_c)$, namely,

$$P_j(\alpha)\,\partial_\alpha u_j(\cdot,\alpha_c) = 2\left(\frac{t^2}{2} - \alpha_c\right) u_j(\cdot,\alpha_c),$$

or equivalently

$$\partial_\alpha u_j(\cdot,\alpha_c) = 2P_j(\alpha_c)^{-1}\left(\left(\frac{t^2}{2} - \alpha_c\right) u_j(\cdot,\alpha_c)\right).$$

As a consequence,

$$\|\partial_\alpha u_j(\cdot,\alpha_c)\|_{L^2(\mathbb{R})} \leq 2\|P_j(\alpha_c)^{-1}\|_{\mathcal{L}(H_j(\alpha_c))} \left\|\left(\frac{t^2}{2} - \alpha_c\right) u_j(\cdot,\alpha_c)\right\|_{L^2(\mathbb{R})}.$$

But $P_j(\alpha)$ is a selfadjoint operator on $H_j(\alpha_c)$ with spectrum

$$\{\dots,\lambda_{j-2}(\alpha) - \lambda_j(\alpha), \lambda_{j+2}(\alpha) - \lambda_j(\alpha), \lambda_{j+4}(\alpha) - \lambda_j(\alpha),\dots\} \text{ if } j \geq 3,$$

and

$$\{\lambda_{j+2}(\alpha) - \lambda_j(\alpha), \lambda_{j+4}(\alpha) - \lambda_j(\alpha),\dots\} \quad \text{if } j \in \{1,2\}.$$

As a consequence,

$$\|P_j(\alpha_c)^{-1}\|_{\mathcal{L}(H_j(\alpha_c))} = \begin{cases} \dfrac{1}{\lambda_{j+2}(\alpha)-\lambda_j(\alpha)}, & \text{if } j \in \{1,2\}, \\ \max\left\{\dfrac{1}{\lambda_j(\alpha)-\lambda_{j-2}(\alpha)}, \dfrac{1}{\lambda_{j+2}(\alpha)-\lambda_j(\alpha)}\right\}, & \text{if } j \geq 3. \end{cases} \tag{5.3}$$

Coming back to (5.1), we have obtained that for any $j \in \mathbb{N}$ and any critical point α_c of λ_j, we have

$$\lambda_j''(\alpha_c) \geq 2 - 8\|P_j(\alpha_c)^{-1}\|_{\mathcal{L}(H_j(\alpha_c))}\left\|\left(\frac{1}{2}t^2-\alpha_c\right)u_j(\cdot,\alpha_c)\right\|^2_{L^2(\mathbb{R})},$$

which, after having used (2.11), can be rewritten as

$$\lambda_j''(\alpha_c) \geq 2 - \frac{8}{3}\|P_j(\alpha_c)^{-1}\|_{\mathcal{L}(H_j(\alpha_c))}\lambda_j(\alpha_c).$$

Recalling the expression of $\|P_j(\alpha_c)^{-1}\|_{\mathcal{L}(H_j(\alpha_c))}$ in (5.3) then concludes the proof of the lemma. ◻

5.3. Upper bounds for $\lambda_1(\alpha)$. In order to apply Lemma 5.1, we need to prove, at a critical point α_c of λ_1, that $\lambda_3(\alpha_c)/\lambda_1(\alpha_c) > \frac{7}{3}$. To this end, we need to have a rough localization of α_c, and to provide with an upper bound for λ_1 lower bound for λ_3 near the place where α_c has been localized. We start with the following lemma.

Lemma 5.4. *Assume α_c is a critical point for λ_1. Then, we have*

$$\alpha_c \in \left(0, \left(\frac{24}{25}\right)^{1/3}\right) \subset (0,1). \tag{5.4}$$

Moreover, we have

$$\lambda_1(\alpha) \leq \frac{4}{5}\left(\frac{9}{5}\right)^{1/3} \approx 0.97315, \quad \text{for all } \alpha \in \left(0, \left(\frac{24}{25}\right)^{1/3}\right). \tag{5.5}$$

Proof. We denote, for $v \in D(\mathfrak{h}_{\mathcal{M}}(0))$, the energy

$$\mathcal{E}(v,\alpha) := (\mathfrak{h}_{\mathcal{M}}(\alpha)v,v)_{L^2(\mathbb{R})} = \|v'\|^2_{L^2(\mathbb{R})} + \left\|\left(\frac{1}{2}t^2-\alpha\right)v\right\|^2_{L^2(\mathbb{R})},$$

and recall that

$$\lambda_1(\alpha) = \mathcal{E}(u_1(\cdot,\alpha),\alpha) = \min\{\mathcal{E}(v,\alpha), v \in D(\mathfrak{h}_{\mathcal{M}}(0)), \|v\|_{L^2(\mathbb{R})} = 1\}. \tag{5.6}$$

