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Quantitative unique continuation for operators with partially analytic coefficients. Application to approximate control for waves

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Abstract. In this article, we first prove quantitative estimates associated to the unique continuation theorems for operators with partially analytic coefficients of Tataru [Tat95, Tat99b], Robbiano–Zuily [RZ98] and Hörmander [Hör97]. We provide local stability estimates that can be propagated, leading to global ones.

Then, we specify those results to the wave operator on a Riemannian manifold $\mathcal{M}$ with boundary. For this operator, we also prove Carleman estimates and local quantitative unique continuation from and up to the boundary $\partial\mathcal{M}$. This allows us to obtain a global stability estimate from any open subset Γ of $\mathcal{M}$ or $\partial\mathcal{M}$, with the optimal time and dependence on the observation.

As a first application, we compute a sharp lower estimate of the intensity of waves in the shadow of an obstacle.

We also provide the cost of approximate controllability on the compact manifold $\mathcal{M}$: for any $T > 2\sup_{x\in\mathcal{M}} \operatorname{dist}(x, \Gamma)$, we can drive any $H_0^1 \times L^2$ data in time T to an ε-neighborhood of zero in $L^2 \times H^{-1}$, with a control located in Γ, at cost $e^{C/\varepsilon}$.

We finally obtain related results for the Schrödinger equation.

Keywords. Unique continuation, stability estimates, wave equation, control theory, Schrödinger equation

Contents

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1. Introduction and main results

In this article, we are interested in the quantification of *global unique continuation* results of the following form: given a differential operator P on an open set $\Omega \subset \mathbb{R}^n$, and given a small subset U of Ω, we have

$$[Pu = 0 \text{ in } \Omega,\ u|_U = 0] \ \Rightarrow\ u = 0 \text{ on } \Omega. \tag{1.1}$$

More generally, in cases where (1.1) is known to hold, we are interested in proving a quantitative version of

$$[Pu \text{ small in } \Omega,\ u \text{ small in } U] \ \Rightarrow\ u \text{ small in } \Omega.$$

A more tractable problem than (1.1) is the so called *local unique continuation* problem: given $x^0 \in \mathbb{R}^n$ and an oriented local hypersurface S containing x^0, do we have the following implication:

$$\begin{aligned}&\text{there is a neighborhood } \Omega \text{ of } x^0 \text{ such that}\\ &[Pu = 0 \text{ in } \Omega,\ u|_{\Omega\cap S^-} = 0] \ \Rightarrow\ x^0 \notin \operatorname{supp} u,\end{aligned} \tag{1.2}$$

where S^- denotes one side of S? It turns out that proving (1.2) for a suitable class of hypersurfaces (with regard to the operator P) is in general a key step in the proof of properties of the type (1.1). The first general unique continuation result of the form (1.2) is the Holmgren theorem (due to Holmgren [Hol01] in a special case, and to John [Joh49] in the general case), stating that, for operators with analytic coefficients, unique continuation holds across any noncharacteristic hypersurface S (see e.g. [Hör90, Theorem 8.6.5] for a precise statement). This local unique continuation result enjoys a global version proved by John [Joh49], where uniqueness is propagated through a family of noncharateristic hypersurfaces.

For operators with (only) smooth (C^∞) coefficients, the most general result was proved by Hörmander [Hör63, Chapter VIII], [Hör94, Chapter XXVIII]. Uniqueness across a hypersurface holds under a strict pseudoconvexity condition (see e.g. Definition 1.7 below). This result uses as a key tool Carleman estimates, which were introduced in [Car39] and developed at first for elliptic operators in [Cal58]. We also refer to [Zui83] for a general presentation of these problems.

A particular motivation comes from geoseismics [Sym83] and control theory [Lio88a, Lio88b]: in these contexts, one is interested in recovering the data/energy of a wave from the observation on a small part of the domain along a time interval. As well, unique continuation results for waves have been useful tools to solve inverse problems, for instance using the boundary control method [Bel87] (see also the review article [Bel07] and the book [KKL01]).

More precisely, consider the wave operator $P = \partial_t^2 - \Delta_g$ on $\Omega = (-T, T) \times \mathcal{M}$, where $(\mathcal{M}, g)$ is a Riemannian manifold (with or without boundary) and Δ_g the associated (negative) Laplace–Beltrami operator. A central question raised by the above applications is that of global unique continuation from sets of the form $(-T, T) \times \omega$, where $\omega \subset \mathcal{M}$ (resp. $\omega \subset \partial\mathcal{M}$) is an observation region.

In this setting and in the context of control theory, the unique continuation property (1.1) is equivalent to approximate controllability (from $(-T, T) \times \omega$); and an associated quantitative estimate (as proved in the present paper) is equivalent to estimating the cost of approximate controls.

If $\mathcal{M}$ is analytic (and connected), the above-mentioned Holmgren theorem applies, which together with the argument of John [Joh49] allows one to prove unique continuation from $(-T, T) \times \omega$ for any nonempty open set ω as soon as $T > \mathcal{L}(\mathcal{M}, \omega)$, where, for $E \subset \mathcal{M}$, we have set

$$\mathcal{L}(\mathcal{M}, E) := \sup_{x \in \mathcal{M}} \operatorname{dist}(x, E), \qquad \operatorname{dist}(x, E) = \inf_{y \in E} \operatorname{dist}(x, y), \tag{1.3}$$

with

$$\operatorname{dist}(x, y) = \inf\{\operatorname{length}(\gamma) : \gamma \in C^1([0, 1]; \mathcal{M}),\ \gamma(0) = x,\ \gamma(1) = y\},$$
$$\operatorname{length}(\gamma) = \int_0^1 \sqrt{g_{\gamma(t)}(\dot\gamma(t), \dot\gamma(t))}\, dt.$$

Due to finite speed of propagation, it is also not hard to prove that unique continuation from $(-T, T) \times \omega$ does not hold if $T < \mathcal{L}(\mathcal{M}, \omega)$ (see also [Rus71a, Rus71b]), so that the result is sharp. Unique continuation from $(-T, T) \times \omega$ may even fail in critical time $T = \mathcal{L}(\mathcal{M}, \omega)$ in certain situations [Rus71b, Theorem 5].

Removing the analyticity condition on $\mathcal{M}$ has led to considerable difficulties, since Hörmander's general uniqueness result does not apply in this setting: time-like surfaces, as $\{x_1 = 0\}$, do not satisfy the pseudoconvexity assumption for the wave operator. The local unique continuation can even fail after adding some smooth lower order terms to the wave operator, as proved by Alinhac–Baouendi [AB79, Ali83, AB95].

This uniqueness problem in the C^∞ setting was first solved by Rauch–Taylor [RT73] and Lerner [Ler88] in the case $T = \infty$ and $\mathcal{M} = \mathbb{R}^d$ (under different assumptions at infinity). Then, Robbiano [Rob91] managed to prove that unique continuation from $(-T, T) \times \omega$ holds in any domain $\mathcal{M}$ as soon as $\omega \neq \emptyset$ and $T \geq C_0 \mathcal{L}(\mathcal{M}, \omega)$, with C_0 sufficiently large. Hörmander [Hör92] improved this result to $T > \sqrt{27/23}\, \mathcal{L}(\mathcal{M}, \omega)$. That these two results fail to hold in time $\mathcal{L}$ translates the fact that the local uniqueness results of these two authors are not valid across any noncharacteristic surface.

The local uniqueness theorem across any noncharacteristic surface for $\partial_t^2 - \Delta_g$ was proved by Tataru [Tat95], leading to the global unique continuation result in optimal time $T > \mathcal{L}(\mathcal{M}, \omega)$. The result of Tataru was not restricted to the wave operator: he considered operators with coefficients that are analytic in part of the variables, interpolating between the Holmgren theorem and the Hörmander theorem. The technical assumptions of that article were successively removed by Robbiano–Zuily [RZ98], Hörmander [Hör97] and Tataru [Tat99b], leading to a very general local unique continuation result for operators with partially analytic coefficients (containing as particular cases both the Holmgren and Hörmander theorems).

Concerning quantitative estimates of unique continuation, when (1.1) holds, one may expect to have an estimate of the form

$$\begin{aligned} &\|u\|_\Omega \leq K\varphi(\|u\|_U, \|Pu\|_{\tilde{\Omega}}, \|u\|_{\tilde{\Omega}}) \\ &\text{with } \varphi(a, b, c) \to 0 \text{ when } (a, b) \to 0 \text{ with } c \text{ bounded,} \end{aligned} \tag{1.4}$$

where $U \subset \Omega \subset \tilde{\Omega}$ are nonempty, K is a constant, and for appropriate norms. In this context, much less seems to be known. Two additional difficulties arise: one needs first to quantify the local unique continuation property (1.2), and then to “propagate” the local estimates obtained to a global one.

In the setting of the Holmgren theorem, local estimates of unique continuation of the form (1.4) were proved by John [Joh60]: they are of Hölder type, i.e. $\varphi(a, b, c) = (a + b)^\delta c^{1-\delta}$, in case P is elliptic, and of logarithmic type, i.e. $\varphi(a, b, c) = c\big(\log\big(1 + \frac{c}{a+b}\big)\big)^{-1}$, in the general case.

In the situation of the Hörmander theorem, it was proved by Bahouri [Bah87] that Hölder stability always holds locally. Such local estimates were propagated, leading to global ones (in the case of elliptic operators P of order 2 with appropriate boundary conditions, even with low regularity assumptions) by Lebeau and Robbiano [Rob95, LR95]. They can also be improved to $\varphi(a, b, c) = a + b$ if boundary conditions are added to close the estimates [Rob95, LR95].

The global problem for the wave operator in the analytic setting was tackled by Lebeau [Leb92]. For $\Omega = \tilde{\Omega} = (-T, T) \times \mathcal{M}$ and $U = (-T, T) \times \omega$ with $\omega \subset \mathcal{M}$ (or more precisely $\Gamma \subset \partial\mathcal{M}$), he proved that the stability estimate (1.4) with $\varphi(a, b, c) = c\big(\log\big(1 + \frac{c}{a+b}\big)\big)^{-1}$ holds for any $T > \mathcal{L}(\mathcal{M}, \omega)$. He also proved that this inequality is optimal if there exists a ray of geometric optics that does not intersect $(-T, T) \times \omega$ (and only has transverse intersection with $\partial\mathcal{M}$). Under this assumption the (stronger) linear observability estimate (i.e. (1.4) with $\varphi(a, b, c) = a + b$) of the Bardos–Lebeau–Rauch–Taylor theorem [RT74, BLR92] is not satisfied. In the C^∞ situation for this problem, the first result is due to Robbiano [Rob95], who proved estimate (1.4) for T sufficiently large with $\varphi(a,b,c) = c\big(\log\big(1+\frac{c}{a+b}\big)\big)^{-1/2}$. This result was improved by Phung [Phu10] to $\varphi(a,b,c) = c\big(\log\big(1+\frac{c}{a+b}\big)\big)^{-(1-\varepsilon)}$ (still in large time, for any $\varepsilon > 0$, with the constant K in (1.4) depending on ε). In his unpublished lecture notes [Tat99a], Tataru proposes a strategy to obtain estimates of the form (1.4) with $\varphi_\varepsilon = c\big(\log\big(1+\frac{c}{a+b}\big)\big)^{-(1-\varepsilon)}$ in the general context of uniqueness theorems for operators with partially analytic coefficients.

In this article, we develop a systematic approach both to quantifying the local uniqueness theorem of Tataru, Robbiano–Zuily and Hörmander, and to propagating the quantitative local uniqueness results to a global one, with an optimal dependence $\varphi = c\big(\log\big(1+\frac{c}{a+b}\big)\big)^{-1}$. When doing so, we face both difficulties of producing quantitative and global estimates. Then, we specify the results to the wave operator on $\mathcal{M}$. For this operator, we also prove appropriate Carleman estimates and local quantitative unique continuation results from and up to the boundary $\partial\mathcal{M}$. This allows us to obtain a global stability estimate from any open subset of $\mathcal{M}$ or $\partial\mathcal{M}$, with the optimal time ($T > \mathcal{L}(\mathcal{M},\omega)$) and dependence on the observation. This generalizes the result of Lebeau [Leb92] to nonanalytic manifolds, and provides the cost of approximate controllability. We also treat the case of the Schrödinger operator.

In the present introduction, we first discuss the case of the wave and Schrödinger equations; in these particular settings, the results are more precise and simpler to state. Moreover, in this context, we are able to deal with the boundary value problem as well. Second, we state a general quantitative uniqueness result for operators with partially analytic coefficients in the setting of Tataru [Tat95, Tat99b], Robbiano–Zuily [RZ98] and Hörmander [Hör97] (used in the proof for the wave and Schrödinger equations).

1.1. The wave and Schrödinger equations

In this section, we describe the motivating applications of our main result, i.e. to the wave equation with Dirichlet boundary conditions. In this very particular setting, we are also able to tackle the boundary value problem. We finally state a related result for the Schrödinger equation.

When dealing with a manifold $\mathcal{M}$ with boundary, we will always assume that the manifold, the boundary and the metric are smooth. Moreover, $\mathrm{Int}(\mathcal{M})$ will denote the set of points in $\mathcal{M}$ which have a neighborhood homeomorphic to an open subset of $\mathbb{R}^n$. The boundary of $\mathcal{M}$, denoted by $\partial\mathcal{M}$, is the complement of $\mathrm{Int}(\mathcal{M})$ in $\mathcal{M}$. All manifolds considered will be assumed to be connected.

Theorem 1.1 (Quantitative unique continuation for waves). *Let $\mathcal{M}$ be a compact Riemannian manifold with (or without) boundary. For any nonempty open subset ω of $\mathcal{M}$ and any $T > 2\mathcal{L}(\mathcal{M},\omega)$, there exist $C, \kappa, \mu_0 > 0$ such that for any $(u_0,u_1) \in H_0^1(\mathcal{M}) \times L^2(\mathcal{M})$ and u the solution of*

$$\begin{cases} \partial_t^2 u - \Delta_g u = 0 & \text{in } (0,T)\times \mathrm{Int}(\mathcal{M}), \\ u = 0 & \text{in } (0,T)\times\partial\mathcal{M}, \\ (u,\partial_t u)_{|t=0} = (u_0,u_1) & \text{in } \mathrm{Int}(\mathcal{M}), \end{cases} \tag{1.5}$$

we have, for any $\mu \geq \mu_0$,

$$\|(u_0, u_1)\|_{L^2\times H^{-1}} \leq Ce^{\kappa\mu}\|u\|_{L^2((0,T);H^1(\omega))} + \frac{1}{\mu}\|(u_0, u_1)\|_{H^1\times L^2}.$$

If $\partial\mathcal{M} \neq \emptyset$ *and* Γ *is a nonempty open subset of* $\partial\mathcal{M}$*, then for any* $T > 2\mathcal{L}(\mathcal{M}, \Gamma)$ *there exist* $C, \kappa, \mu_0 > 0$ *such that for any* $(u_0, u_1) \in H_0^1(\mathcal{M}) \times L^2(\mathcal{M})$ *and* u *the solution of* (1.5)*, we have*

$$\|(u_0, u_1)\|_{L^2\times H^{-1}} \leq Ce^{\kappa\mu}\|\partial_\nu u\|_{L^2((0,T)\times\Gamma)} + \frac{1}{\mu}\|(u_0, u_1)\|_{H^1\times L^2}.$$

Theorem 1.1 remains valid if Δ_g is perturbed by lower order terms that are analytic in time but may have low regularity in space. In the special case where they are time-independent, the constants in the previous estimates may be chosen uniformly with respect to these perturbations (in appropriate norms). We refer to Theorem 6.1 for a precise statement. Note also that the statement of Theorem 1.1 remains valid for all $\mu > 0$ (not only $\mu \geq \mu_0$), the estimate for μ bounded being trivial (see Lemma A.3). However, we preferred to keep the formulation with $\mu \geq \mu_0$ to stress that only large values of μ are of interest. This result can also be formulated in the following way, closer to (1.4) (see again Lemma A.3). We only give the boundary observation case, the internal observation case being similar.

Corollary 1.2. *Assume* $\partial\mathcal{M} \neq \emptyset$ *and* Γ *is a nonempty open subset of* $\partial\mathcal{M}$*. Then, for any* $T > 2\mathcal{L}(\mathcal{M}, \Gamma)$*, there exists* $C > 0$ *such that for any* $(u_0, u_1) \in H_0^1(\mathcal{M}) \times L^2(\mathcal{M}) \setminus \{(0, 0)\}$ *and* u *the solution of* (1.5)*, we have*

$$\|(u_0, u_1)\|_{L^2\times H^{-1}} \leq C\frac{\|(u_0, u_1)\|_{H^1\times L^2}}{\log\left(1 + \frac{\|(u_0,u_1)\|_{H^1\times L^2}}{\|\partial_\nu u\|_{L^2((0,T)\times\Gamma)}}\right)},$$

$$\|(u_0, u_1)\|_{H^1\times L^2} \leq Ce^{C\Lambda}\|\partial_\nu u\|_{L^2((0,T)\times\Gamma)} \quad \textit{with} \quad \Lambda = \frac{\|(u_0, u_1)\|_{H^1\times L^2}}{\|(u_0, u_1)\|_{L^2\times H^{-1}}}.$$

In the first estimate, the function on the right hand side is to be understood as being $(\log(1 + 1/x))^{-1}$ for $x > 0$ and 0 for $x = 0$.

In the second estimate, Λ has to be considered as the typical frequency of the initial data. So, the estimate states the cost of observability of the order of an exponential of the typical frequency. As an illustration, taking for initial data $(u_0, u_1) = (\psi_\lambda, 0)$ with ψ_λ a normalized eigenfunction of the Laplace–Dirichlet operator on $\mathcal{M}$, associated to the eigenvalue λ, one has $\Lambda \sim \sqrt{\lambda}$ and Corollary 1.2 recovers the tunneling estimate $\|\partial_\nu\psi_\lambda\|_{L^2(\Gamma)} \geq C^{-1}e^{-C\sqrt{\lambda}}$ (see [LR95]).

As proved by Lebeau [Leb92] in the analytic context, this exponential dependence is sharp in general. More precisely, the form of the estimates in Theorem 1.1 and Corollary 1.2 is optimal as soon as there is a ray of geometric optics (traveling at speed 1) which does not intersect the region $\overline{\Gamma}$ (resp. $\overline{\omega}$ in the internal observation case) in the time interval $[0, T]$ (and only has transverse intersection with the boundary). See [Leb92, Section 2, pp. 5 and 6].

As a consequence of the previous theorem, we can obtain approximate controllability results. For brevity, we only state the case of boundary control.

Theorem 1.3 (Cost of boundary approximate control). *For any $T > 2\mathcal{L}(\mathcal{M}, \Gamma)$, there exist $C, c > 0$ such that for any $\varepsilon > 0$ and any $(u_0, u_1) \in H_0^1(\mathcal{M}) \times L^2(\mathcal{M})$, there exists $g \in L^2((0,T) \times \Gamma)$ with*

$$\|g\|_{L^2((0,T)\times\Gamma)} \leq Ce^{c/\varepsilon}\|(u_0, u_1)\|_{H_0^1(\mathcal{M})\times L^2(\mathcal{M})}$$

such that the solution of

$$\begin{cases} (\partial_t^2 - \Delta)u = 0 & \text{in } (0,T) \times \operatorname{Int}(\mathcal{M}), \\ u = \mathbb{1}_\Gamma g & \text{in } (0,T) \times \partial\mathcal{M}, \\ (u, \partial_t u)_{|t=0} = (u_0, u_1) & \text{in } \operatorname{Int}(\mathcal{M}), \end{cases}$$

satisfies $\|(u, \partial_t u)_{|t=T}\|_{L^2(\mathcal{M})\times H^{-1}(\mathcal{M})} \leq \varepsilon\|(u_0, u_1)\|_{H_0^1(\mathcal{M})\times L^2(\mathcal{M})}$.

That this result is a consequence of Theorem 1.1 is proved in [Rob95, proof of Theorem 2, Section 3]. The solution of the nonhomogeneous boundary value problem is defined in the sense of transposition [Lio88a].

Another application of Theorem 1.1, given in [LL16] and which was at the origin of the present work, is concerned with the *exact* observability/controllability problem. This property was completely characterized (with optimal geometric conditions) in the seminal paper [BLR92]. The proof there proceeds in two steps: first dealing with high frequencies (propagation of wavefront sets), and then reducing the low frequency problem to a unique continuation property. Both steps are nonconstructive (i.e. rely e.g. on contradiction arguments). In [LL16], we explain how Theorem 1.1 allows one to give a completely constructive proof of the second step. We also provide a constructive proof of the first step on a compact manifold. As an application, we estimate the dependence of the observability constant on the observation time T or on the addition of a potential $V(x)$ in the wave operator.

The estimates of Theorem 1.1 and Corollary 1.2 can actually be stated more locally, and interpreted in a different physical context (motivated by [RT73]). The following theorem shows that they are independent of the global geometry, and in particular do not require that $\mathcal{M}$ is compact if one only wants to recover data supported in a given compact set.

Theorem 1.4 (Penetration into shadow for waves). *Let $\mathcal{M}$ be a complete Riemannian manifold with (possibly empty) compact boundary $\partial\mathcal{M}$. Let ω_0 be an open subset of $\mathcal{M}$ and ω_1 a compact subset of $\mathcal{M}$. Then, for any*

$$T > \mathcal{L}(\omega_1, \omega_0) := \sup_{x\in\omega_1} \operatorname{dist}(x, \omega_0),$$

there exists $C > 0$ such that for any $(u_0, u_1) \in H_0^1(\mathcal{M}) \times L^2(\mathcal{M}) \setminus \{(0,0)\}$ supported in ω_1 and u the solution of (1.5) (taken on the time interval $(-T, T)$ instead of $(0,T)$), we have

$$\|(u_0, u_1)\|_{H^1\times L^2} \leq Ce^{C\Lambda}\|u\|_{L^2((-T,T);H^1(\omega_0))} \quad \text{with} \quad \Lambda = \frac{\|(u_0, u_1)\|_{H^1\times L^2}}{\|(u_0, u_1)\|_{L^2\times H^{-1}}}.$$

Roughly speaking, the theorem describes the following physical situation: take a noise creating an initial data compactly supported in ω_1, and suppose an observer is located in a zone ω_0. Then, by observing during the time interval $(-\mathcal{L}(\omega_1,\omega_0)-\varepsilon, \mathcal{L}(\omega_1,\omega_0)+\varepsilon)$, $\varepsilon > 0$, the observer will be able to recover at least a proportion of the initial energy of the order $e^{-C\Lambda}$ where Λ is the typical frequency of the data. This result is particularly interesting if the zone ω_1 is in the "shadow" of an obstacle when seen from ω_0, that is, no rays of geometric optics starting from ω_1 ever reach ω_0. In that case, the classical geometric optics approximation would predict that the observer does not receive any information. We refer to [RT73] for a qualitative result in infinite time; here, Theorem 1.4 provides a quantitative result in finite time, which is optimal with respect to the time and the form of the estimate if ω_1 is indeed in the "shadow" region when observed from ω_0. More precisely, [Leb92, Section 2] implies that the $e^{C\Lambda}$ is optimal as soon as there is a ray of geometric optics (having only transverse intersections with $\partial\mathcal{M}$) starting from the interior of ω_1 at time zero and not intersecting $\overline{\omega_0}$ during the time interval $[-T, T]$. Such an estimate in the shadow region is reminiscent of the tunneling effect for waves (see e.g. [Leb96, LR97, Bur98]). It is of course also related to the tunneling effect in semiclassical analysis [Zwo12, Chapter 7].

Note that it could be desirable to make the observation in positive time only, that is, on the interval $(0, T)$. This can be easily seen to be impossible in general, for instance in dimension one by looking at solutions of the form $u(x+t)$. Yet, a classical parity argument in the time variable allows one to obtain (in any dimension) a similar result with observation on $(0, T)$, $T > \mathcal{L}(\omega_1,\omega_0)$, for all initial data of the form $(u_0, 0)$ or $(0, u_1)$.

We also obtain related results for the Schrödinger equation. We only state here the counterpart of Theorem 1.1 in this setting.

Theorem 1.5. *Let $\mathcal{M}$ be a compact Riemannian manifold with (or without) boundary. For any nonempty open subset ω of $\mathcal{M}$ and any $T > 0$, there exist $C, \kappa, \mu_0 > 0$ such that for any $u_0 \in H^2 \cap H^1_0$ and u the solution of*

$$\begin{cases} i\partial_t u + \Delta_g u = 0 & \text{in } (0,T)\times \mathrm{Int}(\mathcal{M}), \\ u = 0 & \text{in } (0,T)\times \partial\mathcal{M}, \\ u(0) = u_0 & \text{in } \mathrm{Int}(\mathcal{M}), \end{cases} \tag{1.6}$$

we have, for any $\mu \geq \mu_0$,

$$\|u_0\|_{L^2} \leq Ce^{\kappa\mu}\|u\|_{L^2((0,T);H^1(\omega))} + \frac{1}{\mu}\|u_0\|_{H^2}.$$

If $\partial\mathcal{M} \neq \emptyset$ and Γ is a nonempty open subset of $\partial\mathcal{M}$, then for any $T > 0$, there exist $C, \kappa, \mu_0 > 0$ such that for any $u_0 \in H^2 \cap H^1_0$ and u the solution of (1.6), we have

$$\|u_0\|_{L^2} \leq Ce^{\kappa\mu}\|\partial_\nu u\|_{L^2((0,T)\times\Gamma)} + \frac{1}{\mu}\|u_0\|_{H^2}.$$

This result still holds with some lower order perturbations, analytic in t; see Theorem 6.6 for a more precise statement. Note that some related results have already been proven in the internal case by Phung [Phu01] with $e^{\kappa\mu}$ replaced by $e^{\kappa\mu^2}$.

1.2. *Quantitative unique continuation for operators with partially analytic coefficients*

Let us now turn to the general stability result and present the class of partial differential operators we will be dealing with. We consider domains $\Omega \subset \mathbb{R}^n = \mathbb{R}^{n_a} \times \mathbb{R}^{n_b}$, where $n_a + n_b = n$. We denote by $x = (x_a, x_b)$ the global variables and by $\xi = (\xi_a, \xi_b)$ the associated dual variables. The variables x_a will be those with respect to which the operator considered is analytic.

Given a bounded domain $\Omega \subset \mathbb{R}^n = \mathbb{R}^{n_a} \times \mathbb{R}^{n_b}$, we say that a smooth function $f : \Omega \to \mathbb{C}$ is *analytic with respect to* x_a if, for any $x^0 = (x_a^0, x_b^0) \in \Omega$, there is $\varepsilon > 0$ such that f extends as a holomorphic function in the variable x_a for $x = (x_a, x_b) \in (B(x_a^0, \varepsilon) + iB(0, \varepsilon)) \times B(x_b^0, \varepsilon)$.

The following definition is due to Tataru [Tat99b, Definition 2.2].

Definition 1.6 (Analytically principally normal operators). Let P be a partial differential operator on an open set $\Omega \subset \mathbb{R}^{n_a} \times \mathbb{R}^{n_b}$, of order $m \in \mathbb{N}^*$, with smooth coefficients and principal symbol $p(x_a, x_b, \xi_a, \xi_b)$. We say that P is an *analytically principally normal operator* in $\{\xi_a = 0\}$ inside Ω if the coefficients of P are real-analytic in the variable x_a and for any $x^0 \in \Omega$ there exist $\Omega_a \subset \mathbb{R}^{n_a}$ and $\Omega_b \subset \mathbb{R}^{n_b}$ such that $x^0 \in \Omega_a \times \Omega_b \subset \Omega$ and there exists a complex neighborhood $\Omega_a^{\mathbb{C}}$ of Ω_a in $\mathbb{C}^{n_a}$ and a constant $C > 0$ such that for all $z_a, \tilde{z}_a \in \Omega_a^{\mathbb{C}}$ and all $(x_b, \xi_b) \in \Omega_b \times \mathbb{R}^{n_b}$, $\xi_b \neq 0$, we have

$$\begin{aligned} &|\{p(z_a, \cdot, 0, \cdot), p(\tilde{z}_a, \cdot, 0, \cdot)\}(x_b, \xi_b)| + \left|\left\{\overline{p(z_a, \cdot, 0, \cdot)}, p(\tilde{z}_a, \cdot, 0, \cdot)\right\}(x_b, \xi_b)\right| \\ &\qquad\qquad\qquad\qquad \leq C|p(z_a, x_b, 0, \xi_b)|\,|\xi_b|^{m-1}, \qquad (1.7) \\ &|\partial_{z_a} p(z_a, x_b, 0, \xi_b)| \leq C|p(z_a, x_b, 0, \xi_b)|. \end{aligned}$$

Note that in this definition, the Poisson brackets are taken only with respect to the (x_b, ξ_b) variables. Yet, the combination of the two conditions (1.7) and (1.8) implies that such operators are in particular principally normal in $\{\xi_a = 0\}$ in the following more usual sense (see [RZ98], [Hör97] or [Tat99b, Definition 2.1]). Given a closed conic subset Γ of $T^*\Omega$, one says that P is *principally normal* in Γ if

$$|\{\overline{p}, p\}(x, \xi)| \leq C|p(x, \xi)|\,|\xi|^{m-1} \quad \text{for all } (x, \xi) \in \Gamma, \qquad (1.9)$$

where (as opposed to (1.7)–(1.8)) $\{\overline{p}, p\}$ is computed with respect to all variables.

Two interesting cases of operators P being analytically principally normal in $\{\xi_a = 0\}$, considered in [RZ98] and [Hör97], are operators with analytic coefficients in x_a satisfying one of the following two assumptions:

(E) transversal ellipticity: $p(x_a, x_b, 0, \xi_b) \geq c|\xi_b|^m$ for $(x_a, x_b) \in \Omega$, $\xi_b \in \mathbb{R}^{n_b}$;

(H) principal normality and invariance with respect to the null bicharacteristic flow in $\{\xi_a = 0\}$:

$$|\{\overline{p}, p\}(x_a, x_b, 0, \xi_b)| \leq C|p(x_a, x_b, 0, \xi_b)|\,|\xi_b|^{m-1} \quad \text{and} \quad \partial_{x_a} p(x_a, x_b, 0, \xi_b) = 0.$$

We now formulate the definition of strongly pseudoconvex surfaces for an operator P (see [Hör94, Definition 28.3.1], [Tat99b, Definitions 2.3 and 2.4] and [Tat99a, Section 1.2]).

Definition 1.7 (Strongly pseudoconvex oriented surface). Let $\Omega \subset \mathbb{R}^n$, Γ be a closed conic subset of $T^*\Omega$, and P be principally normal in Γ inside Ω (in the sense of (1.9)) with principal symbol p. Let S be a C^2 oriented hypersurface of Ω and $x^0 \in S \cap \Omega$. We say that S is *strongly pseudoconvex* in Γ at x^0 for P if there exists $\phi \in C^2(\Omega; \mathbb{R})$ such that $S = \{\phi = 0\}$, $\nabla\phi(x^0) \neq 0$, and

$$\operatorname{Re}\{\overline{p}, \{p, \phi\}\}(x^0, \xi) > 0 \quad \text{if } p(x^0, \xi) = \{p, \phi\}(x^0, \xi) = 0 \text{ and } \xi \in \Gamma_{x^0}, \xi \neq 0, \tag{1.10}$$

$$\frac{1}{i\tau}\{\overline{p}_\phi, p_\phi\}(x^0, \xi) > 0 \quad \text{if } p_\phi(x^0, \xi) = \{p_\phi, \phi\}(x^0, \xi) = 0 \text{ and } \xi \in \Gamma_{x^0}, \tau > 0, \tag{1.11}$$

where $p_\phi(x, \xi) = p(x, \xi + i\tau\nabla\phi)$.

Note that this is a property of the *oriented* surface S solely, and not of the defining function ϕ (see [Hör94, beginning of Section 28.3]). If $\Gamma = T^*\Omega$, it is the usual condition of the Hörmander theorem (see [Hör94, Section 28.3]), that is, under which uniqueness holds for P at x^0 across the hypersurface S, i.e. from $\phi > 0$ to $\phi < 0$.

Below, this condition will always be used for $\Gamma = \{\xi_a = 0\}$. In this case, and using the homogeneity of p in ξ, assumption (1.11) may be rephrased as

$$\frac{1}{i}\{\overline{p}(x, \xi - i\nabla\phi), p(x, \xi + i\nabla\phi)\}(x^0, 0, \xi_b) > 0 \quad \text{if } p(\zeta) = \{p, \phi\}(\zeta) = 0,\ \xi_b \in \mathbb{R}^{n_b},$$

where $\zeta = (x^0, i\nabla_a\phi(x^0), \xi_b + i\nabla_b\phi(x^0))$. An important feature of this definition is that it is invariant by changes of coordinates.

Note also that in the case $\Gamma = \{\xi_a = 0\}$, condition (1.10) is the limit as $\tau \to 0^+$ of (1.11) on the subset

$$\{p_\phi(x^0, \xi) = \{p_\phi, \phi\}(x^0, \xi) = 0\} \cap \Gamma_{x^0},$$

thanks to the principal normality assumption (1.9) (see Remark 3.5 below).

Before stating our main result, let us discuss some cases of operators of particular interest.

Remark 1.8 (Hörmander case). If $n_a = 0$, there is no analytic variable. In this case, Definition 1.6 coincides with the definition of principally normal operators [Hör94, Chapter XXVIII] and Definition 1.7 with $\Gamma = T^*\Omega$ of strongly pseudoconvex functions. The unique continuation result under consideration is the classical Hörmander theorem [Hör94, Chapter XXVIII].

Remark 1.9 (Holmgren case). If $n_a = n$, that is, the operator is analytic in all the variables, we have $x_a = x$, $\xi_a = \xi$, and hence $\Gamma = \Omega \times \{\xi_a = 0\} = \Omega \times \{\xi = 0\}$. In this situation, conditions (1.7), (1.8) are empty since all the terms vanish.

Next, concerning the conditions on the surface $\{\phi = 0\}$, notice that (1.10) is also empty since $\Gamma_{x^0} \cap \{\xi \neq 0\} = \emptyset$. For (1.11), if $\xi \in \Gamma_{x^0}$, that is, $\xi = 0$, we have $p_\phi(x^0, \xi) = p(x^0, i\tau\nabla\phi(x^0)) = (i\tau)^m p(x^0, \nabla\phi(x^0))$: any noncharacteristic surface at x^0 (i.e. satisfying $p(x^0, \nabla\phi(x^0)) \neq 0$) is a strongly pseudoconvex oriented surface. The unique continuation result under consideration is the classical Holmgren theorem.

Note that, in the case $n_a = n$, the results presented here hold under condition (1.11), namely

$$p(x^0, \nabla\phi(x^0)) = \{p, \phi\}(x^0, \nabla\phi(x^0)) = 0$$
$$\Rightarrow \frac{1}{i}\{\overline{p}(x, \xi - i\nabla\phi), p(x, \xi + i\nabla\phi)\}(x^0, 0) > 0,$$

which is weaker than the noncharacteristicity condition $p(x^0, \nabla\phi(x^0)) \neq 0$ of the Holmgren theorem.

Remark 1.10 (Wave type and Schrödinger type operators). Let us now consider the case of operators P with principal symbol of the form $p_2(x, \xi) = Q_x(\xi)$, where Q_x is a smooth x-family of real quadratic forms in ξ such that $Q_x(0, \xi_b)$ is positive (or negative) definite on $\mathbb{R}^{n_b}$. This is the case of the wave operator or Schrödinger type operators. We remark first that for such operators:

- condition (E) is fulfilled thanks to the positive definiteness of $Q_x(0, \xi_b)$,
- condition (H) is also fulfilled in case the (real-valued) coefficients of Q_x are independent of x_a.

Then, assumption (1.10) holds (uniformly with respect to $x \in \Omega$) again according to the positive definiteness of $Q_x(0, \xi_b)$. It is indeed empty since $p_2(x, (0, \xi_b))$ does not vanish for $\xi_b \neq 0$. Moreover, $\{p_2, \phi\}(x, \xi) = 2\tilde{Q}_x(\xi, \nabla\phi)$, where $\tilde{Q}_x$ is the polar form of Q_x, and

$$\{p_2, \phi\}(x, \xi + i\nabla\phi) = 2\tilde{Q}_x(\xi, \nabla\phi) + 2i Q_x(\nabla\phi).$$

As a consequence (Q being real), $\mathrm{Im}\{p_2, \phi\}(x, \xi + i\nabla\phi) = 2Q_x(\nabla\phi)$ so that (1.11) is also empty (and thus satisfied) for any noncharacteristic hypersurface.

In conclusion, for real quadratic forms which are positive (or negative) definite on $\mathbb{R}^{n_b}$ at $\xi_a = 0$, any noncharacteristic hypersurface is strongly pseudoconvex in the sense of Definition 1.7. In the case $n_a = 1$, this includes the following operators of particular interest:

- $P = D_{x_a}^2 - \sum_{i,j=1}^{n-1} \alpha_{ij}(x) D_{x_b^j} D_{x_b^i} + \ell.o.t.$ (wave operator) with $p = \xi_a^2 - \sum_{i,j=1}^{n-1} \alpha_{ij}(x)\xi_b^j \xi_b^i$,
- $P = D_{x_a} - \sum_{i,j=1}^{n-1} \alpha_{ij}(x) D_{x_b^j} D_{x_b^i} + \ell.o.t.$ (Schrödinger operator) with $p = -\sum_{i,j=1}^{n-1} \alpha_{ij}(x)\xi_b^j \xi_b^i$,

where the quadratic form with coefficients $\alpha_{i,j}$ is positive definite.

We are now prepared to formulate our main result in the general framework. We first describe the geometric context and then state the theorem.

Geometric setting (see Figure 1). We first fix two splittings $\mathbb{R}^n = \mathbb{R}^{n-1}_{x'} \times \mathbb{R}_{x_n}$ and $\mathbb{R}^n = \mathbb{R}^{n_a}_{x_a} \times \mathbb{R}^{n_b}_{x_b}$, possibly in two different bases. We let D be a bounded open subset of $\mathbb{R}^{n-1}$

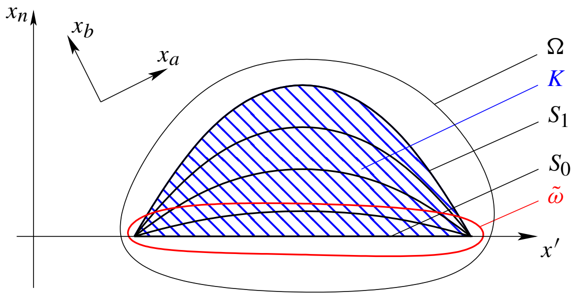

Fig. 1. Geometric setting of Theorem 1.11.

with smooth boundary and $G = G(x', \varepsilon)$ a C^2 function defined in a neighborhood of $\overline{D} \times [0, 1]$ such that

- for all $\varepsilon \in (0, 1]$, we have $\{x' \in \mathbb{R}^{n-1} : G(x', \varepsilon) \geq 0\} = \overline{D}$;
- for all $x' \in D$, the function $\varepsilon \mapsto G(x', \varepsilon)$ is strictly increasing;
- for all $\varepsilon \in (0, 1]$, we have $\{x' \in \mathbb{R}^{n-1} : G(x', \varepsilon) = 0\} = \partial D$;
- $G(x', 0) = 0$.

We set, $S_0 = \overline{D} \times \{0\}$ and, for $\varepsilon \in (0, 1]$,

$$\begin{aligned} S_\varepsilon &= \{(x', x_n) \in \mathbb{R}^n : x_n \geq 0 \text{ and } G(x', \varepsilon) = x_n\} \\ &= (\overline{D} \times \mathbb{R}) \cap \{(x', x_n) \in \mathbb{R}^n : G(x', \varepsilon) = x_n\}, \\ K &= \{x \in \mathbb{R}^n : 0 \leq x_n \leq G(x', 1)\}. \end{aligned}$$

Theorem 1.11. *In the above geometric setting, let Ω be a bounded open neighborhood of K, and P be a differential operator of order m, which is analytically principally normal on Ω in $\{\xi_a = 0\}$. Assume also that, for any $\varepsilon \in [0, 1+\eta)$, the oriented surfaces $S_\varepsilon = \{\phi_\varepsilon = 0\}$ with $\phi_\varepsilon(x', x_n) := G(x', \varepsilon) - x_n$ are strongly pseudoconvex in $\{\xi_a = 0\}$ for P on the whole S_ε, in the sense of Definition 1.7.*

Then, for any open neighborhood $\tilde{\omega} \subset \Omega$ of S_0, there exists a neighborhood U of K and constants $\kappa, C, \mu_0 > 0$ such that for all $\mu \geq \mu_0$ and $u \in C_0^\infty(\mathbb{R}^n)$, we have

$$\|u\|_{L^2(U)} \leq Ce^{\kappa\mu}\left(\|u\|_{H_b^{m-1}(\tilde{\omega})} + \|Pu\|_{L^2(\Omega)}\right) + \frac{C}{\mu^{m-1}}\|u\|_{H^{m-1}(\Omega)},$$

where we have denoted $\|u\|_{H_b^{m-1}(\tilde{\omega})} = \sum_{|\beta|\leq m-1} \|D_b^\beta u\|_{L^2(\tilde{\omega})}$.