We compute[5] the energy $\mathcal{E}(u_\rho, \alpha)$ of the L^2-normalized Gaussian

$$u_\rho = c_\rho \exp\left(-\frac{\rho}{2}t^2\right), \quad c_\rho = \sqrt{\frac{2\pi}{\rho}},\ \rho > 0.$$

Using that

$$||tu_\rho||^2_{L^2(\mathbb{R})} = \frac{1}{2\rho}, \quad ||t^2 u_\rho||^2_{L^2(\mathbb{R})} = \frac{3}{(2\rho)^2},$$

we obtain

$$\mathcal{E}(u_\rho, \alpha) = \frac{\rho}{2} + \frac{3}{16}\rho^{-2} - \frac{\alpha}{2\rho} + \alpha^2.$$

From (5.6), we deduce that, for any $\alpha \in \mathbb{R}$ and $\rho > 0$,

$$\lambda_1(\alpha) \leq \mathcal{E}(u_\rho, \alpha) = \frac{\rho}{2} + \frac{3}{16}\rho^{-2} - \frac{\alpha}{2\rho} + \alpha^2. \tag{5.7}$$

We now write this inequality for particular values of ρ.

First upper bound. We first choose $\rho = \rho_0 := (3/4)^{1/3}$ in (5.7) (corresponding to the minimizing ρ for $\alpha = 0$). The corresponding energy is

$$\mathcal{E}(u_{\rho_0}, \alpha) = \frac{1}{2}\left(\frac{3}{4}\right)^{1/3} + \frac{3}{16}\left(\frac{3}{4}\right)^{-2/3} + \alpha^2 - \frac{1}{2}\alpha\left(\frac{3}{4}\right)^{-1/3}.$$

Hence, we obtain, for this specific ρ_0, for all $\alpha \in \mathbb{R}$,

$$\lambda_1(\alpha) \leq \mathcal{E}(u_{\rho_0}, \alpha) = \alpha^2 - 6^{-1/3}\alpha + \left(\frac{3}{4}\right)^{4/3}.$$

As in [19], we shall use this upper bound for $\alpha > 0$ small enough, namely, for $\alpha \in (0, 0.7)$. The minimum of the function $\alpha \mapsto \mathcal{E}(u_{\rho_0}, \alpha)$ is attained at $\alpha = 6^{-1/3}/2 \approx 0.28 \in (0, 0.7)$. Thus, on the interval $\alpha \in (0, \frac{7}{10})$, we have

$$\begin{aligned} \lambda_1(\alpha) &\leq \max\left(\mathcal{E}(u_{\rho_0}, 0), \mathcal{E}\left(u_{\rho_0}, \frac{7}{10}\right)\right) \\ &= \mathcal{E}\left(u_{\rho_0}, \frac{7}{10}\right) \approx 0.79 \quad \text{for all } \alpha \in (0, 0.7). \end{aligned} \tag{5.8}$$

Second upper bound. We next choose $\rho = 1/(2\alpha) > 0$ in (5.7) (corresponding to the cancellation of the last two terms in that inequality). This leads to

$$\lambda_1(\alpha) \leq \mathcal{E}(u_{1/(2\alpha)}, \alpha) = \frac{1}{4\alpha} + \frac{3}{4}\alpha^2, \tag{5.9}$$

[5] This idea was rather efficient for giving an upper bound of the analog problem for the De Gennes model.

for all $\alpha > 0$. Implementing this in (2.15) we deduce in particular that if α_c is a critical point of λ_1, then

$$\alpha_c^2 < \frac{1}{4\alpha_c} + \frac{3}{4}\alpha_c^2.$$

According to Corollary 2.2, we have $\alpha_c > 0$, so that this inequality is equivalent to $\alpha_c < 1$.

We then note that

$$\alpha \mapsto \mathcal{E}(u_{1/(2\alpha)}, \alpha) = \frac{1}{4\alpha} + \frac{3}{4}\alpha^2$$

reaches its minimum on $\mathbb{R}_+^*$ at $\alpha = 6^{-1/3} \approx 0.55$. In particular, this function is increasing on $(0.7, 1)$, and thus

$$\mathcal{E}(u_{1/(2\alpha)}, \alpha) \leq \mathcal{E}(u_{1/2}, 1) \leq 1, \quad \text{for all } \alpha \in (0.7, 1). \tag{5.10}$$

Using (5.8), (5.9) and (5.10), we also deduce that $\lambda_1(\alpha) \leq 1$ for all $\alpha \in (0, 1)$.