If $n_a = n$ (Holmgren case), then, for any $\widetilde{\varphi} \in C_0^\infty(\tilde{\omega})$ with $\widetilde{\varphi} = 1$ on a neighborhood of S_0, and for any $s \in \mathbb{R}$, there exist $\kappa, C, \mu_0 > 0$ such that for all $\mu \geq \mu_0$ and $u \in C_0^\infty(\mathbb{R}^n)$, we have

$$\|u\|_{L^2(U)} \leq Ce^{\kappa\mu}\left(\|\widetilde{\varphi}u\|_{H^{-s}(\mathbb{R}^n)} + \|Pu\|_{L^2(\Omega)}\right) + \frac{C}{\mu^{m-1}}\|u\|_{H^{m-1}(\Omega)}.$$

If $n_a = 0$ (Hörmander case), there are $c, \kappa, C, \mu_0 > 0$ such that for all $\mu \geq \mu_0$ and $u \in C_0^\infty(\mathbb{R}^n)$, we have

$$\|u\|_{H^{m-1}(U)} \leq Ce^{\kappa\mu}\left(\|u\|_{H^{m-1}(\tilde{\omega})} + \|Pu\|_{L^2(\Omega)}\right) + Ce^{-c\mu}\|u\|_{H^{m-1}(\Omega)}.$$

Note that in the first two cases, we obtain a result of the type (1.4) with a logarithmic function φ, whereas in the framework of the Hörmander theorem, we obtain the stronger Hölder type dependence (see [Bah87, Rob95, LR95, LRL12]):

$$\|u\|_{H^{m-1}(U)} \leq C\big(\|u\|_{H^{m-1}(\tilde{\omega})} + \|Pu\|_{L^2(\Omega)}\big)^{\delta}\|u\|_{H^{m-1}(\Omega)}^{1-\delta}$$

for some $\delta \in (0, 1)$.

The formulation of the above result using a foliation by hypersurfaces is inspired by that of [Joh49, Theorem, p. 224] in the context of the Holmgren theorem. The statement describing the hypersurfaces by graphs could look rigid. We will give later, in Theorem 4.11, a slight variant where the partial analyticity and the foliation by graphs can be described in different coordinates: the linear change of coordinates between the two different splittings $\mathbb{R}^n = \mathbb{R}^{n-1}_{x'} \times \mathbb{R}_{x_n}$ and $\mathbb{R}^n = \mathbb{R}^{n_a} \times \mathbb{R}^{n_b}$ may be replaced by a diffeomorphism. We have chosen not to present this more general result here for the sake of exposition. Most of the global theorems for the wave and Schrödinger equations on a manifold are proved in the setting of Theorem 1.11, after some suitable change of coordinates. In a forthcoming paper [LL17], we apply the more invariant result of Theorem 4.11 to the case of the hypoelliptic wave operator, for which we are not able to construct appropriate coordinates to apply Theorem 1.11 directly.

1.3. Idea of the proof

As already mentioned, unique continuation theorems (e.g. the Hörmander theorem) are often proved with *Carleman estimates*, that is, weighted L^2 estimates of the form

$$\|e^{\tau\psi}u\|_{L^2} \leq C\|e^{\tau\psi}Pu\|_{L^2}, \tag{1.12}$$

where τ is a large parameter and ψ a weight function having level sets appropriately situated with respect to the surface S. Such inequalities are already quantitative, and hence furnish a good starting point towards local quantitative unique continuation results. This strategy has already been followed in [Rob95, LR95] in the case of elliptic operators (see also [Bah87]). Starting from the Carleman inequality (1.12), the idea is to apply the estimates to some function $\chi(x)u$ where χ is a cutoff function according to the level sets of ψ. The exponential weight $e^{\tau\psi(x)}$ in (1.12) (giving an exponentially large/small strength to the large/small values of ψ) naturally leads to inequalities of the form

$$\|u\|_{V_2} \leq e^{\kappa\mu}(\|u\|_{V_1} + \|Pu\|_{V_3}) + e^{-\kappa\mu}\|u\|_{V_3}, \tag{1.13}$$

uniformly for $\mu \geq \mu_0$ and for small open sets $V_1 \subset V_2 \subset V_3$ depending on the local geometry (namely, on the cutoff function χ, the support of $[P, \chi]$, and hence on the level sets of ψ). By optimizing in μ (see [Rob95] or [LRL12, Lemma 5.2]) this can then be written as an interpolation estimate

$$\|u\|_{V_2} \leq (\|u\|_{V_1} + \|Pu\|_{V_3})^{\delta}\|u\|_{V_3}^{1-\delta}$$

for some $\delta \in (0, 1)$. The interest of these interpolation estimates (or directly of estimates like (1.13)) is that they can be easily iterated, leading to some global ones. This procedure ends up with a Hölder type dependence, i.e. (1.4) with $\varphi = (a+b)^{\delta}c^{1-\delta}$. We refer for instance to the survey article [LRL12] for a description of these estimates in the elliptic case, with application to spectral estimates and control results for the heat equation.

Yet, in the context of the unique continuation theorem for partially analytic operators, the Carleman estimates proved in [Tat95, RZ98, Hör97, Tat99b] contain a "microlocal" weight of the form $e^{-\frac{\varepsilon}{2\tau}|D_a|^2}e^{\tau\psi(x)}$ instead of $e^{\tau\psi(x)}$. Whereas the usual $e^{\tau\psi}$ is still here to give strength to the level sets of ψ, the additional term $e^{-\frac{\varepsilon}{2\tau}|D_a|^2}$ is now aimed at localizing in the low frequencies in the variable x_a. In this context, the proof of unique continuation proceeds via a (qualitative) complex-analytic argument (maximum principle). Here, this additional argument in the proof of unique continuation also requires to be quantified. As in [Rob95], this procedure naturally leads to local logarithmic (instead of Hölder) stability estimates. The main issue one has to face when quantifying unique continuation is that such estimates cannot be iterated (or would yield dependence estimates of the type (1.4) with a function φ being a composition of as many "log" as steps needed in the iteration).

One idea to overcome this difficulty, proposed by Tataru in his unpublished lecture notes [Tat99a], was to propagate some low frequency estimates of the form

$$\left\{ \begin{array}{rl} \|u\|_{H^{m-1}} &= 1 \\ \left\|m\left(\frac{D_a}{\mu}\right)\sigma\left(\frac{x}{R}\right)Pu\right\|_{L^2} &\leq e^{-\mu^{\alpha}} \end{array} \right\} \Rightarrow \left\|m\left(\frac{D_a}{\tau}\right)\sigma\left(\frac{x}{r}\right)u\right\|_{H^{m-1}} \leq e^{-\tau},\ \forall \tau < c\mu^{\alpha},$$

for functions u supported in $\{\phi < \phi(x_0)\}$, for appropriate compactly supported cutoff functions σ and $m(\xi)$ of Gevrey class $1/\alpha$, $\alpha < 1$, and for some $r < R$. Such estimates could be propagated and would lead to some global stability estimates of the form (1.4) with $\varphi_\varepsilon(a,b,c) = c\left(\log\left(1+\frac{c}{a+b}\right)\right)^{-(1-\varepsilon)}$.

The loss $1-\varepsilon$ in the power of log is due to the use of functions of Gevrey class $1/\alpha$ with compact support. The optimal case $\alpha = 1$ would correspond to analytic functions. Yet, analytic functions cannot have compact support, which is a key ingredient in the usual application of Carleman estimates.

Let us now explain our strategy to solve this problem.

1.3.1. Obtaining local information at low frequency. Part of the proof of the present paper is inspired by this idea of propagating only low frequency estimates (in the analytic variable x_a). However, we replace the Gevrey cutoff functions by some analytic "almost cutoff" functions of the form

$$\chi_\lambda := e^{-|D_a|^2/\lambda}\chi, \tag{1.14}$$

where χ is smooth with the expected compact support, being convolved/regularized with a heat kernel in the variable x_a, hence analytic in this variable. It turns out that the right choice of the regularization parameter λ is $\lambda = C\mu$ where μ is the frequency where we want to measure our solution. That such functions are not compactly supported makes all commutator estimates (e.g. when applying the Carleman estimate to functions like $\chi_\lambda u$ instead of χu, as explained above) much more intricate and requires a careful study of the dependence on the regularization parameter λ, the local frequency μ and the parameter τ in the Carleman inequality. All estimates are carried out up to an exponentially small remainder (in terms of these parameters).

When following this procedure, the local estimate we prove (which we are in addition able to propagate) is a generalization of (1.13), but truncated at low frequencies in the analytic variable x_a. In a neighborhood of a point x^0, it is of the form

$$\left\| m_\mu\left(\frac{D_a}{\beta\mu}\right)\chi_{2,\mu}u\right\|_{H^{m-1}} \le Ce^{\kappa\mu}\left(\left\| m_\mu\left(\frac{D_a}{\mu}\right)\chi_{1,\mu}u\right\|_{H^{m-1}} + \|Pu\|_{L^2(B(x^0,R))}\right) + Ce^{-\kappa'\mu}\|u\|_{H^{m-1}}, \tag{1.15}$$

uniformly for $\mu \ge \mu_0$. See the beginning of Section 3 for a more precise statement and remarks on this result. Here, χ_1 and χ_2 are some cutoff functions in the physical space that localize respectively to the place where the information is taken (locally in $\{\phi > 0\}$) and where it is propagated to (a small neighborhood of x^0). These functions respectively correspond to $\mathbb{1}_{V_1}$ and $\mathbb{1}_{V_2}$ in (1.13). The Fourier multiplier m_μ cuts off (analytically) the ξ_a frequencies (m has to be thought of as $\mathbb{1}_{B_{\mathbb{R}^{n_a}}(0,1)}$). All these cutoff functions are used only with their analytic regularization according to (1.14) with $\lambda = \mu$. They never localize exactly. Using such regularized cutoff functions and Fourier multipliers follows the spirit of analytic semiclassical analysis [Sjö82] (see also [Mar02]). However, we do not make use of that theory and rather construct the relevant mollifiers by hand, making the proof self-contained in this respect.

The proof of estimates like (1.15), stated more precisely in Theorem 3.1, is the object of Section 3. It proceeds in three steps. First, as in the usual proofs of unique continuation results, starting from the hypersurface $\{\phi = 0\}$, one needs to construct a weight function ψ with two properties:

- satisfying the assumptions required to apply the Carleman estimate (ψ should be a strongly pseudoconvex *function* in the sense of Definition 2.1 below);
- having level sets appropriately located with respect to those of ϕ (so that propagating uniqueness across level sets of ψ still corresponds to propagating zero locally from $\phi > 0$ to $\phi < 0$).

This corresponds to the so called "convexification process" [Hör94, Chapter XXVIII].

Second, we apply as a black box the Carleman estimates of [Tat95, RZ98, Hör97, Tat99b] (or some similar ones that we prove in the presence of boundary) to χu, where χ is a particular cutoff function (localizing near the point of interest, and according to level sets of ψ), containing both rough cutoffs and mollified ones. We then need to estimate all terms arising from the commutator $e^{-\frac{\varepsilon}{2\tau}|D_a|^2}e^{\tau\psi}[P,\chi]$, which are either well localized or yield an exponentially small contribution.

Finally, we need to transfer the information given by the Carleman inequality to some estimate like (1.15) on the low frequencies of the function. This is done through a complex analysis argument, the Carleman parameter τ playing the role of complex variable, as in [Tat95]. If ζ is the complex variable, the Carleman estimate corresponds to an estimate for $\zeta = i\tau \in i\mathbb{R}_+$. Combined with a priori estimates, a Phragmén–Lindelöf type theorem allows us to extend this estimate to part of the real domain, where it corresponds to estimating $\|m(\frac{D_a}{\beta\mu})\chi u\|$. To obtain estimates that are uniform with respect to the frequency (and regularization) parameter μ, we also need, following [Tat99a], a scaling argument, replacing τ by τ/μ.

1.3.2. Propagating local information to global one. Once the local estimates are proved, we need to iterate them to obtain a global estimate. This is the object of Section 4. At first, we define some tools that will allow propagating our local estimate (1.15) easily in an abstract way. Estimate (1.15) says essentially that, for a solution of $Pu = 0$, information can be transferred from the support of χ_1 to the support of χ_2. We formalize that with the notion of *zone of dependence*. Roughly speaking, we say that an open set O_2 depends on O_1 if (1.15) holds for every χ_1 equal to 1 on O_1 and any χ_2 supported in O_2. This part allows formulating the proof of Theorem 1.11 as a completely geometric one. Even if quite different in definition, it is close in spirit to the interpolation theory developed by Lebeau [Leb92] to propagate globally the local information obtained by the Cauchy–Kowalevski theorem. Moreover, it should adapt to some more general kinds of foliations. Note that at each step of this propagation argument, we have a loss in the range of frequency: from information on frequencies $\leq \mu$, we obtain from (1.15) information on frequencies $\leq \beta\mu$, with β small. This is overcome by the fact that we only have a finite number of steps in this iterative procedure.

Once this propagation result is obtained, we are left with low frequency information on the solution u. Since we have no information about the high frequency part, the only thing to do is to use some trivial bound of the type

$$\left\|\left(1 - m\left(\frac{D_a}{\mu}\right)\right)u\right\|_{L^2} \leq \frac{C}{\mu^{m-1}}\|u\|_{m-1}.$$

This is actually much worse than the negative exponential that we already had. But it turns out to be the best we can do without any more information.

In Section 6, we specify our general result to the case of the wave and Schrödinger equations. The main task is to construct appropriate noncharacteristic hypersurfaces that fit in the geometric setting of Theorem 1.11. This part is quite classical and was already present for instance in [Leb92]. We recall the argument in the present context.

1.3.3. Carleman estimates for the Dirichlet boundary value problem. Finally, to prove the results of Section 1.1, it remains to deal with the boundary value problem. This is the object of Section 5. As far as (qualitative) unique continuation is concerned, there is no need to prove quantitative estimates up to the boundary. As a consequence, we need here to carry over the analysis of [Tat95, RZ98, Hör97, Tat99b] at the boundary. In this context, we consider only a particular class of operators and a particular boundary condition. We assume that the operator belongs to the class described in Remark 1.10 (hence encompassing wave and Schrödinger type operators), that is, with symbols of the form $p_2(x, \xi) = Q_x(\xi)$ where Q_x is a smooth family of real quadratic forms. We further assume that the analytic variables x_a are tangent to the boundary, and that the functions satisfy *Dirichlet boundary conditions*. Recall that this situation is of particular interest for the wave/Schrödinger equations, for which x_a is the time variable, which is always tangent to the boundary of the cylindrical domain $\mathbb{R}^1_{x_a} \times \mathcal{M}_{x_b}$.

The proofs of the quantitative unique continuation results up to and from the boundary rely on Carleman estimates for these operators at the boundary. As such, the estimates interpolate between the "boundary elliptic Carleman estimates" of Lebeau and Robbiano [LR95], and the "partially analytic Carleman estimates" of Tataru [Tat95] (see also [RZ98, Hör97]). Then, we obtain the counterpart of the local estimate of Theorem 3.1 for this boundary value problem. All local, semiglobal and global results will then follow as in the boundaryless case. We only need to be careful when performing changes of variables.

2. Preliminaries

The preliminary results presented in this section are mainly used in Section 3 for the local estimate. Some are also used independently in Section 4 for the semiglobal estimate. They concern:

- the Carleman estimate adapted to operators with partially analytic coefficients, as stated in [Tat95, RZ98, Hör97, Tat99b];
- the regularization procedure for cutoff functions and Fourier multipliers (which is a key part in the proofs);
- some preliminary commutator-type estimates.

2.1. Notation

First, let us recall basic notation, used all along the article.

Throughout, dist stands for the Euclidean distance in $\mathbb{R}^n$, $\mathbb{R}$ or $\mathbb{R}^{n_a}$, or the Riemannian distance on $(\mathcal{M}, g)$. For $K \subset \mathbb{R}^n$ (resp. $\mathbb{R}$, $\mathbb{R}^{n_a}$) and $d > 0$, we define the *d-neighborhood* of K by

$$\mathrm{Nhd}(K, d) := \bigcup_{x \in K} B(x, d),$$

where the balls are taken according to the distance dist. For open sets U, U', we write $U \Subset U'$ if $\overline{U}$ is compact and $\overline{U} \subset U'$.

We denote by $\mathcal{F}$ the Fourier transform in all variables, and by $\mathcal{F}_a$ that in the variables $x_a \in \mathbb{R}^{n_a}$ only. When no confusion is possible, we shall write $\hat{u} = \mathcal{F}_a(u)$ or $\hat{u} = \mathcal{F}(u)$.

We write $\langle \xi \rangle = (1 + |\xi|^2)^{1/2}$, and denote by $\| \cdot \|_m$ the classical H^m norm on $\mathbb{R}^n$: $\|u\|_m := \|\langle \xi \rangle^m \mathcal{F}(u)\|_{L^2(\mathbb{R}^n)}$. Similarly,

$$\|u\|_{m,\tau} = (2\pi)^{n/2} \|(\tau^2 + |D|^2)^{m/2} u\|_0 = \|(\tau^2 + |\xi|^2)^{m/2} \mathcal{F}(u)\|_0$$

will denote the weighted (semiclassical) H^m norm for $\tau \geq 1$. In the main part of this article, τ will be a large parameter. Finally, we use the notation $\| \cdot \|_{H^k \to H^\ell}$ for the operator norm from $H^k(\mathbb{R}^n)$ to $H^\ell(\mathbb{R}^n)$.

2.2. The Carleman estimate

Before stating the Carleman estimate used in the main part of paper, we need to introduce appropriate weight functions ψ.

Definition 2.1 (Strongly pseudoconvex function). Let Γ be a closed conic subset of $T^*\Omega$, and let P be a principally normal operator in Γ (in the sense of (1.9)), with principal symbol p. Let $\psi \in C^2(\Omega; \mathbb{R})$ and let $x^0 \in \Omega$. We say that ψ is *strongly pseudoconvex* in Γ at x^0 for P if

$$\mathrm{Re}\,\{\overline{p}, \{p, \psi\}\}(x^0, \xi) > 0 \quad \text{if } p(x^0, \xi) = 0 \text{ and } \xi \in \Gamma_{x^0},\ \xi \neq 0, \tag{2.1}$$

$$\frac{1}{i\tau}\{\overline{p}_\psi, p_\psi\}(x^0, \xi) > 0 \quad \text{if } p_\psi(x^0, \xi) = 0 \text{ and } \xi \in \Gamma_{x^0},\ \tau > 0, \tag{2.2}$$

where $p_\psi(x, \xi) = p(x, \xi + i\tau\nabla\psi)$.

Note that in the case $\Gamma = T^*\Omega$, this property is the usual one for proving a Carleman estimate with weight function ψ. It is classical that a strongly pseudoconvex surface S (in the sense of Definition 1.7) is a level surface for some strongly pseudoconvex function in the sense of Definition 2.1 (see e.g. [Hör94, Proposition 28.3.3] or [Tat99a, Theorem 1.5]), and that both definitions are stable with respect to small C^2 perturbations. In what follows, a more precise link (adapted to our needs) between these two notions will be made in Section 3.1.

In this paper (just as in [Tat95, RZ98, Hör97, Tat99b]), Definitions 1.7 and 2.1 will always be used with $\Gamma = \Omega \times \{\xi_a = 0\}$.

For $\varepsilon, \tau > 0$ we define the operator

$$Q^\psi_{\varepsilon,\tau} u = e^{-\frac{\varepsilon}{2\tau}|D_a|^2}(e^{\tau\psi} u), \tag{2.3}$$

introduced in [Tat95].

The following result is due to Tataru [Tat99b, Theorem 2]. A proof in cases (E) and (H) can be found in [Hör97, (5.15), and the last equation before Section 7]. Some closely related estimates are also proved in [RZ98, Proposition 4.6].

In Section 5, when studying the boundary value problem for wave equations, we include a proof of this result in case (H) assuming that P has a real principal part, is of order $m = 2$, and under the additional assumption that the coefficients of P do not depend on x_a.

Theorem 2.2. *Let $x^0 \in \Omega = \Omega_a \times \Omega_b \subset \mathbb{R}^{n_a} \times \mathbb{R}^{n_b}$ and P be a partial differential operator on Ω of order m. Assume that*

- *P is analytically principally normal in $\{\xi_a = 0\}$ inside Ω (in the sense of Definition 1.6);*
- *ψ is a quadratic polynomial in $x = (x_a, x_b)$, strongly pseudoconvex in $\Omega \times \{\xi_a = 0\}$ at x^0 for P (in the sense of Definition 2.1).*

Then there exist $\varepsilon, \mathsf{R}, \mathsf{d}, C, \tau_0 > 0$ *such that* $B(x^0, \mathsf{R}) \subset \Omega$ *and for any* $\tau > \tau_0$, *we have*

$$\tau \| Q^{\psi}_{\varepsilon,\tau} u\|^2_{m-1,\tau} \leq C\big(\|Q^{\psi}_{\varepsilon,\tau} Pu\|_0^2 + \|e^{\tau(\psi-\mathsf{d})} Pu\|_0^2 + \|e^{\tau(\psi-\mathsf{d})} u\|^2_{m-1,\tau}\big) \tag{2.4}$$

for any $u \in C_0^\infty(B(x^0, \mathsf{R}))$.

Note that, compared to usual Carleman estimates of the form (1.12), there are two additional remainder terms in (2.4) due to the introduction of the frequency localization operator $e^{-\frac{\varepsilon}{2\tau}|D_a|^2}$. Moreover, most Carleman estimates in [Tat95, RZ98, Hör97, Tat99b] do not contain the term $\|e^{\tau(\psi-\mathsf{d})} Pu\|_0^2$ on the right hand side. Also, this result was stated in [Tat99b] under the assumption that pseudoconvexity holds on all of Ω. Yet, pseudoconvexity at one point implies pseudoconvexity in a small neighborhood (see [Tat99b, Lemmata 2.5 and 2.6]), so it implies the local Carleman estimate for functions supported close to x^0.

2.3. *Regularization of cutoff functions and Fourier multipliers*

All along the paper, we shall use several cutoff functions and need to regularize them. Here, we explain the regularization procedure we use, give some of its basic properties, and define some (appropriately regularized) Fourier multipliers.

2.3.1. Regularization of functions. Before describing the regularization operators, let us collect some basic facts about Gaussian integrals. Note first that (differentiate with respect to z or see e.g. [Le72, (2.1.7), p. 17]), for $z \geq 0$,

$$\int_z^{+\infty} e^{-s^2}\, ds = \frac{e^{-z^2}}{\sqrt{\pi}} \int_0^{+\infty} \frac{e^{-z^2 s^2}}{1+s^2}\, ds \leq \frac{\sqrt{\pi}}{2} e^{-z^2}.$$

As a consequence,

$$\int_r^{+\infty} e^{-s^2/t}\, ds \leq \frac{\sqrt{\pi}}{2}\sqrt{t}\, e^{-r^2/t}, \qquad \int_r^{+\infty} \langle s\rangle^m e^{-s^2/t} ds \leq C_m \langle r\rangle^m \langle t\rangle^{(m+1)/2} e^{-r^2/t}$$

for all $r \geq 0$, $t > 0$, $m \in \mathbb{N}$, where the second estimate is obtained by iterated integration by parts. Hence,

$$\int_{x_a \in \mathbb{R}^{n_a},\, |x_a| \geq r} e^{-|x_a|^2/t}\, dx_a \leq C_{n_a} \langle r\rangle^{n_a-1} \langle t\rangle^{n_a/2} e^{-r^2/t} \quad \text{for all } r \geq 0,\ t > 0. \tag{2.5}$$

Moreover, for any measurable set $E \subset \mathbb{R}^{n_a}$, any $x_a \in \mathbb{R}^{n_a}$, and any $t > 0$,

$$\int_E e^{-\frac{1}{t}|x_a - y_a|^2}\, dy_a \leq \int_{\mathbb{R}^{n_a}} e^{-\frac{1}{t}|x_a-y_a|^2}\, dy_a = (\pi t)^{n_a/2}.$$

In addition, according to (2.5), there exists $C_{n_a} > 0$ such that for any closed set $E \subset \mathbb{R}^{n_a}$, any $x_a \notin E$, and any $t > 0$, we obtain

$$\begin{aligned}\int_E e^{-|x_a-y_a|^2/t}\, dy_a &\leq \int_{B(x_a,\mathrm{dist}(x_a,E))^c} e^{-|x_a-y_a|^2/t}\, dy_a \\ &\leq C_{n_a}\langle \mathrm{dist}(x_a,E)\rangle^{n_a-1}\langle t\rangle^{n_a/2} e^{-\mathrm{dist}(x_a,E)^2/t}\end{aligned}$$

Hence there exists $C_{n_a} > 0$ such that for any closed set $E \subset \mathbb{R}^{n_a}$, any $x_a \in \mathbb{R}^{n_a}$, and any $t > 0$, we have

$$\int_E e^{-|x_a-y_a|^2/t}\, dy_a \leq C_{n_a}\langle \mathrm{dist}(x_a,E)\rangle^{n_a-1}\langle t\rangle^{n_a/2} e^{-\mathrm{dist}(x_a,E)^2/t}. \tag{2.6}$$

We are now prepared to define the appropriate regularization process, used all along the article. We shall use f_λ to denote

- $f_\lambda := e^{-|D|^2/\lambda} f$ for $f \in L^\infty(\mathbb{R})$;
- or (more often)

$$f_\lambda := e^{-|D_a|^2/\lambda} f$$

  for $f \in L^\infty(\mathbb{R}^n)$, and a fortiori for $f \in L^\infty(\mathbb{R}^{n_a})$.

We hope that this will not be confusing. We now discuss in more detail the basic properties of this regularization process in the second case only (the first case can be seen as the particular situation $n_a = 1, n_b = 0$).

The definition can be rewritten as

$$\begin{aligned}f_\lambda(x_a,x_b) &= \left(\frac{\lambda}{4\pi}\right)^{n_a/2}\left(e^{-\frac{\lambda}{4}|\cdot|^2} *_{\mathbb{R}^{n_a}} f(\cdot,x_b)\right)(x_a) \\ &= \left(\frac{\lambda}{4\pi}\right)^{n_a/2}\int_{\mathbb{R}^{n_a}} f(y_a,x_b) e^{-\frac{\lambda}{4}|x_a-y_a|^2}\, dy_a.\end{aligned}$$

Note that similar smoothing of functions is used systematically in analytic microlocal analysis (see [Sjö82] or [Mar02]). In this context, it is related to the Fourier–Bros–Iagolnitzer transform. In applications to unique continuation, it has been used in [RT73, Ler88, Rob91, Hör92, Leb92, Rob95, Tat95, RZ98, Hör97, Tat99b]. In particular, the operator $Q^\psi_{\varepsilon,\tau}$ defined in (2.3) contains such a regularization (the regularizing parameter λ being linked to the Carleman large parameter τ).

Several times in the proofs we will use

$$\|f_\lambda\|_{L^2(\mathbb{R}^n)} \leq (2\pi)^{-n_a/2}\|e^{-|\cdot|^2/\lambda}\|_{L^\infty(\mathbb{R}^{n_a})}\|\mathcal{F}_a(f)(\xi_a,x_b)\|_{L^2(\mathbb{R}^n)} = \|f\|_{L^2(\mathbb{R}^n)} \tag{2.7}$$

and

$$\|f_\lambda\|_{L^\infty} \leq \left(\frac{\lambda}{4\pi}\right)^{n_a/2}\left\|e^{-\frac{\lambda}{4}|\cdot|^2}\right\|_{L^1(\mathbb{R}^{n_a})}\|f\|_{L^\infty(\mathbb{R}^n)} = \|f\|_{L^\infty(\mathbb{R}^n)}. \tag{2.8}$$

Notice also that

$$f \geq 0 \ \Rightarrow\ f_\lambda \geq 0, \quad \text{and hence} \quad f \geq g \ \Rightarrow\ f_\lambda \geq g_\lambda.$$

Moreover, the function f_λ may be extended as an entire function in the variable x_a by

$$f_\lambda(z_a, x_b) = \left(\frac{\lambda}{4\pi}\right)^{n_a/2} \int_{\mathbb{R}^{n_a}} f(y_a, x_b) e^{-\frac{\lambda}{4}(z_a - y_a)^2}\, dy_a, \qquad z_a \in \mathbb{C}^{n_a},\ x_b \in \mathbb{R}^{n_b},$$

(where $\zeta_a^2 = \zeta_a \cdot \zeta_a = |\operatorname{Re} \zeta_a|^2 - |\operatorname{Im} \zeta_a|^2 + 2i \operatorname{Re} \zeta_a \cdot \operatorname{Im} \zeta_a$ is the real inner product) with the uniform bound

$$\begin{aligned}
|f_\lambda(z_a, x_b)| &\leq \left(\frac{\lambda}{4\pi}\right)^{n_a/2} \|f\|_{L^\infty} \int_{y_a \in \operatorname{supp} f(\cdot, x_b)} \left| e^{-\frac{\lambda}{4}(z_a - y_a)^2} \right| dy_a \\
&\leq \left(\frac{\lambda}{4\pi}\right)^{n_a/2} \|f\|_{L^\infty} e^{\frac{\lambda}{4}|\operatorname{Im} z_a|^2} \int_{y_a \in \operatorname{supp} f(\cdot, x_b)} e^{-\frac{\lambda}{4}|\operatorname{Re} z_a - y_a|^2}\, dy_a \\
&\leq C \langle\lambda\rangle^{n_a/2} \|f\|_{L^\infty} e^{\frac{\lambda}{4}|\operatorname{Im} z_a|^2} \\
&\quad \times \langle \operatorname{dist}(\operatorname{Re} z_a, \operatorname{supp} f(\cdot, x_b))\rangle^{n_a - 1} e^{-\frac{\lambda}{4}\operatorname{dist}(\operatorname{Re} z_a, \operatorname{supp} f(\cdot, x_b))^2} \qquad (2.9)
\end{aligned}$$

where the last estimate comes from (2.6) applied with $t = 4/\lambda$ (observe that $\lambda\langle 1/\lambda\rangle = \langle\lambda\rangle$). Note that $\operatorname{supp} f(\cdot, x_b)$ is well-defined for every $x_b \in \mathbb{R}^{n_b}$ if f is a continuous function; however, strictly speaking, this is not the case if f is only in $L^\infty(\mathbb{R}^n)$. In this situation, $\operatorname{supp} f(\cdot, x_b)$ in (2.9) can simply be replaced by

$$S_f(x_b) := \{x_a \in \mathbb{R}^{n_a} : (x_a, x_b) \in \operatorname{supp} f\} \subset \mathbb{R}^{n_a},$$

where $\operatorname{supp} f \subset \mathbb{R}^n$ is the support of f (in the distributional sense). In case f is continuous, $\operatorname{supp} f(\cdot, x_b) \subset S_f(x_b)$ and both statements are correct (the first one being slightly more precise). We will not discuss this subtlety anymore and will continue to write some expressions similar to (2.9). The estimate then makes sense by taking an element of L^∞ that is zero outside of $\operatorname{supp} f$ and is bounded by $\|f\|_{L^\infty}$.

For functions compactly supported in the x_a variable, we have the simpler estimate

$$|f_\lambda(z_a, x_b)| \leq C\lambda^{n_a/2} \|f\|_{L^\infty} |\operatorname{supp} f(\cdot, x_b)| e^{\frac{\lambda}{4}|\operatorname{Im} z_a|^2} e^{-\frac{\lambda}{4}\operatorname{dist}(\operatorname{Re} z_a, \operatorname{supp} f(\cdot, x_b))^2}. \qquad (2.10)$$

2.3.2. Fourier multipliers. Finally, we also need to introduce frequency localization functions, i.e. appropriately smoothed Fourier multipliers. Let $m(\xi_a)$ be a smooth radial function (i.e. depending only on $|\xi_a|$), compactly supported (in $|\xi_a| < 1$) with values in $[0, 1]$ and such that $m(\xi_a) = 1$ for $|\xi_a| < 3/4$. We shall denote by M^μ the Fourier multiplier $M^\mu u = m(D_a/\mu)u$, that is,

$$(M^\mu u)(x_a, x_b) = \mathcal{F}_a^{-1}\big(m(\xi_a/\mu)\mathcal{F}_a(u)(\xi_a, x_b)\big)(x_a),$$

where $\mathcal{F}_a$ denotes the Fourier transform in the variable x_a only. Given $\lambda, \mu > 0$, we shall denote by M_λ^μ the Fourier multiplier of symbol $m_\lambda^\mu(\xi_a) = m_\lambda(\xi_a/\mu)$, i.e. $M_\lambda^\mu = m_\lambda(D_a/\mu)$ or

$$(M_\lambda^\mu u)(x_a, x_b) = \mathcal{F}_a^{-1}\big(m_\lambda(\xi_a/\mu)\mathcal{F}_a(u)(\xi_a, x_b)\big)(x_a),$$

with, according to the above notation for the subscript λ,

$$m_\lambda(\xi_a) = \left(\frac{\lambda}{4\pi}\right)^{n_a/2} \int_{\mathbb{R}^{n_a}} m(\eta_a) e^{-\frac{\lambda}{4}|\xi_a-\eta_a|^2}\, d\eta_a.$$

Note that in this definition, the symbol is *first regularized and then dilated*. We hope the notation (with the subscript for regularization and the exponent for dilation) will not be confusing. Note also that these Fourier multipliers only act in the variable x_a.

2.4. Some preliminary estimates

In this section, we state several technical lemmata of commutator type, needed to prove the main local result formulated in Theorem 3.1. The proofs can certainly be omitted by the hurried reader. The spirit is that all the estimates that we would expect for exact cutoff functions remain true for their analytically regularized version, up to some exponentially small remainders in terms of λ. So, the important fact in all the estimates below is the uniformity with respect to λ and μ as large parameters.

2.4.1. Some basic preliminary estimates

Lemma 2.3. (1) *For any $d > 0$, there exist $C, c > 0$ such that for any $f_1, f_2 \in L^\infty(\mathbb{R}^n)$ such that* $\operatorname{dist}(\operatorname{supp} f_1, \operatorname{supp} f_2) \geq d$ *and all $\lambda \geq 0$, we have*

$$\|f_{1,\lambda} f_2\|_{L^\infty} \leq Ce^{-c\lambda}\|f_1\|_{L^\infty}\|f_2\|_{L^\infty}, \qquad \|f_{1,\lambda} f_{2,\lambda}\|_{L^\infty} \leq Ce^{-c\lambda}\|f_1\|_{L^\infty}\|f_2\|_{L^\infty}.$$

(2) *If moreover $f_1, f_2 \in C^\infty(\mathbb{R}^n)$ have bounded derivatives, then for all $k \in \mathbb{N}$, there exist $C, c > 0$ such that for all $\lambda \geq 1$, we have*

$$\|f_{1,\lambda} f_2\|_{H^k(\mathbb{R}^n)\to H^k(\mathbb{R}^n)} \leq Ce^{-c\lambda}.$$

(3) *Let $f_1, f_2 \in L^\infty(\mathbb{R}^{n_a})$ with* $\operatorname{dist}(\operatorname{supp} f_1, \operatorname{supp} f_2) > 0$. *Then there exist $C, c > 0$ such that for all $\lambda, \mu \geq 1$ and all $k \in \mathbb{N}$, we have*

$$\|f_{1,\lambda}(D_a/\mu) f_2(D_a/\mu)\|_{H^k(\mathbb{R}^n)\to H^k(\mathbb{R}^n)} \leq Ce^{-c\lambda},$$
$$\|f_{1,\lambda}(D_a/\mu) f_{2,\lambda}(D_a/\mu)\|_{H^k(\mathbb{R}^n)\to H^k(\mathbb{R}^n)} \leq Ce^{-c\lambda}.$$

Proof. We have

$$|f_{1,\lambda}(x_a, x_b)| \leq C\lambda^{n_a/2}\|f_1\|_{L^\infty} \int_{y_a \in \operatorname{supp}_{x_a} f_1(\cdot, x_b)} e^{-\frac{\lambda}{4}|y_a-x_a|^2}\, dy_a.$$

Moreover, for all $x_b \in \mathbb{R}^{n_b}$ we have

$$\operatorname{dist}_{\mathbb{R}^{n_a}}\big(\operatorname{supp}_{x_a} f_1(\cdot, x_b), \operatorname{supp}_{x_a} f_2(\cdot, x_b)\big) \geq \operatorname{dist}(\operatorname{supp} f_1, \operatorname{supp} f_2) \geq d,$$

so that for all $x = (x_a, x_b) \in \operatorname{supp} f_2$, we have $|y_a - x_a| \geq d$ in the above integral. As a consequence, for all $x = (x_a, x_b) \in \operatorname{supp} f_2$,

$$\begin{aligned} |f_{1,\lambda}(x_a, x_b)| &\leq C\lambda^{n_a/2}\|f_1\|_{L^\infty} \int_{|y_a - x_a| \geq d} e^{-\frac{\lambda}{4}|y_a - x_a|^2}\, dy_a \\ &\leq C\|f_1\|_{L^\infty}\lambda^{n_a/2} \int_{|y_a| \geq d} e^{-\frac{\lambda}{4}|y_a|^2}\, dy_a \leq Ce^{-c\lambda}\|f_1\|_{L^\infty}, \end{aligned}$$

which provides the first estimate in (1).

The second estimate is obtained by decomposing

$$f_{1,\lambda} f_{2,\lambda} = f_{1,\lambda} f_{2,\lambda} \mathbb{1}_{\mathrm{Nhd}(\operatorname{supp} f_2, d/3)} + f_{1,\lambda} f_{2,\lambda} \mathbb{1}_{\mathrm{Nhd}(\operatorname{supp} f_2, d/3)^c},$$

and applying the previous result to $f_{1,\lambda}\mathbb{1}_{\mathrm{Nhd}(\operatorname{supp} f_2, d/3)}$ and $f_{2,\lambda}\mathbb{1}_{\mathrm{Nhd}(\operatorname{supp} f_2, d/3)^c}$, where all the supports are disjoint as required.

(2) is proved by induction on $k \in \mathbb{N}$. For $k = 0$, it is precisely the first estimate of (1). Now assume that it holds for $k - 1$ and write $\|f_{1,\lambda} f_2 u\|_{H^k} \leq \|f_{1,\lambda} f_2 u\|_{H^{k-1}} + \|\nabla(f_{1,\lambda} f_2 u)\|_{H^{k-1}}$. It remains to estimate $\|\nabla(f_{1,\lambda} f_2 u)\|_{H^{k-1}}$: for this, it suffices to write

$$\nabla(f_{1,\lambda} f_2 u) = (\nabla f_1)_\lambda f_2 u + f_{1,\lambda}\nabla(f_2)u + f_{1,\lambda} f_2 \nabla(u),$$

where all functions have the appropriate support properties to apply the case $k - 1$. This finally yields $\|\nabla(f_{1,\lambda} f_2 u)\|_{H^{k-1}} \leq Ce^{-c\lambda}\|u\|_{H^{k-1}} + Ce^{-c\lambda}\|\nabla u\|_{H^{k-1}}$ proving (2).

The proof of (3) only relies on the fact that for any $k \in \mathbb{N}$,

$$\|f_{1,\lambda}(D_a/\mu) f_2(D_a/\mu)\|_{H^k(\mathbb{R}^n) \to H^k(\mathbb{R}^n)} = \|f_{1,\lambda}(\xi_a/\mu) f_2(\xi_a/\mu)\|_{L^\infty} = \|f_{1,\lambda} f_2\|_{L^\infty}$$

(and similarly for the other term), and on the use of (1). $\square$

Similarly, we have the following variant.