Third upper bound. We refine slightly the second upper bound by taking $\rho = 3/(5\alpha)$ in (5.7). We obtain

$$\begin{aligned}\lambda_1(\alpha) \leq \mathcal{E}(u_{3/(5\alpha)}, \alpha) &= \frac{3}{10\alpha} + \frac{3 \times 25}{16 \times 9}\alpha^2 - \frac{5}{6}\alpha^2 + \alpha^2 \\ &= \frac{3}{10\alpha} - \frac{5}{16}\alpha^2 + \alpha^2 = \frac{3}{10\alpha} + \frac{11}{16}\alpha^2.\end{aligned}$$

Using again (2.15), we deduce that if α_c is a critical point of λ_1, then

$$\alpha_c^2 < \frac{3}{10\alpha_c} + \frac{11}{16}\alpha_c^2$$

which, since $\alpha_c > 0$ (according to Corollary 2.2), is equivalent to the small improvement

$$\alpha_c < \left(\frac{24}{25}\right)^{1/3} \approx 0.98645,$$

which proves (5.4).

Moreover, $\alpha \mapsto \mathcal{E}(u_{3/(5\alpha)}, \alpha)$ reaches its minimum on $\mathbb{R}_+^*$ at $\alpha = (\frac{12}{55})^{1/3} \approx 0.60$. In particular, this function is increasing on $(0.7, 1)$, and thus

$$\begin{aligned}\lambda_1(\alpha) &\leq \mathcal{E}(u_{3/(5\alpha)}, \alpha) \\ &\leq \mathcal{E}(u_{3/(5(24/25)^{1/3})}, (24/25)^{1/3}) \\ &= \frac{4}{5}\left(\frac{9}{5}\right)^{1/3} \approx 0.97315,\end{aligned}$$

for all $\alpha \in (0.7, (24/25)^{1/3})$. Together with (5.8) we finally deduce (5.5), which concludes the proof of the lemma. ◻

5.4. Lower bounds for $\lambda_3(\alpha)$.

Lemma 5.5. *For all* $\alpha \in (0, (24/25)^{1/3}]$,

$$\lambda_3(\alpha) \geq \sqrt{15} - \frac{6}{5}\left(\frac{24}{25}\right)^{1/3} - \frac{9}{25} \approx 2.3292. \tag{5.11}$$

Proof. We push the idea of comparing our operator with an harmonic oscillator. We first notice that, for any $\gamma \geq 0$, $\alpha \in \mathbb{R}$, $t \in \mathbb{R}$, we have

$$\gamma t^2 + \alpha^2 - (\alpha + \gamma)^2 \leq \left(\frac{1}{2}t^2 - \alpha\right)^2. \tag{5.12}$$

Indeed, the polynomial $X^2/4 - (\alpha + \gamma)X + (\alpha + \gamma)^2$ is nonnegative on $\mathbb{R}$ for all $\alpha, \gamma \in \mathbb{R}$, and thus $\gamma X - (\alpha+\gamma)^2 \leq X^2/4 - \alpha X$, which implies (5.12) for $X = t^2$. From this we can compare the operator $\mathfrak{h}_{\mathcal{M}}(\alpha)$ to the Harmonic Oscillator

$$\mathfrak{H}_{\alpha,\gamma} := -\frac{\mathrm{d}}{\mathrm{d}t} + V_\gamma(t), \quad V_\gamma(t) = \gamma t^2 + \alpha^2 - (\alpha + \gamma)^2.$$

According to the minimax principle (see, e.g., [3, Lemma B.6] or Chapter 11 and discussion on top of p. 148 in [20]), denoting

$$\mathrm{Sp}(\mathfrak{H}_{\alpha,\gamma}) := \{\tilde{\lambda}_j(\alpha, \gamma) \mid j \in \mathbb{N}^*\},$$

(5.12) implies

$$\lambda_j(\alpha) \geq \tilde{\lambda}_j(\alpha, \gamma), \quad \text{for all } j \in \mathbb{N}^*.$$

Recalling that the eigenvalues of the harmonic oscillator $-\mathrm{d}^2/\mathrm{d}t^2 + \gamma t^2$ are given by $\{(2k + 1)\sqrt{\gamma} \mid k \in \mathbb{N}\}$, we deduce that, for any $\gamma > 0$,

$$\lambda_3(\alpha) \geq \tilde{\lambda}_3(\alpha, \gamma) = 5\gamma^{1/2} - 2\gamma\alpha - \gamma^2.$$