Lemma 2.4. *Let $f_2 \in C^\infty(\mathbb{R}^n)$ with all derivatives bounded, and $d > 0$. Then for every $k \in \mathbb{N}$, there exist $C, c > 0$ such that for all $f_1 \in H^k(\mathbb{R}^n)$ with $\operatorname{dist}(\operatorname{supp} f_1, \operatorname{supp} f_2) \geq d$ and all $\lambda \geq 0$, we have*

$$\|f_{1,\lambda} f_2\|_{H^k} \leq Ce^{-c\lambda}\|f_1\|_{H^k}.$$

Proof. We have

$$f_{1,\lambda} f_2(x_a, x_b) = \left(\frac{\lambda}{4\pi}\right)^{n_a/2} \int_{\mathbb{R}^{n_a}} f_2(x_a, x_b) f_1(y_a, x_b) e^{-\frac{\lambda}{4}|x_a - y_a|^2}\, dy_a,$$

so that

$$\begin{aligned}|f_{1,\lambda}f_2|(x_a,x_b) &\le \left(\frac{\lambda}{4\pi}\right)^{n_a/2}\int_{|x_a-y_a|\ge d}|f_2(x_a,x_b)f_1(y_a,x_b)|e^{-\frac{\lambda}{4}|x_a-y_a|^2}\,dy_a\\ &\le \|f_2\|_{L^\infty(\mathbb{R}^n)}\left(\frac{\lambda}{4\pi}\right)^{n_a/2}\left(\mathbb{1}_{|\cdot|\ge d}e^{-\frac{\lambda}{4}|\cdot|^2}*_{\mathbb{R}^{n_a}}|f_1|(\cdot,x_b)\right)(x_a).\end{aligned}$$

As a consequence, using the Young inequality, we have

$$\|f_{1,\lambda}f_2\|_{L^2}\le\|f_2\|_{L^\infty(\mathbb{R}^n)}\left(\frac{\lambda}{4\pi}\right)^{n_a/2}\|\mathbb{1}_{|\cdot|\ge d}e^{-\frac{\lambda}{4}|\cdot|^2}\|_{L^1(\mathbb{R}^{n_a})}\|f_1\|_{L^2(\mathbb{R}^n)},$$

and by (2.5) (with $\lambda\langle 1/\lambda\rangle=\langle\lambda\rangle$),

$$\|f_{1,\lambda}f_2\|_{L^2}\le C\langle\lambda\rangle^{n_a/2}e^{-\lambda d^2/4}\|f_2\|_{L^\infty(\mathbb{R}^n)}\|f_1\|_{L^2(\mathbb{R}^n)},$$

which implies the result in the case $k=0$. We obtain the case $k>0$ by differentiating and applying the same result (see e.g. the proof of Lemma 2.3). □

Lemma 2.5. *Let $\psi:\mathbb{R}^n\to\mathbb{R}$ be a C^∞ function, $f_1\in C^\infty(\mathbb{R})$ with bounded derivatives and $f_2\in C_0^\infty(\mathbb{R}^n)$ such that* $\operatorname{dist}(\operatorname{supp} f_1\circ\psi,\operatorname{supp} f_2)>0$ *. Then, for all $k\in\mathbb{N}$, there exist $C,c>0$ such that for all $\lambda>0$, we have*

$$\|f_{1,\lambda}(\psi)f_2\|_{H^k(\mathbb{R}^n)\to H^k(\mathbb{R}^n)}\le Ce^{-c\lambda}.$$

Proof. We prove $\|f_{1,\lambda}(\psi)f_2\|_{L^\infty(\mathbb{R}^n)}\le Ce^{-c\lambda}$, which implies the result for $k=0$. We obtain the case $k>0$ by differentiating and applying the same result (see e.g. the proof of Lemma 2.3).

Since $f_2\in C_0^\infty(\mathbb{R}^n)$, the set $K:=\psi(\operatorname{supp} f_2)$ is a compact subset of $\mathbb{R}$. Moreover, the assumption $\operatorname{dist}(\operatorname{supp} f_1(\psi),\operatorname{supp} f_2)>0$ implies that $\operatorname{dist}(\operatorname{supp} f_1,K)>0$. Indeed, otherwise $\operatorname{supp} f_1\cap\psi(\operatorname{supp} f_2)\neq\emptyset$; taking t in this intersection, there would be $x\in\operatorname{supp} f_2$ such that $\psi(x)=t\in\operatorname{supp} f_1$, i.e. $x\in\operatorname{supp} f_1(\psi)$, which contradicts the assumption. Now, note that $x\in\operatorname{supp} f_2$ implies $\psi(x)\in K$, so that we have the pointwise estimate $|f_2|\le\|f_2\|_{L^\infty}\mathbb{1}_K\circ\psi$ on $\mathbb{R}^n$. As a consequence,

$$\|f_{1,\lambda}(\psi)f_2\|_{L^\infty(\mathbb{R}^n)}\le C\|f_{1,\lambda}(\psi)\mathbb{1}_K(\psi)\|_{L^\infty(\mathbb{R}^n)}\le C\|f_{1,\lambda}\mathbb{1}_K\|_{L^\infty(\mathbb{R})}\le Ce^{-c\lambda},$$

where we have used Lemma 2.3 together with $\operatorname{dist}(\operatorname{supp} f_1,K)>0$. □

Lemma 2.6. *Let $f_1,f_2\in C_0^\infty(\mathbb{R}^n)$ with $f_1=1$ in a neighborhood of* $\operatorname{supp} f_2$*. Then for all $k\in\mathbb{N}$ there exist $C,c>0$ such that for all $\lambda>0$ and all $u\in H^k(\mathbb{R}^n)$, we have*

$$\begin{aligned}\|f_{2,\lambda}\partial^\alpha u\|_0 &\le C\|f_{1,\lambda}u\|_k+Ce^{-c\lambda}\|u\|_k\quad \text{for } |\alpha|\le k,\\ \|f_{2,\lambda}u\|_k &\le C\|f_{1,\lambda}u\|_k+Ce^{-c\lambda}\|u\|_k.\end{aligned}$$

Proof. Let $d = \mathrm{dist}(\mathrm{supp}\, f_2, \mathrm{supp}(1 - f_1)) > 0$. Thanks to Lemma 2.3(1), we have

$$\| f_{2,\lambda} \mathbb{1}_{\mathrm{Nhd}(\mathrm{supp}\, f_2, d/3)^c} \partial^\alpha u\|_0 \leq C e^{-c\lambda} \|u\|_k.$$

Concerning the other term, we use again Lemma 2.3 applied to $\mathbb{1}_{\mathrm{Nhd}(\mathrm{supp}\, f_2, d/3)}$ and some $\partial^\alpha(1 - f_1)$ (using $\partial^\alpha(f_{1,\lambda}) = (\partial^\alpha f_1)_\lambda$) to obtain

$$\begin{aligned}\| f_{2,\lambda} \mathbb{1}_{\mathrm{Nhd}(\mathrm{supp}\, f_2, d/3)} \partial^\alpha u\|_0 &\leq \| f_{2,\lambda} \mathbb{1}_{\mathrm{Nhd}(\mathrm{supp}\, f_2, d/3)} \partial^\alpha (f_{1,\lambda} u)\|_0 \\ &\quad + \| f_{2,\lambda} \mathbb{1}_{\mathrm{Nhd}(\mathrm{supp}\, f_2, d/3)} \partial^\alpha ((1 - f_{1,\lambda}) u)\|_0 \\ &\leq \| f_{2,\lambda} \mathbb{1}_{\mathrm{Nhd}(\mathrm{supp}\, f_2, d/3)} \partial^\alpha (f_{1,\lambda} u)\|_0 + C e^{-c\lambda} \|u\|_k.\end{aligned}$$

Writing then

$$\| f_{2,\lambda} \mathbb{1}_{\mathrm{Nhd}(\mathrm{supp}\, f_2, d/3)} \partial^\alpha (f_{1,\lambda} u)\|_0 \leq C \|\partial^\alpha (f_{1,\lambda} u)\|_0 \leq C \| f_{1,\lambda} u\|_k$$

concludes the proof of the first estimate of the lemma.

The second inequality follows by noticing that $\partial^\alpha(f_{2,\lambda} u)$ is a sum of terms of the form $(\partial^\beta f_2)_\lambda \partial^{\alpha-\beta} u$ to which we can apply the first part of the lemma. ◻

Lemma 2.7. *Assume $m_1, m_2 \in L^\infty(\mathbb{R}^{n_a})$ are bounded by 1, and satisfy*

$$\mathrm{dist}(\mathrm{supp}\, m_1, \mathrm{supp}\, m_2) \geq d > 0.$$

Then there exists $C > 0$ such that for all $f \in L^\infty(\mathbb{R}^{n_b}; L^\infty(\mathbb{R}^{n_a}))$ satisfying $\mathcal{F}_a(f) \in L^\infty(\mathbb{R}^{n_b}; L^1(\mathbb{R}^{n_a}))$ and all $\mu, \lambda > 0$, we have

$$\begin{aligned}&\|m_{1,\lambda}(D_a/\mu) f(x) m_{2,\lambda}(D_a/\mu)\|_{L^2(\mathbb{R}^n)\to L^2(\mathbb{R}^n)} \\ &\qquad \leq \|\mathcal{F}_a(f)\|_{L^\infty_{x_b} L^1(|\xi_a| \geq d\mu/3)} + C e^{-c\lambda} \|\mathcal{F}_a(f)\|_{L^\infty(\mathbb{R}^{n_b}; L^1(\mathbb{R}^{n_a}))},\end{aligned}$$

and the same estimate with m_2 in place of $m_{2,\lambda}$.

Proof. We begin with the first estimate, the second being simpler to handle. We denote $m^\mu_{j,\lambda}(\xi_a) = m_{j,\lambda}(\xi_a/\mu)$ for $j = 1, 2$, and, to lighten notation, set $\hat{f} = \mathcal{F}_a(f)$. We set $f_L = \mathbb{1}_{|D_a| \leq d\mu/3} f$ (that is, $\hat{f}_L(\xi_a) = \mathbb{1}_{|\xi_a| \leq d\mu/3} \hat{f}(\xi_a)$) and $f_H = \mathbb{1}_{|D_a| \geq d\mu/3} f$. We first have

$$\begin{aligned}\|m^\mu_{1,\lambda}(D_a) f_H(x) m^\mu_{2,\lambda}(D_a)\|_{L^2\to L^2} &\leq \| f_H\|_{L^\infty(\mathbb{R}^n)} \leq \|\hat{f}_H\|_{L^\infty(\mathbb{R}^{n_b}; L^1(\mathbb{R}^{n_a}))} \\ &\leq \|\hat{f}\|_{L^\infty_{x_b} L^1(|\xi_a| \geq d\mu/3)}.\end{aligned}$$

It remains to estimate $\|m^\mu_{1,\lambda}(D_a) f_L(x) m^\mu_{2,\lambda}(D_a)\|_{L^2\to L^2}$. We work in the Fourier domain: for $u \in L^2(\mathbb{R}^n)$, we have

$$\mathcal{F}_a(m^\mu_{1,\lambda}(D_a) f_L(x) m^\mu_{2,\lambda}(D_a) u)(\xi_a, x_b) = m^\mu_{1,\lambda}(\xi_a)\big[\hat{f}_L(\xi_a, x_b) * [m^\mu_{2,\lambda}(\xi_a)\hat{u}(\xi_a, x_b)]\big],$$

where $*$ denotes convolution in the variable ξ_a only. Now, we set $\widetilde{m}_1 = \mathbb{1}_{\mathrm{Nhd}(\mathrm{supp}\, m_1, d/3)}$ and $\widetilde{m}_2 = \mathbb{1}_{\mathrm{Nhd}(\mathrm{supp}\, m_2, d/3)}$, which satisfy $\|\widetilde{m}_j\|_{L^\infty} \leq 1$ together with

$$\mathrm{dist}(\mathrm{supp}\, \widetilde{m}_1, \mathrm{supp}\, \widetilde{m}_2) \geq d/3.$$

We write

$$m^{\mu}_{1,\lambda}(\xi_a)\big[\hat f_L(\xi_a, x_b) * \big(m^{\mu}_{2,\lambda}(\xi_a)\hat u(\xi_a, x_b)\big)\big] = Y_1 + Y_2 + Y_3,$$

with

$$\begin{aligned}
Y_1 &= \widetilde m^{\mu}_1 m^{\mu}_{1,\lambda}(\xi_a)\big[\hat f_L(\xi_a, x_b) * \big(\widetilde m^{\mu}_2 m^{\mu}_{2,\lambda}(\xi_a)\hat u(\xi_a, x_b)\big)\big],\\
Y_2 &= (1 - \widetilde m_{1,\mu})m^{\mu}_{1,\lambda}(\xi_a)\big[\hat f_L(\xi_a, x_b) * \big(\widetilde m^{\mu}_2 m^{\mu}_{2,\lambda}(\xi_a)\hat u(\xi_a, x_b)\big)\big],\\
Y_3 &= m^{\mu}_{1,\lambda}(\xi_a)\big[\hat f_L(\xi_a, x_b) * \big((1 - \widetilde m^{\mu}_2)m^{\mu}_{2,\lambda}(\xi_a)\hat u(\xi_a, x_b)\big)\big].
\end{aligned}$$

The term Y_1 vanishes since $\widetilde m^{\mu}_2 m^{\mu}_{2,\lambda}(\xi_a)u(\xi_a, x_b)$ is supported in the set where $\xi_a/\mu \in \mathrm{Nhd}(\operatorname{supp} m_2, d/3)$; hence, as $\operatorname{supp} \hat f_L \subset \{|\xi_a|/\mu \le d/3\}$, the convolution

$$\hat f_L(\xi_a, x_b) * \big(\widetilde m^{\mu}_2 m^{\mu}_{2,\lambda}(\xi_a)u(\xi_a, x_b)\big)$$

is supported in $\xi_a/\mu \in \mathrm{Nhd}(\operatorname{supp} m_2, 2d/3)$ which does not intersect the support (in ξ_a/μ) of $\widetilde m^{\mu}_1$, that is, $\mathrm{Nhd}(\operatorname{supp} m_1, d/3)$.

Concerning Y_2, Lemma 2.3 implies $\|(1-\widetilde m^{\mu}_1)m^{\mu}_{1,\lambda}\|_{L^\infty_{\xi_a}} \le Ce^{-c\lambda}$. This, together with the Young inequality in ξ_a and the uniform boundedness of $\widetilde m^{\mu}_2 m^{\mu}_{2,\lambda}$, yields

$$\begin{aligned}
&\big\|(1 - \widetilde m^{\mu}_1)m^{\mu}_{1,\lambda}(\xi_a)\big[\hat f_L(\xi_a, x_b) * \big(\widetilde m^{\mu}_2 m^{\mu}_{2,\lambda}(\xi_a)\hat u(\xi_a, x_b)\big)\big]\big\|_{L^2(\mathbb{R}^n)}\\
&\quad\le \|(1 - \widetilde m^{\mu}_1)m^{\mu}_{1,\lambda}\|_{L^\infty_{\xi_a}}\|\hat f_L\|_{L^\infty_{x_b}L^1_{\xi_a}}\|\mathcal{F}_a(u)\|_{L^2(\mathbb{R}^{n_a}\times\mathbb{R}^{n_b})} \le Ce^{-c\lambda}\|\hat f\|_{L^\infty_{x_b}L^1_{\xi_a}}\|u\|_{L^2(\mathbb{R}^n)}.
\end{aligned}$$

The term Y_3 is treated similarly and the proof of the first estimate is complete.

The second estimate has the same proof and is actually simpler because the term $(1 - \widetilde m^{\mu}_2)m^{\mu}_2$ is zero. □

Lemma 2.8. *Assume $f_1, f_2 \in L^\infty(\mathbb{R}^n)$ are bounded by 1 and satisfy*

$$\operatorname{dist}(\operatorname{supp} f_1, \operatorname{supp} f_2) \ge d > 0.$$

Then there exists $C > 0$ such that for all $m \in L^\infty(\mathbb{R}^{n_a})$ satisfying $\hat m \in L^1(\mathbb{R}^{n_a})$ and all $\lambda > 0$, we have

$$\|f_{1,\lambda}(x)m(D_a)f_{2,\lambda}(x)\|_{L^2(\mathbb{R}^n)\to L^2(\mathbb{R}^n)} \le \|\hat m\|_{L^1(|\eta_a|\ge d/3)} + Ce^{-c\lambda}\|\hat m\|_{L^1(\mathbb{R}^{n_a})},$$

and the same estimate with $f_{2,\lambda}$ replaced by f_2.

Proof. This is essentially the same proof as for the previous lemma except that we have to take care of the fact that the functions f_i depend on all variables, while m only depends on $x_a \in \mathbb{R}^{n_a}$. Again, we set $m_L = \mathbb{1}_{|D_a|\le d/3}m$ (that is, $\hat m_L(\eta_a) = \mathbb{1}_{|\eta_a|\le d/3}\hat m(\eta_a)$) and $m_H = \mathbb{1}_{|D_a|\ge d/3}m$. First, we have

$$\begin{aligned}
\|f_{1,\lambda}(x)m_H(D_a)f_{2,\lambda}(x)\|_{L^2(\mathbb{R}^n)\to L^2(\mathbb{R}^n)} &\le \|m_H(D_a)\|_{L^2(\mathbb{R}^n)\to L^2(\mathbb{R}^n)} \le \|m_H\|_{L^\infty(\mathbb{R}^{n_a})}\\
&\le \|\hat m_H\|_{L^1(\mathbb{R}^{n_a})} \le \|\hat m\|_{L^1(|\eta_a|\ge d/3)}.
\end{aligned}$$

Concerning the second term, and denoting $\check{m}_L = \mathcal{F}_a^{-1}(m_L)$, i.e. $\check{m}_L(\eta_a) = (2\pi)^{-n_a}\hat{m}_L(-\eta_a)$, we have

$$f_{1,\lambda}(x)m_L(D_a)f_{2,\lambda}(x)u = f_{1,\lambda}(x)\check{m}_L *_{\mathbb{R}^{n_a}} \big(f_{2,\lambda}(\cdot, x_b)u(\cdot, x_b)\big).$$

Now we can finish the proof as in the previous lemma: introducing $\tilde{f}_j := \mathbb{1}_{\mathrm{Nhd}(\operatorname{supp} f_j, d/3)}$, $j = 1, 2$, we notice that

$$\begin{aligned}\operatorname{supp}(\check{m}_L *_{\mathbb{R}^{n_a}} [\tilde{f}_2 f_{2,\lambda}u]) &\subset \mathrm{Nhd}(\operatorname{supp} f_2, d/3) + \{(x_a, 0) : |x_a| \le d/3\}\\ &\subset \mathrm{Nhd}(\operatorname{supp} f_2, 2d/3).\end{aligned}$$

Moreover, Lemma 2.3 still yields

$$\|(1 - \tilde{f}_j)f_{j,\lambda}\|_{L^\infty(\mathbb{R}^n)} \le Ce^{-c\lambda}, \qquad j = 1, 2,$$

so that the proof then follows exactly that of Lemma 2.7. We obtain the second inequality similarly. □

Lemma 2.9. *Let $k \in \mathbb{N}$ and $f \in C_0^\infty(\mathbb{R}^n)$. Then there exist C, c such that, for any $\lambda, \mu > 0$, we have*

$$\begin{aligned}\|M_\lambda^\mu f_\lambda(1 - M_\lambda^{2\mu})\|_{H^k(\mathbb{R}^n)\to H^k(\mathbb{R}^n)} &\le Ce^{-c\mu^2/\lambda} + Ce^{-c\lambda},\\ \|(1 - M_\lambda^{2\mu})f_\lambda M_\lambda^\mu\|_{H^k(\mathbb{R}^n)\to H^k(\mathbb{R}^n)} &\le Ce^{-c\mu^2/\lambda} + Ce^{-c\lambda}.\end{aligned}$$

Recall that the Fourier multipliers M_λ^μ are defined in Section 2.3.2.

Proof. Note first that $\mathcal{F}_a(\partial_{x_a}^\alpha \partial_{x_b}^\beta f_\lambda)(\xi_a, x_b) = (i\xi_a)^\alpha e^{-|\xi_a|^2/\lambda}\partial_{x_b}^\beta \mathcal{F}_a(f)(\xi_a, x_b)$. Hence, for $k = 0$, the result is a direct consequence of (the first estimate in) Lemma 2.7 since $\operatorname{supp} m \subset \{|\xi_a| \le 1\}$ and $\operatorname{supp}(1 - m(\cdot/2)) \subset \{|\xi_a| \ge 3/2\}$. Note that we also use the fact that $(1 - m)_\lambda = 1 - m_\lambda$.

For $k \ge 1$, we proceed by induction, noticing that

$$\nabla[(1 - M_\lambda^{2\mu})f_\lambda M_\lambda^\mu u] = (1 - M_\lambda^{2\mu})(\nabla f)_\lambda M_\lambda^\mu u + (1 - M_\lambda^{2\mu})f_\lambda M_\lambda^\mu \nabla u$$

(see e.g. the proof of Lemma 2.3). □

Lemma 2.10. *Let $f_1, f_2 \in C^\infty(\mathbb{R}^n)$ be bounded together with all their derivatives, with* $\operatorname{dist}(\operatorname{supp} f_1, \operatorname{supp} f_2) \ge d > 0$. *Then for every $k \in \mathbb{N}$, there exist $C, c > 0$ such that for all $\mu, \lambda > 0$, we have*

$$\begin{aligned}\|f_{1,\lambda}M_\lambda^\mu f_{2,\lambda}\|_{H^k(\mathbb{R}^n)\to H^k(\mathbb{R}^n)} &\le Ce^{-c\mu^2/\lambda} + Ce^{-c\lambda},\\ \|f_{1,\lambda}M_\lambda^\mu f_2\|_{H^k(\mathbb{R}^n)\to H^k(\mathbb{R}^n)} &\le Ce^{-c\mu^2/\lambda} + Ce^{-c\lambda}.\end{aligned}$$

Proof. We first prove both estimates for $k = 0$, by using Lemma 2.8 with m replaced by $m_b = m_\lambda(\cdot/\mu)$. The Fourier transform of m_b is given by

$$\hat{m}_b(\eta_a) = \mu^{n_a}\mathcal{F}_a(m_\lambda)(\mu\eta_a) = \mu^{n_a}e^{-|\eta_a|^2\mu^2/\lambda}\hat{m}(\mu\eta_a).$$

As a consequence,

$$\|\hat{m}_b\|_{L^1(|\eta_a|\geq d/3)} \leq e^{-\frac{d^2\mu^2}{9\lambda}}\|\hat{m}\|_{L^1(\mathbb{R}^{n_a})}$$

and $\|\hat{m}_b\|_{L^1(\mathbb{R}^{n_a})} \leq \|\hat{m}\|_{L^1(\mathbb{R}^{n_a})}$, so that

$$\|\hat{m}_b\|_{L^1(|\eta_a|\geq d/3)} + Ce^{-c\lambda}\|\hat{m}_b\|_{L^1(\mathbb{R}^{n_a})} \leq Ce^{-\frac{d^2\mu^2}{9\lambda}} + Ce^{-c\lambda}.$$

Lemma 2.8 then yields the sought result for $k = 0$.

Again, for $k \geq 1$, we argue by induction noticing that

$$\nabla[f_{1,\lambda}M_\lambda^\mu f_{2,\lambda}u] = (\nabla f_1)_\lambda M_\lambda^\mu f_{2,\lambda}u + f_{1,\lambda}M_\lambda^\mu(\nabla f_2)_\lambda u + f_{1,\lambda}M_\lambda^\mu f_{2,\lambda}\nabla u,$$

and using the fact that the relevant support properties of ∇f_i are preserved (see e.g. the proof of Lemma 2.3). □

Lemma 2.11. *Let $k \in \mathbb{N}$ and $f \in C_0^\infty(\mathbb{R}^n)$. Then there exist $C, c > 0$ such that for all $\mu, \lambda > 0$ and $u \in H^k(\mathbb{R}^n)$, we have*

$$\|M_\lambda^\mu f_\lambda u\|_k \leq \|f_\lambda M_\lambda^{2\mu}u\|_k + C\big(e^{-c\mu^2/\lambda} + e^{-c\lambda}\big)\|u\|_k. \tag{2.11}$$

Moreover, for any $f_1 \in C^\infty(\mathbb{R}^n)$ bounded together with all its derivatives, such that $f_1 = 1$ on a neighborhood of supp f, *for any $k \in \mathbb{N}$, there exist $C, c > 0$ such that for all $\mu, \lambda > 0$ and $u \in H^k(\mathbb{R}^n)$, we have*

$$\|f_\lambda M_\lambda^\mu u\|_k \leq C\|M_\lambda^\mu f_{1,\lambda}u\|_k + C\big(e^{-c\mu^2/\lambda} + e^{-c\lambda}\big)\|u\|_k. \tag{2.12}$$

Proof. We write

$$\|M_\lambda^\mu f_\lambda u\|_k \leq \|M_\lambda^\mu f_\lambda M_\lambda^{2\mu}u\|_k + \|M_\lambda^\mu f_\lambda(1 - M_\lambda^{2\mu})u\|_k.$$

According to Lemma 2.9, we have $\|M_\lambda^\mu f_\lambda(1-M_\lambda^{2\mu})u\|_k \leq C\big(e^{-c\mu^2/\lambda}+e^{-c\lambda}\big)\|u\|_k$. The first term above is simply estimated by $\|M_\lambda^\mu f_\lambda M_\lambda^{2\mu}u\|_k \leq \|f_\lambda M_\lambda^{2\mu}u\|_k$, which proves (2.11).

Concerning the “moreover” part, we write

$$\|f_\lambda M_\lambda^\mu u\|_k \leq \|f_\lambda M_\lambda^\mu f_{1,\lambda}u\|_k + \|f_\lambda M_\lambda^\mu(1 - f_1)_\lambda u\|_k.$$

For the first term, we only have to remark that $\|f_\lambda M_\lambda^\mu f_{1,\lambda}u\|_k \leq C\|M_\lambda^\mu f_{1,\lambda}u\|_k$ uniformly in λ. Then, since dist(supp f, supp $1 - f_1$) > 0, Lemma 2.10 yields

$$\|f_\lambda M_\lambda^\mu(1 - f_1)_\lambda u\|_k \leq C\big(e^{-c\mu^2/\lambda} + e^{-c\lambda}\big)\|u\|_k,$$

which eventually proves (2.12). □

Lemma 2.12. *Let $k \in \mathbb{N}$ and $f \in C_0^\infty(\mathbb{R}^n)$. Assume* $\operatorname{supp} f \subset \bigcup_{i\in I} U_i$ *where $(U_i)_{i\in I}$ is a finite family of bounded open sets. Let $b_i \in C_0^\infty(\mathbb{R}^n)$ with $b_i = 1$ on a neighborhood of $\overline{U_i}$. Then, for any $k \in \mathbb{N}$, there exist $C, c > 0$ such that for all $\mu, \lambda > 0$ and $u \in H^k(\mathbb{R}^n)$, we have*

$$\|M_\lambda^\mu f_\lambda u\|_k \le C \sum_{i\in I} \|M_\lambda^{2\mu}(b_i)_\lambda u\|_k + C\big(e^{-c\mu^2/\lambda} + e^{-c\lambda}\big)\|u\|_k.$$

Proof. Applying the first statement of Lemma 2.11 to f, we obtain

$$\|M_\lambda^\mu f_\lambda u\|_k \le \|f_\lambda M_\lambda^{2\mu} u\|_k + C\big(e^{-c\mu^2/\lambda} + e^{-c\lambda}\big)\|u\|_k. \tag{2.13}$$

Let now $(f_i)_{i\in I}$ be a smooth partition of unity such that

$$\sum_{i\in I} f_i = 1 \text{ in a neighborhood of } \operatorname{supp} f, \qquad \operatorname{supp} f_i \subset U_i, \qquad 0 \le f_i \le 1.$$

Note that in particular $b_i = 1$ in a neighborhood of $\operatorname{supp} f_i$. Using the second estimate of Lemma 2.6, we have

$$\begin{aligned}\|f_\lambda M_\lambda^{2\mu} u\|_k &\le C\Big\|\sum_i (f_i)_\lambda M_\lambda^{2\mu} u\Big\|_k + Ce^{-c\lambda}\|M_\lambda^{2\mu} u\|_k \\ &\le C\sum_i \|(f_i)_\lambda M_\lambda^{2\mu} u\|_k + Ce^{-c\lambda}\|u\|_k. \end{aligned}\tag{2.14}$$

Using the second estimate in Lemma 2.11, we then obtain

$$\|(f_i)_\lambda M_\lambda^{2\mu} u\|_k \le C\|M_\lambda^{2\mu}(b_i)_\lambda u\|_k + C\big(e^{-c\mu^2/\lambda} + e^{-c\lambda}\big)\|u\|_k,$$

which, combined with (2.13) and (2.14) concludes the proof of the lemma. □

Lemma 2.13. *There exists $C > 0$ such that for all $D \in \mathbb{R}$ and $\widetilde{\chi} \in L^\infty(\mathbb{R})$ such that* $\operatorname{supp} \widetilde{\chi} \subset (-\infty, D]$*, and all $\lambda, \tau > 0$, we have*

$$|e^{\tau z}\widetilde{\chi}_\lambda(z)| \le C\|\widetilde{\chi}\|_{L^\infty(\mathbb{R})}\langle\lambda\rangle^{1/2} e^{\frac{\lambda}{4}|\operatorname{Im} z|^2} e^{D\tau} e^{\tau^2/\lambda} \qquad \textit{for all } z \in \mathbb{C},$$

$$\|e^{\tau\psi}\widetilde{\chi}_\lambda(\psi)\|_{L^\infty(\mathbb{R}^n)} \le C\|\widetilde{\chi}\|_{L^\infty(\mathbb{R})}\langle\lambda\rangle^{1/2} e^{D\tau} e^{\tau^2/\lambda} \qquad \textit{for all } \psi \in C^0(\mathbb{R}^n;\mathbb{R}).$$

Proof. First, according to (2.9), we have the estimate

$$|\widetilde{\chi}_\lambda(z)| \le C\|\widetilde{\chi}\|_{L^\infty(\mathbb{R})}\langle\lambda\rangle^{1/2} e^{\frac{\lambda}{4}|\operatorname{Im} z|^2} e^{-\frac{\lambda}{4}\operatorname{dist}(\operatorname{Re} z, \operatorname{supp}\widetilde{\chi})^2} \qquad \text{for all } z \in \mathbb{C}.$$

Now, if $\operatorname{Re} z \le D$, we use the bound $|e^{\tau z}| \le e^{D\tau}$, which yields

$$|e^{\tau z}\widetilde{\chi}_\lambda(z)| \le C\|\widetilde{\chi}\|_{L^\infty(\mathbb{R})}\langle\lambda\rangle^{1/2} e^{\frac{\lambda}{4}|\operatorname{Im} z|^2} e^{D\tau}.$$

Next, for $\operatorname{Re} z \ge D$, we have $\operatorname{dist}(\operatorname{Re} z, \operatorname{supp}\widetilde{\chi}) \ge \operatorname{Re} z - D \ge 0$, and

$$\begin{aligned}|e^{\tau z}\widetilde{\chi}_\lambda(z)| &\le e^{\tau \operatorname{Re} z} C\|\widetilde{\chi}\|_{L^\infty(\mathbb{R})}\langle\lambda\rangle^{1/2} e^{\frac{\lambda}{4}|\operatorname{Im} z|^2} e^{-\frac{\lambda}{4}(\operatorname{Re} z - D)^2} \\ &\le C\|\widetilde{\chi}\|_{L^\infty(\mathbb{R})}\langle\lambda\rangle^{1/2} e^{\frac{\lambda}{4}|\operatorname{Im} z|^2} \sup_{s\ge D} e^{\tau s} e^{-\frac{\lambda}{4}(D-s)^2}.\end{aligned}$$

Finally,

$$\sup_{s\geq D} e^{\tau s}e^{-\frac{\lambda}{4}(D-s)^2} = \sup_{t\geq 0} e^{\tau(D+t)}e^{-\frac{\lambda}{4}t^2} = e^{\tau D}\sup_{t\geq 0} e^{t(\tau-\lambda t/4)} = e^{D\tau}e^{\tau^2/\lambda},$$

which concludes the proof of the first estimate of the lemma. The second estimate follows from the first for $z = s \in \mathbb{R}$ combined with

$$\|e^{\tau\psi}\widetilde{\chi}_\lambda(\psi)\|_{L^\infty(\mathbb{R}^n)} \leq \|e^{\tau s}\widetilde{\chi}_\lambda(s)\|_{L^\infty(\mathbb{R})}. \qquad \square$$

Lemma 2.14. *There exist C, c such that, for any $\varepsilon, \tau, \lambda, \mu > 0$ and any $k \in \mathbb{N}$, we have*

$$\left\| e^{-\frac{\varepsilon|D_a|^2}{2\tau}}(1 - M_\lambda^\mu)\right\|_{H^k(\mathbb{R}^n)\to H^k(\mathbb{R}^n)} \leq e^{-\frac{\varepsilon\mu^2}{8\tau}} + Ce^{-c\lambda}.$$

Proof. Since the operator $e^{-\frac{\varepsilon|D_a|^2}{2\tau}}(1 - M_\lambda^\mu)$ is a Fourier multiplier, it suffices to estimate $\sup_{\xi_a\in\mathbb{R}^{n_a}}\left|e^{-\frac{\varepsilon|\xi_a|^2}{2\tau}}(1 - m_\lambda(\xi_a/\mu))\right|$. Recall that $m \in C_0^\infty(\mathbb{R}^{n_a}; [0, 1])$ is a radial function that we identify with a function $m = m(s) \in C_0^\infty(\mathbb{R}_+)$ satisfying $\operatorname{supp} m \subset [0, 1)$ and $m = 1$ on $[0, 3/4)$. We distinguish the following two cases:

- if $|s| \leq \mu/2$, Lemma 2.3 applied with $f_1 = 1 - m$ (and hence $f_{1,\lambda}(s) = 1 - m_\lambda(s)$) and $f_2 = \mathbb{1}_{|s|\leq 1/2}$ implies $|\mathbb{1}_{|s|\leq\mu/2}(1 - m_\lambda(s/\mu))| \leq Ce^{-c\lambda}$ uniformly with respect to $\lambda, \mu > 0$;
- if $|s| \geq \mu/2$, we simply have $\left|\mathbb{1}_{|s|\geq\mu/2}e^{-\frac{\varepsilon|s|^2}{2\tau}}(1 - m_\lambda(s/\mu))\right| \leq e^{-\frac{\varepsilon\mu^2}{8\tau}}$.

Combining these two estimates concludes the proof. $\square$

2.4.2. Some more involved preliminary estimates. We will need the estimate of the following lemma.

Lemma 2.15. *Let ψ be a smooth real valued function on $\mathbb{R}^n$, which is a quadratic polynomial in $x_a \in \mathbb{R}^{n_a}$, let $R_\sigma > 0$, and $\sigma \in C_0^\infty(B_{\mathbb{R}^n}(0, R_\sigma))$. Let $\chi \in C_0^\infty(\mathbb{R})$ with* $\operatorname{supp}\chi \subset (-\infty, 1)$*, and $\tilde{\chi} \in C^\infty(\mathbb{R})$ with $\tilde{\chi} = 1$ on a neighborhood of $(-\infty, 3/2)$ and* $\operatorname{supp}\tilde{\chi} \subset (-\infty, 2)$*. Define $\chi_\delta(s) := \chi(s/\delta)$, $\tilde{\chi}_\delta(s) := \tilde{\chi}(s/\delta)$. Let $f \in C_0^\infty(\mathbb{R}^n)$ be* real analytic *in the variable x_a in a neighborhood of $\overline{B}_{\mathbb{R}^n}(0, R_\sigma)$ and set*

$$g := e^{\tau\psi}\chi_{\delta,\lambda}(\psi)\tilde{\chi}_\delta(\psi)f\sigma_\lambda \in C_0^\infty(\mathbb{R}^n).$$

Then there exist $c_0, c_1 > 0$ such that for all $N \in \mathbb{N}$ and $\beta \in \mathbb{N}^{n_b}$, there exists $C > 0$ such that for all $\delta > 0$, there is $\varepsilon_0 > 0$ such that for any $\lambda \geq 1$, $\tau > 0$, and $0 < \varepsilon < \varepsilon_0$, we have

$$\begin{aligned}|\partial_{x_b}^\beta\mathcal{F}_a(g)(\xi_a, x_b)| \leq{}& C\langle\xi_a\rangle^{-N}(\tau + \delta^{-1} + 1)^{N+|\beta|}\lambda^{(n_a+1)/2}e^{\delta\tau}\\ &\times\left(e^{\tau^2/\lambda}e^{c_1\varepsilon^2\lambda}e^{-c_0\varepsilon|\xi_a|} + e^{\tau^2/\lambda}e^{-c_0\lambda} + e^{c_1\lambda\varepsilon^2}e^{\delta\tau}e^{-c_0\delta^2\lambda}\right).\end{aligned}$$

In particular, for all $\delta > 0$, $N \in \mathbb{N}$, $\beta \in \mathbb{N}^{n_b}$, there are $C, c, \varepsilon_0 > 0$ such that for any $\lambda, \tau \geq 1$ and $0 < \varepsilon < \varepsilon_0$, we have

$$|\partial_{x_b}^\beta\mathcal{F}_a(g)(\xi_a, x_b)| \leq C\langle\xi_a\rangle^{-N}\tau^{N+|\beta|}\lambda^{(n_a+1)/2}e^{\delta\tau}e^{\tau^2/\lambda}\left(e^{C\varepsilon^2\lambda}e^{-c\varepsilon|\xi_a|} + e^{\delta\tau}e^{-c\lambda}\right).$$

Proof. First, we prove the result for $N = 0$ and $\beta = 0$ (the other cases will be obtained by differentiating g).

Also, we notice that since the regularization of σ only occurs in the variable x_a, we have $\operatorname{supp} g \subset \operatorname{supp} \sigma_\lambda \subset \mathbb{R}^{n_a} \times \overline{B}_{\mathbb{R}^{n_b}}(0, R_\sigma)$. Hence, estimates are only needed in the region $x_b \in K_b := \overline{B}_{\mathbb{R}^{n_b}}(0, R_\sigma)$.

Since f is real analytic in x_a in a neighborhood of the compact set $\overline{B}_{\mathbb{R}^n}(0, R_\sigma)$, there exists $R_f > 0$ such that f can be extended in an analytic way in a neighborhood of $z_a \in \overline{B}_{\mathbb{R}^{n_a}}(0, R_a(x_b)) + i\overline{B}_{\mathbb{R}^{n_a}}(0, R_f)$, uniformly for $x_b \in K_b$, where we have set $R_a(x_b)^2 = (R_\sigma + R_f)^2 - |x_b|^2$. Note that z_a denotes the complex variable associated to x_a.

Notice also that we can extend $\tilde{\chi}$ by 1 (hence analytically) on a neighborhood of $(-\infty, 3/2) + i\mathbb{R}$. Moreover, since ψ is quadratic in x_a, there exists $\varepsilon_0 = \varepsilon_0(\delta) > 0$ such that

$$\big(\psi(\operatorname{Re} z_a, x_b) \le \tfrac{4}{3}\delta < \tfrac{3}{2}\delta,\ |\operatorname{Im} z_a| \le \varepsilon_0 R_f,\ x_b \in K_b\big) \Rightarrow \operatorname{Re}\psi(z_a, x_b) \le \tfrac{3}{2}\delta, \quad (2.15)$$

$$\big(\psi(\operatorname{Re} z_a, x_b) = \tfrac{4}{3}\delta,\ |\operatorname{Im} z_a| \le \varepsilon_0 R_f,\ x_b \in K_b\big) \Rightarrow \operatorname{Re}\psi(z_a, x_b) \ge \tfrac{5}{4}\delta. \quad (2.16)$$

In particular, $\tilde{\chi}(\psi(z_a, x_b)) = 1$ on

$$\big(\psi(\operatorname{Re} z_a, x_b) \le \tfrac{4}{3}\delta < \tfrac{3}{2}\delta,\ |\operatorname{Im} z_a| \le \varepsilon_0 R_f,\ x_b \in K_b\big).$$

As a consequence, given $x_b \in K_b$, the function

$$z_a \mapsto \chi_{\delta,\lambda}(\psi(z_a, x_b))\tilde{\chi}_\delta(\psi(z_a, x_b))$$

is analytic on a neighborhood of $\big\{x_a \in \mathbb{R}^{n_a} : \psi(x_a, x_b) \le \frac{4}{3}\delta\big\} + i\overline{B}_{\mathbb{R}^{n_a}}(0, \varepsilon_0 R_f)$. Hence, for $x_b \in K_b$, $z_a \mapsto g(z_a, x_b)$ is holomorphic in a neighborhood of

$$\mathcal{A}_{x_b}(\varepsilon_0) := \Big(\big\{\psi(x_a, x_b) \le \tfrac{4}{3}\delta\big\} \cap \overline{B}_{\mathbb{R}^{n_a}}(0, R_a(x_b))\Big) + i\overline{B}_{\mathbb{R}^{n_a}}(0, \varepsilon_0 R_f).$$

The plan of the proof is to first estimate g in the complex domain, and then bound its Fourier transform using a complex deformation. We use the analyticity inside of $\mathcal{A}_{x_b}(\varepsilon_0)$ and the smallness elsewhere on the real domain.