We finally choose a particular value of γ. Choosing $\gamma = \frac{3}{5}$, we deduce that

$$\lambda_3(\alpha) \geq \tilde{\lambda}_3\left(\alpha, \frac{3}{5}\right) = \sqrt{15} - \frac{6}{5}\alpha - \frac{9}{25},$$

and in particular that for all $\alpha \in (0, 1]$,

$$\begin{aligned}\lambda_3(\alpha) &\geq \tilde{\lambda}_3\left(\alpha, \frac{3}{5}\right) \geq \tilde{\lambda}_3\left(1, \frac{3}{5}\right)\\ &= \sqrt{15} - \frac{6}{5} - \frac{9}{25} \approx 2.313.\end{aligned}$$

More precisely, for all $\alpha \in (0, (24/25)^{1/3}]$,

$$\lambda_3(\alpha) \geq \tilde{\lambda}_3\left(\alpha, \frac{3}{5}\right) \geq \tilde{\lambda}_3\left(\left(\frac{24}{25}\right)^{1/3}, \frac{3}{5}\right)$$
$$= \sqrt{15} - \frac{6}{5}\left(\frac{24}{25}\right)^{1/3} - \frac{9}{25} \approx 2.3292,$$

which concludes the proof of the lemma. □

5.5. End of the proof of Theorem 1.2. From Lemmas 5.1, 5.4, and 5.5, we may now conclude the proof of Theorem 1.2.

End of the proof of Theorem 1.2. Let α_c be a critical point of $\alpha \mapsto \lambda_1(\alpha)$. As a consequence of Lemmas 5.4 and 5.5, namely, equations (5.4), (5.5), and (5.11), we have necessarily

$$\lambda_1(\alpha_c) \leq \frac{4}{5}\left(\frac{9}{5}\right)^{1/3} \approx 0.97315,$$
$$\lambda_3(\alpha_c) \geq \sqrt{15} - \frac{6}{5}\left(\frac{24}{25}\right)^{1/3} - \frac{9}{25} \approx 2.3292.$$

As a consequence, we have

$$\frac{\lambda_3(\alpha_c)}{\lambda_1(\alpha_c)} \geq \frac{5}{4}\left(\frac{9}{5}\right)^{-1/3}\left(\sqrt{15} - \frac{6}{5}\left(\frac{24}{25}\right)^{1/3} - \frac{9}{25}\right) \approx 2.3935 > \frac{7}{3} \approx 2.3333.$$

Using Lemma 5.1, we deduce that $\lambda_1''(\alpha_c) > 0$ for any critical point α_c. According to (2.1), $\lambda_1(\alpha) \to +\infty$ as $\alpha \to +\infty$. These two properties imply uniqueness (and nondegeneracy) of the critical point α_c, which is necessarily a minimum, and concludes the proof of the lemma. □

5.6. Numerics. In the present subsection, we collect numerical data and figures communicated to us by Mikael Persson-Sundqvist. The latter are obtained (by using the function NDEigenvalues of Matematica) by computing numerically the first eigenvalues for the Dirichlet problem on the bounded interval $[-20, 20]$.

On Figure 5.1 is represented the graph of the quotient $\alpha \mapsto \lambda_3(\alpha)/\lambda_1(\alpha)$ for $\alpha \in (0,1)$. This illustrates the proof of Theorem 1.2 relying on Lemmas 5.1, 5.4, and 5.5, in which we prove that

$$\frac{\lambda_3(\alpha)}{\lambda_1(\alpha)} \geq 2.39 \quad \text{for } \alpha \in \left(0, \left(\frac{24}{25}\right)^{1/3}\right).$$

Here, Figure 5.1 suggests that $\lambda_3(\alpha)/\lambda_1(\alpha) \geq 4$ for all $\alpha \in (0,1)$. Matematica actually gives $\lambda_3(\alpha)/\lambda_1(\alpha) \geq 4.07529$ (with the accuracy of the calculator) with a great stability when exchanging from Dirichlet to Neumann boundary conditions at the endpoints of the bounded interval $[-20, 20]$[6]. Note here that in the limit $\alpha \to +\infty$, we have $\lambda_3(\alpha)/\lambda_1(\alpha) \to 3$ (as a consequence of (2.2)). In particular, the condition $\lambda_3(\alpha)/\lambda_1(\alpha) > \frac{7}{3}$ holds for α large. This gives another way to see that there are no critical points for large α, although this does not furnish any explicitly computable bound.