Step 1: uniform estimates of g. We separately estimate the functions $f\sigma_\lambda$ and $e^{\tau\psi}\chi_{\delta,\lambda}(\psi)\tilde{\chi}_\delta(\psi)$, and then deduce estimates for g.

According to the basic estimate (2.10) for σ_λ, we have, uniformly for $x_b \in K_b$,

$$|(f\sigma_\lambda)(z_a, x_b)| \le C\lambda^{n_a/2} e^{\frac{\lambda}{4}|\operatorname{Im} z_a|^2} e^{-\frac{\lambda}{4}\operatorname{dist}(\operatorname{Re} z_a, \operatorname{supp}\sigma(\cdot, x_b))^2},$$
$$z_a \in B_{\mathbb{R}^{n_a}}(0, R_a(x_b)) + iB_{\mathbb{R}^{n_a}}(0, R_f),$$

where the constant C depends only on $\|f\|_{L^\infty}$ (on the previous complex domain) and $\|\sigma\|_{L^\infty}$.

In particular, for any $\varepsilon \in [0, 1]$, uniformly for $x_b \in K_b$,

$$|(f\sigma_\lambda)(z_a, x_b)| \le C\lambda^{n_a/2} e^{\frac{\lambda}{4}\varepsilon^2 R_f^2}, \qquad z_a \in B_{\mathbb{R}^{n_a}}(0, R_a(x_b)) + iB_{\mathbb{R}^{n_a}}(0, \varepsilon R_f). \quad (2.17)$$

We now notice that

$$\operatorname{dist}(x_a, \operatorname{supp}\sigma(\cdot, x_b)) \geq \operatorname{dist}((x_a, x_b), B(0, R_\sigma)) \geq R_f \quad \text{for } |x_a| \geq R_a(x_b). \tag{2.18}$$

As a first consequence, $\operatorname{dist}(x_a, \operatorname{supp}\sigma(\cdot, x_b)) \geq R_f$ if $|x_a| = R_a(x_b)$, so that for any $\varepsilon \in [0, 1]$ we obtain, uniformly for $x_b \in K_b$,

$$\begin{aligned}|(f\sigma_\lambda)(z_a, x_b)| \leq C\lambda^{n_a/2} e^{\frac{\lambda}{4}\varepsilon^2 R_f^2} 4e^{-\frac{\lambda}{4}R_f^2} \leq C\lambda^{n_a/2} e^{\frac{\lambda}{4}(\varepsilon^2-1)R_f^2},\\ |\operatorname{Im} z_a| \leq \varepsilon R_f,\ |\operatorname{Re} z_a| = R_a(x_b).\end{aligned} \tag{2.19}$$

Using now estimate (2.10) for σ_λ on the real domain, together with the boundedness of f and (2.18), we obtain, uniformly for $x_b \in K_b$,

$$\begin{aligned}|(f\sigma_\lambda)(x_a, x_b)| &\leq C\lambda^{n_a/2} e^{-\frac{\lambda}{4}\operatorname{dist}(x_a, \operatorname{supp}\sigma(\cdot, x_b))^2}\\ &\leq C\lambda^{n_a/2} e^{-\frac{\lambda}{4}R_f^2}, \quad x_a \in \mathbb{R}^{n_a},\ |x_a| \geq R_a(x_b).\end{aligned} \tag{2.20}$$

We now estimate the term $e^{\tau\psi}\chi_{\delta,\lambda}(\psi)\tilde{\chi}_\delta(\psi)$ in parts of the complex domain.
First, on the real domain, we have

$$|e^{\tau s}\chi_{\delta,\lambda}(s)\tilde{\chi}_\delta(s)| \leq e^{2\delta\tau}|\chi_{\delta,\lambda}(s)\tilde{\chi}_\delta(s)| \leq C\lambda^{1/2}e^{2\delta\tau}e^{-c\delta^2\lambda}, \quad s \geq \tfrac{4}{3}\delta,$$

after having used (2.6), where c is a numerical constant. As a consequence,

$$|e^{\tau\psi(x_a,x_b)}\chi_{\delta,\lambda}(\psi(x_a, x_b))\tilde{\chi}_\delta(\psi(z_a, x_b))| \leq C\lambda^{1/2}e^{2\delta\tau}e^{-c\delta^2\lambda} \quad \text{if } \psi(x_a, x_b) \geq \tfrac{4}{3}\delta. \tag{2.21}$$

Next, for $z \in \mathbb{C}$, by Lemma 2.13, there is $C > 0$ such that for all $\delta \in \mathbb{R}$ and all $\lambda \geq 1$, $\tau > 0$, we have

$$|e^{\tau z}\chi_{\delta,\lambda}(z)| \leq C\lambda^{1/2}e^{\frac{\lambda}{4}(\operatorname{Im} z)^2}e^{\delta\tau}e^{\tau^2/\lambda} \quad \text{for all } z \in \mathbb{C}. \tag{2.22}$$

Since ψ is a quadratic polynomial in x_a with real coefficients, we have

$$|\operatorname{Im}\psi(z_a, x_b)| \leq C|\operatorname{Re} z_a|\,|\operatorname{Im} z_a| + C(K_b)|\operatorname{Im} z_a|, \quad (z_a, x_b) \in \mathbb{C}^{n_a} \times K_b,$$

where we have used the fact that K_b is compact. As a consequence, there is a constant $C_0 = C_0(\psi, R_\sigma, R_f) > 0$ such that

$$|\operatorname{Im}\psi(z_a, x_b)| \leq \varepsilon C_0 \quad \text{for } z_a \in B_{\mathbb{R}^{n_a}}(0, R_a(x_b)) + iB_{\mathbb{R}^{n_a}}(0, \varepsilon R_f),\ x_b \in K_b.$$

Hence, using (2.22), we obtain, for all $\varepsilon \in (0, \varepsilon_0)$,

$$\begin{aligned}|e^{\tau\psi(z_a,x_b)}\chi_{\delta,\lambda}(\psi(z_a, x_b))\tilde{\chi}_\delta(\psi(z_a, x_b))| \leq C\lambda^{1/2}e^{\lambda C_0^2\varepsilon^2/4}e^{\delta\tau}e^{\tau^2/\lambda},\\ x_b \in K_b,\ z_a \in \mathcal{A}_{x_b}(\varepsilon).\end{aligned} \tag{2.23}$$

According to (2.9), we also have

$$|\chi_{\delta,\lambda}(z)| \le C\lambda^{1/2} e^{\frac{\lambda}{4}|\operatorname{Im} z|^2} e^{-\frac{\lambda}{4}\operatorname{dist}(\operatorname{Re} z, \operatorname{supp}\chi_\delta)^2}$$
$$\le C\lambda^{1/2} e^{\frac{\lambda}{4}|\operatorname{Im} z|^2} e^{-c\delta^2\lambda} \quad \text{on } \operatorname{Re} z \ge \tfrac{5}{4}\delta,$$

where c is a numerical constant. Using (2.16) yields

$$|\chi_{\delta,\lambda}(\psi(z_a, x_b))| \le C\lambda^{1/2} e^{C_0^2\varepsilon^2\lambda/4} e^{-c\delta^2\lambda},$$
$$x_b \in K_b,\ z_a \in \mathcal{A}_{x_b}(\varepsilon),\ \psi(\operatorname{Re} z_a, x_b) = \tfrac{4}{3}\delta,$$

and, with (2.15), this implies

$$|e^{\tau\psi(z_a,x_b)}\chi_{\delta,\lambda}(\psi(z_a, x_b))\tilde{\chi}_\delta(\psi(z_a, x_b))| \le C\lambda^{1/2} e^{C_0^2\varepsilon^2\lambda/4} e^{3\delta\tau/2} e^{-c\delta^2\lambda},$$
$$x_b \in K_b, z_a \in \mathcal{A}_{x_b}(\varepsilon),\ \psi(\operatorname{Re} z_a, x_b) = \tfrac{4}{3}\delta. \tag{2.24}$$

Let us finally gather all estimates obtained on the function g. Multiplying (2.23) by (2.17) and (2.19), we find that there is a constant $C_1 > 0$ independent of $\lambda, \mu, \tau, \delta, \varepsilon$ such that, for any $\varepsilon \in (0, \varepsilon_0)$,

$$|g(z_a, x_b)| \le C\lambda^{(n_a+1)/2} e^{C_1\lambda\varepsilon^2} e^{\delta\tau} e^{\tau^2/\lambda}, \quad x_b \in K_b,\ z_a \in \mathcal{A}_{x_b}(\varepsilon), \tag{2.25}$$
$$|g(z_a, x_b)| \le C\lambda^{(n_a+1)/2} e^{\lambda(-R_f^2/4 + C_1\varepsilon^2)} e^{\delta\tau} e^{\tau^2/\lambda},$$
$$x_b \in K_b,\ z_a \in \mathcal{A}_{x_b}(\varepsilon),\ |\operatorname{Re} z_a| = R_a(x_b). \tag{2.26}$$

Next, multiplying (2.24) and (2.17) we also have

$$|g(z_a, x_b)| \le C\lambda^{(n_a+1)/2} e^{C_1\varepsilon^2\lambda} e^{3\delta\tau/2} e^{-c\delta^2\lambda},$$
$$x_b \in K_b, z_a \in \mathcal{A}_{x_b}(\varepsilon),\ \psi(\operatorname{Re} z_a, x_b) = \tfrac{4}{3}\delta. \tag{2.27}$$

Combining (2.20) with (2.22), and rewriting (2.21), we also have on the real domain

$$|g(x_a, x_b)| \le C\lambda^{(n_a+1)/2} e^{\delta\tau} e^{\tau^2/\lambda} e^{-\frac{\lambda}{4}R_f^2}, \quad x_a \in \mathbb{R}^{n_a},\ |x_a| \ge R_a(x_b),\ x_b \in K_b, \tag{2.28}$$

$$|g(x_a, x_b)| \le C\lambda^{1/2} e^{2\delta\tau} e^{-c\delta^2\lambda}, \quad x_a \in \mathbb{R}^{n_a},\ x_b \in \mathbb{R}^{n_b},\ \psi(x_a, x_b) \ge \tfrac{4}{3}\delta. \tag{2.29}$$

Step 2: estimating the Fourier transform using a deformation of contour in the complex domain. We now want to estimate $\mathcal{F}_a(g)(\xi_a, x_b)$ uniformly with respect to x_b. We split the integral as

$$\mathcal{F}_a(g)(\xi_a, x_b) = \int_{\mathbb{R}^{n_a}} e^{-ix_a\cdot\xi_a} g(x_a, x_b)\, dx_a = I_0 + I_1 + I_2$$

with $I_j = I_j(\xi_a, x_b)$ defined by

$$I_0 := \int_{|x_a|\le R_a(x_b),\ \psi(x_a,x_b)\le\frac{4}{3}\delta}, \qquad I_1 := \int_{|x_a|\le R_a(x_b),\ \psi(x_a,x_b)>\frac{4}{3}\delta}, \qquad I_2 := \int_{|x_a|>R_a(x_b)}.$$

Using (2.28) together with the compactness of $\operatorname{supp} g \subset \operatorname{supp} f$, we obtain, for all $\delta, \tau > 0$ and $\lambda > 1$,

$$|I_2| \leq C\lambda^{(n_a+1)/2} e^{\delta\tau} e^{\tau^2/\lambda} e^{-R_f^2\lambda/4}. \tag{2.30}$$

Using (2.29), we obtain, for all $\delta, \tau > 0$ and $\lambda > 1$,

$$|I_1| \leq C\lambda^{1/2} e^{2\delta\tau} e^{-c\delta^2\lambda}. \tag{2.31}$$

We now want to estimate the integral $I_0(\xi_a, x_b)$: for $\xi_a \in \mathbb{R}^{n_a} \setminus \{0\}$, we write $x_a = x_1 \frac{\xi_a}{|\xi_a|} + x_a'$ for $x_1 = x_a \cdot \frac{\xi_a}{|\xi_a|}$ and x_a' such that $x_a' \cdot \xi_a = 0$ and make the (ξ_a-dependent) orthogonal change of coordinates to (x_1, x_a') (preserving the ball $B_{\mathbb{R}^{n_a}}(0, R_a(x_b))$). This yields

$$\begin{aligned} I_0(\xi_a, x_b) &= \int_{B_{\mathbb{R}^{n_a}}(0, R_a(x_b)) \cap \{\psi(\cdot, x_b) \geq \frac{4}{3}\delta\}} e^{-ix_1|\xi_a|} g(x_1, x_a')\, dx_a'\, dx_1 \\ &= \int_{B_{\mathbb{R}^{n_a-1}}(0, R_a(x_b))} \mathcal{I}_{\xi_a, x_b}(x_a')\, dx_a', \end{aligned}$$

with

$$\mathcal{I}_{\xi_a, x_b}(x_a') = \int_{|x_1|^2 \leq R_a(x_b)^2 - |x_a'|^2,\ \psi(x_1, x_a', x_b) \leq \frac{4}{3}\delta} e^{-ix_1|\xi_a|} g(x_1, x_a')\, dx_1,$$

so that

$$|I_0(\xi_a, x_b)| \leq C \sup_{x_a' \in B_{\mathbb{R}^{n_a-1}}(0, R_a(x_b))} |\mathcal{I}_{\xi_a, x_b}(x_a')|.$$

Hence, it only remains to estimate $|\mathcal{I}_{\xi_a, x_b}(x_a')|$ uniformly. Now, g being analytic in a neighborhood of $\mathcal{A}_{x_b}(\varepsilon_0)$, and given any $x_a' \in B_{\mathbb{R}^{n_a-1}}(0, R_a(x_b))$, the function $z_1 \mapsto e^{-iz_1|\xi_a|} g(z_1, x_a')$ is holomorphic in a neighborhood of the set

$$|\operatorname{Re} z_1|^2 \leq R_a(x_b)^2 - |x_a'|^2, \quad \psi(\operatorname{Re} z_1, x_a', x_b) \leq \tfrac{4}{3}\delta, \quad |\operatorname{Im} z_1| \leq \varepsilon R_f,$$

for $\varepsilon \in (0, \varepsilon_0)$.

Now, we have

$$\left\{x_1 \in \mathbb{R} : |x_1|^2 \leq R_a(x_b)^2 - |x_a'|^2,\ \psi(x_1, x_a', x_b) \leq \tfrac{4}{3}\delta\right\} = \bigcup_{k \in J} [\alpha_k^1, \alpha_k^2], \tag{2.32}$$

where $J = J(x_a', x_b)$ has 0, 1 or 2 elements since ψ is quadratic. Moreover,

$$\text{either} \quad |\alpha_k^i|^2 + |x_a'|^2 = R_a(x_b)^2, \quad \text{or} \quad \psi(\alpha_k^i, x_a', x_b) = \tfrac{4}{3}\delta \tag{2.33}$$

for $k \in J$ and $i = 1, 2$, together with

$$\mathcal{I}_{\xi_a, x_b}(x_a') = \sum_{k \in J} \int_{[\alpha_k^1, \alpha_k^2]} e^{-ix_1|\xi_a|} g(x_1, x_a')\, dx_1.$$

To estimate $\mathcal{I}_{\xi_a,x_b}(x_a')$, we now make a change of contour in the complex variable z_1 as follows:

$$\int_{[\alpha_k^1,\alpha_k^2]} e^{-ix_1|\xi_a|} g(x_1, x_a')\, dx_1 = I_L + I_T + I_R,$$

where

$$I_\star = \int_{\gamma_\star} e^{-iz_1|\xi_a|} g(z_1, x_a')\, dz_1 \qquad \text{for } \star = L, T, R,$$

and

$$\begin{aligned}\gamma_L &= [\alpha_k^1, \alpha_k^1 - i\varepsilon R_f],\\ \gamma_T &= [\alpha_k^1 - i\varepsilon R_f, \alpha_k^2 - i\varepsilon R_f],\\ \gamma_R &= [\alpha_k^2 - i\varepsilon R_f, \alpha_k^2]\end{aligned}$$

are three oriented segments in $\mathbb{C}$ (see Figure 2). We have

$$|I_\star| \le \int_{\gamma_\star} e^{\mathrm{Im}(z_1)|\xi_a|} |g(z_1, x_a')|\, dz_1 \qquad \text{for } \star = L, T, R.$$

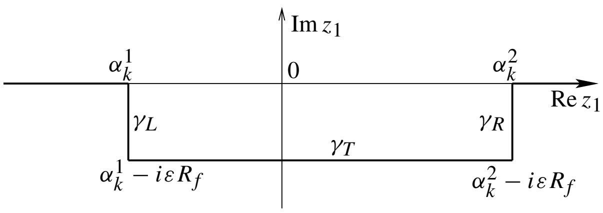

Fig. 2. Oriented contours.

On γ_L and γ_R, using (2.33) and $\mathrm{Im}\, z_1 \le 0$, we can use either estimate (2.26) or (2.27) and obtain, uniformly in $x_a', \xi_a, x_b, \delta, \tau > 0, \lambda > 1$, and $\varepsilon \in (0, \varepsilon_0(\delta))$,

$$|I_L| + |I_R| \le C\varepsilon\lambda^{(n_a+1)/2} e^{C_1\lambda\varepsilon^2}\big(e^{\delta\tau} e^{-\lambda R_f^2/4} e^{\tau^2/\lambda} + e^{3\delta\tau/2} e^{-c\delta^2\lambda}\big).$$

On γ_T, we have $(z_1, x_a') \in \mathcal{A}_{x_b}(\varepsilon)$ and $\mathrm{Im}\, z_1 = -\varepsilon R_f$, and thus using (2.25) we obtain, uniformly in $x_a', \xi_a, x_b, \delta, \tau > 0, \lambda > 1$, and $\varepsilon \in (0, \varepsilon_0(\delta))$,

$$|I_T| \le C\lambda^{(n_a+1)/2} e^{C_1\lambda\varepsilon^2} e^{\delta\tau} e^{\tau^2/\lambda} e^{-\varepsilon R_f|\xi_a|}.$$

Combining the estimates on I_L, I_R, I_T now proves that there is $C > 0$ such that for any $\xi_a \in \mathbb{R}^{n_a} \setminus \{0\}$ (and, by continuity, for all $\xi_a \in \mathbb{R}^{n_a}$), $x_b \in \mathbb{R}^{n_b}$, $\delta, \tau > 0$, $\lambda > 1$, and $\varepsilon \le \min\big(\varepsilon_0(\delta), \frac{R_f}{2\sqrt{2C_1}}\big)$,

$$|I_0| \le C\lambda^{(n_a+1)/2} e^{\delta\tau} e^{\tau^2/\lambda}\big(e^{C_1\lambda\varepsilon^2} e^{-\varepsilon R_f|\xi_a|} + e^{-R_f^2\lambda/8}\big) + C\lambda^{(n_a+1)/2} e^{C_1\lambda\varepsilon^2} e^{3\delta\tau/2} e^{-c\delta^2\lambda},$$

which, in view of estimates (2.30) and (2.31), implies the result for $N = 0$ and $\alpha = 0$.