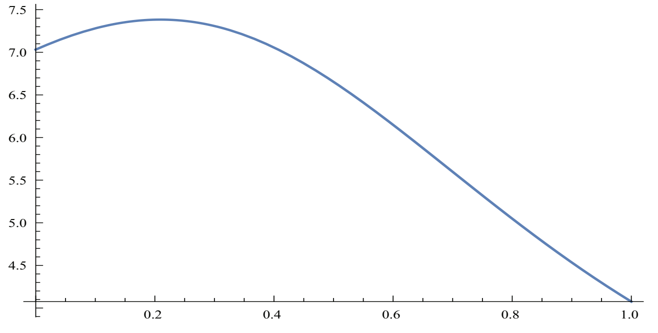

FIGURE 5.1. Graph of the quotient λ_3/λ_1 over $(0, 1)$

On Figure 1.1, the graph of the first six eigenvalues $\alpha \mapsto \lambda_j(\alpha)$ is represented on the interval $[-2.5, 6.5]$, which contains the change of variation of these six functions. This figure shows a unique critical point, which looks nondegenerate, and therefore contributes to motivate Conjecture 1.6. Note that the asymptotic behavior as $\alpha \to \pm\infty$ is also seen on the figure: we have $\lambda_{2k+1}(\alpha) \sim \lambda_{2k+2}(\alpha) \sim \sqrt{2}(2k+1)\alpha^{1/2}$ as $\alpha \to +\infty$ (see (2.2)–(2.3)), and a single-well analysis [18, 30] would yield a precise asymptotic expansion as $\alpha \to -\infty$ as well. Note that, as $\alpha \to +\infty$, Figure 1.1 exhibits pairs of asymptotically exponentially close curves, which is predicted by the double well analysis (2.3). On Table 5.1, we can see the numerical value of the critical point $\alpha_{c,j}$ of λ_j. In particular, we see that $\alpha_{c,1} \approx 0.35$, which is consistent with Lemma 5.4 which proves that $\alpha_{c,1} < (24/25)^{1/3}$. Also on Table 5.1, we see the value of the quotient $\lambda_j(\alpha_{c,j})/\alpha_{c,j}^2$ which converges very rapidly towards $E_c \approx 2.35$, as predicted by Theorem 1.4.

5.7. "Proof" of Statement 1.3. We here provide a completely numerical proof. Hence, our statement cannot be considered as a mathematical theorem. We could have proved rigorously some reduction to an explicitly finite interval of values for α, hence excluding the presence of more than one minimum outside this interval. But at the moment, the remaining interval is still very large ($[0, 45]$), and this does not seem sufficiently decisive. Note (see (2.2) and (2.3)) that

$$\frac{\lambda_4(\alpha)}{\lambda_2(\alpha)} \to 3 \quad \text{as } \alpha \to +\infty.$$

[6]Because of Agmon estimates, it is rather clear that the error is very small.

Value of j	$j=1$	$j=2$	$j=3$	$j=4$	$j=5$	$j=6$
Value of the critical point $\alpha_{c,j}$ of λ_j	0.35	1.13	1.14	1.55	1.78	2.06
Value of the quotient $\lambda_j(\alpha_{c,j})/\alpha_{c,j}^2$	4.78	1.27	2.69	2.25	2.41	2.34

TABLE 5.1. Values of the critical point $\alpha_{c,j}$ and the quotient $\lambda_j(\alpha_{c,j})/\alpha_{c,j}^2$ for $j=1,\dots,6$.

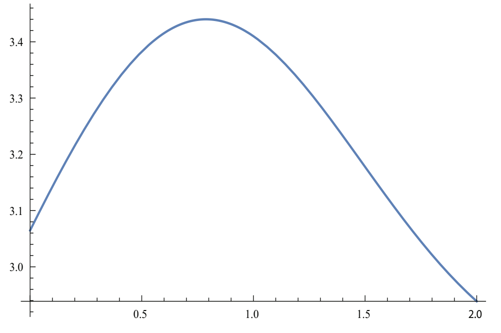

FIGURE 5.2. Graph of λ_4/λ_2 for $\alpha \in (0,2)$

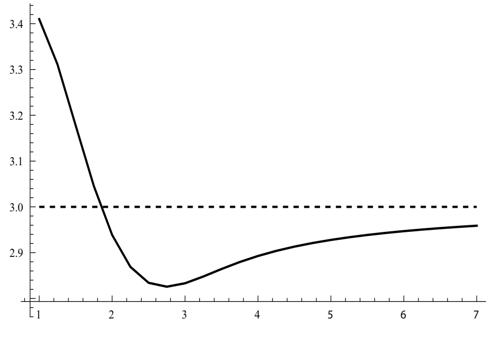

FIGURE 5.3. Graph of λ_4/λ_2 for $\alpha > 1$

The values of the quotient $\alpha \mapsto \lambda_4(\alpha)/\lambda_2(\alpha)$ are plotted on Figures 5.2–5.3. Figures 5.2–5.3 numerically show that $\min\{\lambda_4(\alpha)/\lambda_2(\alpha) \mid \alpha \in \mathbb{R}\} \approx 2.82$ and thus $\lambda_4/\lambda_2 \geq \frac{7}{3}$ on $\mathbb{R}$. According to Lemma 5.1, this (numerically) proves that $\lambda_2''(\alpha_c) > 0$, and consequently shows uniqueness of a critical point for λ_2, hence proving Statement 1.3.

Acknowledgments. Many thanks to F. Nicoleau and M. Persson-Sundqvist for their precious help for numerical computations with Matematica. The authors also wish to thank D. Robert for useful discussions on the possible use of coherent states in semiclassical analysis, and Y. Colin de Verdière for mentioning to us the results of [7]. The authors are also grateful to the referee for remarks that helped to improve the presentation of the paper. The second author is partially supported by the Agence Nationale de la Recherche under grants SALVE (ANR-19-CE40-0004) and ADYCT (ANR-20-CE40-0017).

References

[1] Y. Almog and B. Helffer, *On the spectrum of some Bloch-Torrey vector operators*, Pure Appl. Anal. **4** (2022), no. 1, 1–48. https://dx.doi.org/10.2140/paa.2022.4.1. MR4419367

[2] Y. Almog, B. Helffer, and X.-B. Pan, *Superconductivity near the normal state in a half-plane under the action of a perpendicular electric current and an induced magnetic field*, Trans. Amer. Math. Soc. **365** (2013), no. 3, 1183–1217. https://dx.doi.org/10.1090/S0002-9947-2012-05572-3. MR3003262

[3] H. Bahouri, D. Barilari, I. Gallagher, and M. Léautaud, *Spectral summability for the quartic oscillator with applications to the Engel group*, J. Spectr. Theory **13** (2023), no. 2, 623–706. https://dx.doi.org/10.4171/jst/464. MR4651735

[4] R. Brummelhuis, T. Paul, and A. Uribe, *Spectral estimates around a critical level*, Duke Math. J. **78** (1995), no. 3, 477–530. https://dx.doi.org/10.1215/S0012-7094-95-07823-5. MR1334204

[5] B. Candelpergher and J.-C. Nosmas, *Propriétés spectrales d'opérateurs différentiels asymptotiques autoadjoints*, Comm. Partial Differential Equations **9** (1984), no. 2, 137–167 (French). https://dx.doi.org/10.1080/03605308408820328. MR0736413

[6] M. Chatzakou, *Quantizations on the Engel and the Cartan groups*, J. Lie Theory **31** (2021), no. 2, 517–542. MR4225041

[7] Y. Colin de Verdière, *Bohr-Sommerfeld rules to all orders*, Ann. Henri Poincaré **6** (2005), no. 5, 925–936. https://dx.doi.org/10.1007/s00023-005-0230-z. MR2219863

[8] Y. Colin de Verdière and C. Letrouit, *Propagation of well-prepared states along Martinet singular geodesics*, J. Spectr. Theory **12** (2022), no. 3, 1235–1253. https://dx.doi.org/10.4171/jst/421. MR4578349

[9] Y. Colin de Verdière and B. Parisse, *Équilibre instable en régime semi-classique I: Concentration microlocale*, Comm. Partial Differential Equations **19** (1994), no. 9–10, 1535–1563 (French, with French summary). https://dx.doi.org/10.1080/03605309408821063. MR1294470

[10] M. Dauge and B. Helffer, *Eigenvalues variation I: Neumann problem for Sturm-Liouville operators*, J. Differential Equations **104** (1993), no. 2, 243–262. https://dx.doi.org/10.1006/jdeq.1993.1071. MR1231468

[11] M. Dimassi and J. Sjöstrand, *Spectral Asymptotics in the Semi-classical Limit*, London Mathematical Society Lecture Note Series, vol. 268, Cambridge University Press, Cambridge, 1999. https://dx.doi.org/10.1017/CBO9780511662195. MR1735654

[12] J. J. Duistermaat, *Oscillatory integrals, Lagrange immersions and unfolding of singularities*, Comm. Pure Appl. Math. **27** (1974), 207–281. https://dx.doi.org/10.1002/cpa.3160270205. MR0405513

[13] S. Dyatlov, *Semiclassical Lagrangian Distributions: Lecture Notes.* https://math.mit.edu/~dyatlov/files/2012/hlagrangians.pdf.

[14] S. Fournais and B. Helffer, *Spectral Methods in Surface Superconductivity*, Progress in Nonlinear Differential Equations and their Applications, vol. 77, Birkhäuser Boston, Inc., Boston, MA, 2010. MR2662319

[15] A. GRIGIS and J. SJÖSTRAND, *Microlocal Analysis for Differential Operators: An Introduction*, London Mathematical Society Lecture Note Series, vol. 196, Cambridge University Press, Cambridge, 1994. https://dx.doi.org/10.1017/CBO9780511721441. MR1269107

[16] B. HELFFER, *Hypoellipticité analytique sur des groupes nilpotents de rang 2 (d'après G. Métivier)*, Séminaire Goulaouic-Schwartz, 1979–1980 (French), École Polytech., Palaiseau, 1980, pp. Exp. No. 1, 14 pp. (French). MR0600685

[17] ______, *Conditions nécessaires d'hypoanalyticité pour des opérateurs invariants à gauche homogènes sur un groupe nilpotent gradué*, J. Differential Equations **44** (1982), no. 3, 460–481 (French). https://dx.doi.org/10.1016/0022-0396(82)90008-0. MR0661164

[18] ______, *Introduction to semi-classical methods for the Schrödinger operator with magnetic field*, Aspects théoriques et appliqués de quelques EDP issues de la géométrie ou de la physique, Sémin. Congr., vol. 17, Soc. Math. France, Paris, 2009, pp. 49–117 (English, with English and French summaries). MR2792093

[19] ______, *The Montgomery model revisited*, Colloq. Math. **118** (2010), no. 2, 391–400. https://dx.doi.org/10.4064/cm118-2-3. MR2602157

[20] ______, *Spectral Theory and its Applications*, Cambridge Studies in Advanced Mathematics, vol. 139, Cambridge University Press, Cambridge, 2013. MR3027462

[21] B. HELFFER and Y. A. KORDYUKOV, *Spectral gaps for periodic Schrödinger operators with hypersurface magnetic wells*, Mathematical Results in Quantum Mechanics, Moeciu, Romania, 10–15 September, 2007, World Sci. Publ., Hackensack, NJ, 2008, pp. 137–154. https://dx.doi.org/10.1142/9789812832382_0010. MR2466683

[22] ______, *Spectral gaps for periodic Schrödinger operators with hypersurface magnetic wells: Analysis near the bottom*, J. Funct. Anal. **257** (2009), no. 10, 3043–3081. https://dx.doi.org/10.1016/j.jfa.2009.04.007. MR2568685

[23] ______, *Semiclassical analysis of Schrödinger operators with magnetic wells*, Spectral and Scattering Theory for Quantum Magnetic Systems, Contemp. Math., vol. 500, Amer. Math. Soc., Providence, RI, 2009, pp. 105–121. https://dx.doi.org/10.1090/conm/500/09824. MR2655146

[24] B. HELFFER, A. MARTINEZ, and D. ROBERT, *Ergodicité et limite semi-classique*, Comm. Math. Phys. **109** (1987), no. 2, 313–326 (French, with English summary). http://dx.doi.org/10.1007/BF01215225. MR0880418

[25] B. HELFFER and A. MOHAMED, *Semiclassical analysis for the ground state energy of a Schrödinger operator with magnetic wells*, J. Funct. Anal. **138** (1996), no. 1, 40–81. https://dx.doi.org/10.1006/jfan.1996.0056. MR1391630

[26] B. HELFFER and A. MORAME, *Magnetic bottles for the Neumann problem: Curvature effects in the case of dimension 3 (general case)*, Ann. Sci. École Norm. Sup. (4) **37** (2004), no. 1, 105–170 (English, with English and French summaries). https://dx.doi.org/10.1016/j.ansens.2003.04.003. MR2050207

[27] B. HELFFER and F. NOURRIGAT, *Hypoellipticité pour des groupes nilpotents de rang de nilpotence* 3, Comm. Partial Differential Equations **3** (1978), no. 8, 643–743 (French). https://dx.doi.org/10.1080/03605307808820075. MR0499352

[28] B. HELFFER and M. PERSSON-SUNDQVIST, *Spectral properties of higher order anharmonic oscillators*, J. Math. Sci. (N.Y.) **165** (2010), no. 1, 110–126, Problems in mathematical analysis. No. 44. https://dx.doi.org/10.1007/s10958-010-9784-5. MR2838999

[29] B. HELFFER and D. ROBERT, *Puits de potentiel généralisés et asymptotique semi-classique*, Ann. Inst. H. Poincaré Phys. Théor. **41** (1984), no. 3, 291–331 (French, with English summary). MR0776281

[30] B. HELFFER and J. SJÖSTRAND, *Multiple wells in the semiclassical limit. I*, Comm. Partial Differential Equations **9** (1984), no. 4, 337–408. https://dx.doi.org/10.1080/03605308408820335. MR0740094

[31] L. HILLAIRET and J. L. MARZUOLA, *Eigenvalue spacing for 1D singular Schrödinger operators*, Asymptot. Anal. **133** (2023), no. 1–2, 267–289. http://dx.doi.org/10.3233/ASY-221814. MR4595196

[32] L. HÖRMANDER, *The Analysis of Linear Partial Differential Operators. IV: Fourier Integral Operators*, Corrected reprint of the 1985 original, Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 275, Springer-Verlag, Berlin, 1994. MR1481433

[33] C. LAURENT and M. LÉAUTAUD, *Uniform observation of semiclassical Schrödinger eigenfunctions on an interval*, Tunis. J. Math. **5** (2023), no. 1, 125–170. https://dx.doi.org/10.2140/tunis.2023.5.125. MR4578538

[34] V. P. MASLOV and M. V. FEDORIUK, *Semiclassical Approximation in Quantum Mechanics*, Contemporary Mathematics, vol. 5, D. Reidel Publishing Co., Dordrecht-Boston, MA, 1981, Translated from the Russian by J. Niederle and J. Tolar; Mathematical Physics and Applied Mathematics, 7. MR0634377

[35] B. MITYAGIN, P. SIEGL, and J. VIOLA, *Concentration of eigenfunctions of Schrödinger operators*, J. Fourier Anal. Appl. **28** (2022), no. 4, Paper No. 67, 34 pp. https://dx.doi.org/10.1007/s00041-022-09961-3. MR4462791

[36] R. MONTGOMERY, *Hearing the zero locus of a magnetic field*, Comm. Math. Phys. **168** (1995), no. 3, 651–675. http://dx.doi.org/10.1007/BF02101848. MR1328258

[37] F. W. J. OLVER, *Asymptotics and Special Functions*, Computer Science and Applied Mathematics, Academic Press [Harcourt Brace Jovanovich, Publishers], New York-London, 1974. MR0435697

[38] X.-B. PAN, *Surface superconductivity in 3 dimensions*, Trans. Amer. Math. Soc. **356** (2004), no. 10, 3899–3937. https://dx.doi.org/10.1090/S0002-9947-04-03530-5. MR2058511

[39] X.-B. PAN and K.-H. KWEK, *Schrödinger operators with non-degenerately vanishing magnetic fields in bounded domains*, Trans. Amer. Math. Soc. **354** (2002), no. 10, 4201–4227. https://dx.doi.org/10.1090/S0002-9947-02-03033-7. MR1926871

[40] PHẠM THE LẠI and D. ROBERT, *Sur un problème aux valeurs propres non linéaire*, Journées: Équations aux Dérivées Partielles (Saint-Cast, 1979), École Polytech., Palaiseau, 1979, pp. Exp. No. 15, 5 pp. (French). MR0544516

[41] M. REED and B. SIMON, *Methods of Modern Mathematical Physics. IV: Analysis of Operators*, Academic Press [Harcourt Brace Jovanovich, Publishers], New York-London, 1978. MR0493421

[42] B. SIMON, *Coupling constant analyticity for the anharmonic oscillator. (With appendix)*, Ann. Physics **58** (1970), 76–136.
https://dx.doi.org/10.1016/0003-4916(70)90240-X. MR0416322

[43] M. ZWORSKI, *Semiclassical Analysis*, Graduate Studies in Mathematics, vol. 138, American Mathematical Society, Providence, RI, 2012. https://dx.doi.org/10.1090/gsm/138. MR2952218

BERNARD HELFFER:
Laboratoire de Mathématiques Jean Leray
Nantes Université, UMR CNRS 6629
2 rue de la Houssinière
44322 Nantes Cedex 03
France

MATTHIEU LÉAUTAUD:
Laboratoire de Mathématiques d'Orsay
UMR 8628, Université Paris-Saclay
CNRS, Bâtiment 307
91405 Orsay Cedex
France

E-MAIL: Bernard.Helffer@univ-nantes.fr
E-MAIL: matthieu.leautaud@universite-paris-saclay.fr

KEY WORDS AND PHRASES: Spectral theory, 1D Schrödinger operator, quartic potential, eigenvalues, eigenfunctions. semiclassical analysis.

2020 MATHEMATICS SUBJECT CLASSIFICATION: 34L40, 35P15, 35P20, 81Q05, 81Q10, 81Q20.

Received: September 27, 2022.