To obtain the result for $N \in \mathbb{N}$ and $\beta \in \mathbb{N}^{n_b}$, we notice that the functions $g_{\alpha,\beta} = \partial_{x_a}^{\alpha}\partial_{x_b}^{\beta} g$ can be written as a finite sum of terms that have the same form as in one of the assumptions of the theorem with some different f, b and χ_δ (with the same support and analyticity properties) and with powers of $\tau^{\alpha'}\delta^{-\beta'}$ for $|\alpha'|+|\beta'| \leq |\alpha|+|\beta|$. The constants in the exponentials do not depend on α, β since they are functions of ψ, R_σ, R_f, K_b only. Noting that $(i\xi_a)^\alpha \partial_{x_b}^{\beta}\mathcal{F}_a(g)(\xi_a, x_b) = \mathcal{F}_a(\partial_{x_a}^{\alpha}\partial_{x_b}^{\beta} g)(\xi_a, x_b)$ concludes the proof. □

Remark 2.16. It is likely that the above lemma works as well if ψ is any real valued function, analytic (or at least polynomial) in the variable x_a (and not only polynomial of degree 2). The main point is to have uniform bounds of the form $\psi(z_a, x_b) = \psi(\operatorname{Re} z_a, x_b) + O(\varepsilon)$ for $(\operatorname{Re} z_a, x_b)$ in a compact set, and $|\operatorname{Im} z_a| \leq \varepsilon R_f$. This could be done by Taylor expansion. Moreover, the decomposition (2.32) in a finite number of intervals should still be possible using analyticity in x_a. This generalization is however not needed below since all weight functions ψ will be quadratic polynomials in x_a (and, most of the time, in all variables).

As a consequence of the previous result, we now have the following lemma.

Lemma 2.17. *Under the assumptions of Lemma 2.15, for all $k \in \mathbb{N}$, $\delta > 0$, there exist $N \in \mathbb{N}$ and $C, c_0, \varepsilon_0 > 0$ such that for any $\lambda, \mu, \tau \geq 1$ and $0 < \varepsilon < \varepsilon_0$, we have*

$$\|M_\lambda^{\mu/2} g(1-M_\lambda^\mu)\|_{H^k(\mathbb{R}^n)\to H^k(\mathbb{R}^n)} \leq C\tau^N \lambda^{(n_a+1)/2} e^{\delta\tau} e^{\tau^2/\lambda}\big(e^{C\varepsilon^2\lambda}e^{-c_0\varepsilon\mu} + e^{\delta\tau}e^{-c_0\lambda}\big),$$
$$\|(1-M_\lambda^\mu) g M_\lambda^{\mu/2}\|_{H^k(\mathbb{R}^n)\to H^k(\mathbb{R}^n)} \leq C\tau^N \lambda^{(n_a+1)/2} e^{\delta\tau} e^{\tau^2/\lambda}\big(e^{C\varepsilon^2\lambda}e^{-c_0\varepsilon\mu} + e^{\delta\tau}e^{-c_0\lambda}\big).$$

The estimates of this lemma will only be used in a weaker form: for all $c, \delta > 0$ and $k \in \mathbb{N}$, there exist $c_0, C, N > 0$ such that for any $\tau, \mu \geq 1$ and $c^{-1}\mu \leq \lambda \leq c\mu$, we have

$$\|M_\lambda^{\mu/2} g(1-M_\lambda^\mu)\|_{H^k(\mathbb{R}^n)\to H^k(\mathbb{R}^n)} \leq C\tau^N e^{\tau^2/\lambda} e^{2\delta\tau} e^{-c_0\mu}, \tag{2.34}$$

with the same estimate for the second term. This form is obtained by taking ε sufficiently small in the regime $c^{-1}\mu \leq \lambda \leq c\mu$.

Proof of Lemma 2.17. The two estimates are proved the same way, so we only prove the first one. First, since M_λ^μ is a Fourier multiplier, hence commutes with differentiation, for any α with $|\alpha| \leq k$, the derivative $\partial^\alpha[M_\lambda^{\mu/2} g(1-M_\lambda^\mu)u]$ is a sum of terms of the form $M_\lambda^{\mu/2}(\partial^\beta g)(1-M_\lambda^\mu)(\partial^\gamma u)$ with $|\beta| + |\gamma| = |\alpha| \leq k$. Hence, Lemma 2.7 gives

$$\begin{aligned}\|M_\lambda^{\mu/2} g(1-M_\lambda^\mu)\|_{H^k(\mathbb{R}^n)\to H^k(\mathbb{R}^n)} \leq C \sum_{|\alpha|+|\beta|\leq k} &\|\xi_a^\alpha \partial_{x_b}^\beta \mathcal{F}_a(g)\|_{L^\infty_{x_b}L^1(|\xi_a|\geq d\mu/3)} \\ &+ Ce^{-c\lambda}\|\xi_a^\alpha \partial_{x_b}^\beta \mathcal{F}_a(g)\|_{L^\infty(\mathbb{R}^{n_b};L^1(\mathbb{R}^{n_a}))}.\end{aligned}$$

Next, Lemma 2.15 with $N \in \mathbb{N}$ so large that $\langle \xi_a\rangle^{-(N+k)}$ is integrable on $\mathbb{R}^{n_a}$ yields

$$\begin{aligned}&\|M_\lambda^{\mu/2} g(1-M_\lambda^\mu)\|_{H^k(\mathbb{R}^n)\to H^k(\mathbb{R}^n)} \\ &\qquad\leq C\tau^{N+k}\lambda^{(n_a+1)/2} e^{\delta\tau} e^{\tau^2/\lambda}\big(e^{c_1\varepsilon^2\lambda}e^{-c_0\varepsilon\mu} + e^{\delta\tau}e^{-c_0\lambda}\big),\end{aligned}$$

which concludes the proof. □

3. The local estimate

The aim of this section is to prove the local quantitative uniqueness result, (analytically) localized in frequency in the analytic variables.

In the following, for any $R > 0$, we shall denote

$$\sigma_R(x) := \sigma(R^{-1}|x - x^0|) \text{ with } \sigma \in C^\infty(\mathbb{R}) \text{ such that}$$
$$\sigma = 1 \text{ in a neighborhood of } (-\infty, 1], \text{ and } \sigma = 0 \text{ in a neighborhood of } [2, +\infty). \tag{3.1}$$

Our main local theorem is the following. See Figure 3 for the geometry of the theorem. An important feature of this local result is that it can be iterated and hence propagated.

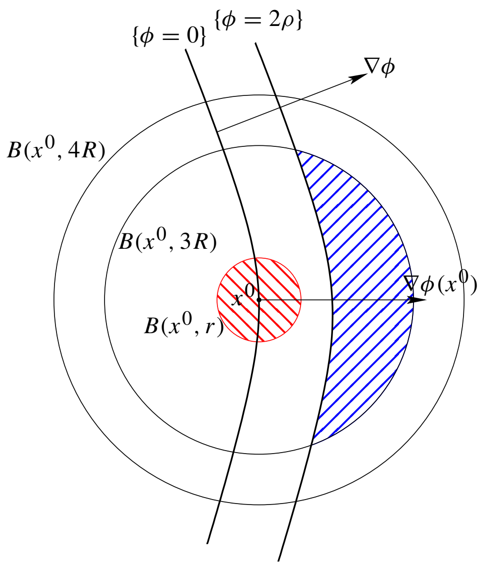

Fig. 3. Geometry of the local uniqueness result. The blue (darker) striped region is the observation region (i.e. where $\vartheta = 1$). The red (lighter) striped region is the observed region (i.e. where $\sigma_r = 1$).

Theorem 3.1. *Let $x^0 \in \Omega \subset \mathbb{R}^{n_a} \times \mathbb{R}^{n_b}$ and P be a partial differential operator on Ω of order m. Assume that*

- *P is analytically principally normal in $\{\xi_a = 0\}$ inside Ω (in the sense of Definition 1.6);*
- *there is a function ϕ defined in a neighborhood of x^0 such that $\phi(x^0) = 0$, and $\{\phi = 0\}$ is a C^2 strongly pseudoconvex oriented surface in $\{\xi_a = 0\}$ at x^0 for P (in the sense of Definition 1.7).*

Then there exists $R_0 > 0$ such that for any $R \in (0, R_0)$ there exist $r, \rho, \tilde{\tau}_0 > 0$ such that for any $\vartheta \in C_0^\infty(\mathbb{R}^n)$ with $\vartheta(x) = 1$ on a neighborhood of $\{\phi \geq 2\rho\} \cap B(x^0, 3R)$, for all $c_1, \kappa > 0$ there exist $C, \kappa', \beta_0 > 0$ such that for all $\beta \leq \beta_0$, we have

$$\|M^{\beta\mu}_{c_1\mu}\sigma_{r,c_1\mu}u\|_{m-1} \leq Ce^{\kappa\mu}\big(\|M^{\mu}_{c_1\mu}\vartheta_{c_1\mu}u\|_{m-1} + \|Pu\|_{L^2(B(x^0,4R))}\big) + Ce^{-\kappa'\mu}\|u\|_{m-1}$$

for all $\mu \geq \tilde{\tau}_0/\beta$ and $u \in C_0^\infty(\mathbb{R}^n)$.

Note that this local result contains in particular the unique continuation result for operators with partially analytic coefficients [Tat95, RZ98, Hör97, Tat99b] (which it is aimed to quantify). The latter is proved by letting $\mu \to +\infty$ in the estimate (and controlling some error terms), yielding: $[Pu = 0 \text{ on } B(x^0, 4R)),\ u = 0 \text{ on } \operatorname{supp} \vartheta] \Rightarrow u = 0$ on $B(x^0, r) \subset \{\sigma_r = 1\}$.

This theorem allows one to systematically quantify this local unique continuation result under partial analyticity conditions (in a way that can be iterated/propagated). As such, it also allows one in particular to systematically quantify both the Hörmander and the Holmgren theorems (again, in a way that can be iterated/propagated). Let us briefly comment on these two extreme situations: $n_a = 0$ (Hörmander case) and $n_a = n$ (Holmgren case).

Remark 3.2. If $n_a = 0$, this inequality takes the form (see also (1.13) in the introduction and the associated discussion):

$$\|\sigma_r u\|_{m-1} \le C e^{\kappa\mu}\big(\|\vartheta u\|_{m-1} + \|Pu\|_{L^2(B(x^0,4R))}\big) + Ce^{-\kappa'\mu}\|u\|_{m-1} \quad \text{for all } \mu \ge \mu_0,$$

or equivalently

$$\|\sigma_r u\|_{m-1} \le C\frac{1}{\varepsilon^{\kappa/\kappa'}}\big(\|\vartheta u\|_{m-1} + \|Pu\|_{L^2(B(x^0,4R))}\big) + C\varepsilon\|u\|_{m-1} \quad \text{for all } \varepsilon \le \varepsilon_0,$$
$$\|\sigma_r u\|_{m-1} \le C\big(\|\vartheta u\|_{m-1} + \|Pu\|_{L^2(B(x^0,4R))}\big)^{\delta}\|u\|_{m-1}^{1-\delta} \quad \text{for some } \delta \in (0,1).$$

This last estimate is an interpolation inequality of Lebeau–Robbiano type [Rob95, LR95], and, as such, propagates well. Here it quantifies the general situation of the Hörmander theorem (see also [Bah87]).

If $n_a = n$, we here describe a systematic way to quantify the Holmgren theorem, which propagates well. See also [Joh60] for a local result and [Leb92] for a global result for waves.

Remark 3.3. The inequality of Theorem 3.1 can be written in the following way: For all $(D, \mu, u) \in \mathbb{R}_+ \times [\tilde{\tau}_0/\beta, +\infty) \times H^{m-1}(\mathbb{R}^n)$ satisfying

$$\|M^{\mu}_{c_1\mu}\vartheta_{c_1\mu} u\|_{m-1} \le e^{-(\kappa+\kappa')\mu}D, \qquad \|Pu\|_{L^2(B(x^0,4R))} \le e^{-(\kappa+\kappa')\mu}D,$$

we have

$$\|M^{\beta\mu}_{c_1\mu}\sigma_{r,c_1\mu} u\|_{m-1} \le C' e^{-\kappa'\mu}(D + \|u\|_{m-1}).$$

This could certainly be written in the framework of propagation of (semiclassical, partially analytic) microsupport with respect to the variable x_a (see [Sjö82] or [Mar02, Section 3.2]). If $n_a = n$, it seems related to microlocal proofs of the Holmgren theorem and the propagation of the analytic wavefront set (see [Sjö82]).

The proof of Theorem 3.1 is divided into three steps, given in Sections 3.1–3.3 respectively.

3.1. Step 1: Geometric setting

The following lemma is a refined version of [RZ98, Lemma 4.1, p. 514] or [Hör97, Lemmata 4.3 and 4.4]. Its proof essentially follows that of [RZ98, Lemma 4.1]. We state the geometric part for balls that are not necessarily Euclidean. This will be useful in Section 5, where we study the boundary value problem and need to make changes of variable.

Lemma 3.4. *Let P be analytically principally normal in $\{\xi_a = 0\}$ inside $\Omega \subset \mathbb{R}^n$, of order m and with principal symbol p. Let $\phi \in C^2(\Omega; \mathbb{R})$ and $S = \{\phi = 0\}$ be a C^2 oriented hypersurface in Ω. Let $x^0 \in S \cap \Omega$ with $\nabla\phi(x^0) \neq 0$. Assume that S is strongly pseudoconvex in $\Omega \times \{\xi_a = 0\}$ at x^0 for P (in the sense of Definition 1.7). Then there exists $A > 0$ such that the function*

$$\psi(x) := (x-x^0)\cdot\nabla\phi(x^0)+A((x-x^0)\cdot\nabla_x\phi(x^0))^2+\frac{1}{2}\phi''(x^0)(x-x^0, x-x^0)-\frac{1}{A}|x-x^0|^2$$

satisfies:

(1) $\psi(x^0) = 0\,,\ \nabla_x\psi(x^0) = \nabla_x\phi(x^0)$*;*
(2) *ψ is strongly pseudoconvex in $\Omega\cap\{\xi_a = 0\}$ at x^0 for P (in the sense of Definition 2.1);*
(3) *let N be a distance function locally equivalent to the Euclidean distance; then there exists $R_0 > 0$ such that for any $R \in (0, R_0)$, there exists $\eta_0 > 0$ and for any $0 < \eta < \eta_0$ and any $\eta_1, \eta_2 > 0$ there exist $\rho, r > 0$ such that*

$$\big(\{\phi \le \rho\} \cap \{\psi \ge -\eta\} \cap B_N(x^0, R)\big) \subset B_N(x^0, R/8), \tag{3.2}$$
$$\big(\{\psi \ge \eta_1\} \cap \overline{B}_N(x^0, R)\big) \subset \{\phi > \rho\}, \tag{3.3}$$
$$B_N(x^0, r) \subset \{-\eta_2 < \psi < \eta_2\}. \tag{3.4}$$

In this statement, the B_N stand for balls with respect to the distance N. Conditions (3.2)–(3.4) are illustrated in Figure 4.

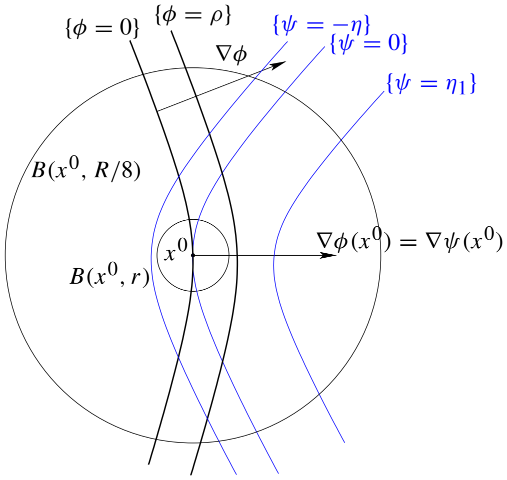

Fig. 4. Local geometry of the level sets of the convexified function ψ (in case N is the Euclidean distance).

Proof of Lemma 3.4. (1) directly follows from the definition of ψ as a second order perturbation of the Taylor expansion of ϕ at x^0.

The proof of the pseudoconvexity in (2) is very similar to that of [RZ98, Lemma 4.1] or [Hör97, Lemma 7.4]. We sketch it for completeness.

Let us compute

$$\operatorname{Re}\{\overline{p},\{p,\psi\}\} = \operatorname{Re}\left(\frac{\partial^2 p}{\partial\xi\partial x}\left[\frac{\partial\bar{p}}{\partial\xi};\nabla\psi\right] + \psi''_{xx}\left[\frac{\partial\bar{p}}{\partial\xi};\frac{\partial p}{\partial\xi}\right] - \frac{\partial^2 p}{\partial\xi^2}\left[\frac{\partial\bar{p}}{\partial x};\nabla\psi\right]\right).$$

Since $\nabla\psi(x^0) = \nabla\phi(x^0)$, we have

$$\begin{aligned}\operatorname{Re}\{\overline{p},\{p,\psi\}\}(x^0,\xi) &= \operatorname{Re}\{\overline{p},\{p,\phi\}\}(x^0,\xi)\\ &\quad + 2A\left|\nabla_x\phi(x^0)\cdot\frac{\partial p}{\partial\xi}(x^0,\xi)\right|^2 - \frac{2}{A}\left|\frac{\partial p}{\partial\xi}(x^0,\xi)\right|^2.\end{aligned}$$

In this identity, all terms are homogeneous of degree $2m-2$ in the variable ξ, so it is enough to prove the estimate for $\xi\in\mathbb{S}^{n-1}$. Hence, applying Lemma A.1 below on the compact set $K = \{\xi\in\mathbb{S}^{n-1} : \xi_a = 0,\ p(x^0,\xi) = 0\}$, together with the first part of the pseudoconvexity assumption, yields, for A large enough,

$$\operatorname{Re}\{\overline{p},\{p,\psi\}\}(x^0,\xi) > 0 \quad \text{if } p(x^0,\xi) = 0 \text{ and } \xi_a = 0,\ \xi_b \neq 0. \tag{3.5}$$

For the second estimate, we compute

$$\begin{aligned}&\frac{1}{i}\{\overline{p}_\phi, p_\phi\}(x,\xi)\\ &= \frac{1}{i}\left(\frac{\partial\bar{p}}{\partial\xi}(x,\xi{-}i\tau\nabla\phi)\frac{\partial p}{\partial x}(x,\xi{+}i\tau\nabla\phi){+}i\tau\phi''_{xx}\left[\frac{\partial\bar{p}}{\partial\xi}(x,\xi{-}i\tau\nabla\phi);\frac{\partial p}{\partial\xi}(x,\xi{+}i\tau\nabla\phi)\right]\right)\\ &\quad -\frac{1}{i}\left(\frac{\partial\bar{p}}{\partial x}(x,\xi{-}i\tau\nabla\phi)\frac{\partial p}{\partial\xi}(x,\xi{+}i\tau\nabla\phi){-}i\tau\phi''_{xx}\left[\frac{\partial\bar{p}}{\partial\xi}(x,\xi{-}i\tau\nabla\phi);\frac{\partial p}{\partial\xi}(x,\xi{+}i\tau\nabla\phi)\right]\right)\\ &= C_{\tau,\phi,1}(x,\xi){+}C_{\tau,\phi,2}(x,\xi)\end{aligned}$$

with

$$\begin{aligned}C_{\tau,\phi,1}(x,\xi) &:= \frac{1}{i}\left(\frac{\partial\bar{p}}{\partial\xi}(x,\bar{\zeta})\frac{\partial p}{\partial x}(x,\zeta) - \frac{\partial\bar{p}}{\partial x}(x,\bar{\zeta})\frac{\partial p}{\partial\xi}(x,\zeta)\right),\\ C_{\tau,\phi,2}(x,\xi) &:= 2\tau\phi''_{xx}\left[\frac{\partial\bar{p}}{\partial\xi}(x,\bar{\zeta});\frac{\partial p}{\partial\xi}(x,\zeta)\right],\end{aligned}$$

where we have denoted $\zeta = \xi + i\tau\nabla\phi(x)$. But we notice that for fixed (x,ξ) (and when ϕ varies), $C_{\tau,\phi,1}(x,\xi)$ only depends on $\nabla\phi(x)$, while $C_{\tau,\phi,2}(x,\xi)$ is linear in $\phi''_{xx}(x^0)$ once $\nabla\phi(x^0)$ is fixed. So, since $\psi(x^0) = 0$, $\nabla\psi(x^0) = \nabla\phi(x^0)$ and

$$\psi''_{xx}(x^0) = \phi''_{xx}(x^0) + 2A\,{}^t\nabla\phi(x^0)\nabla\phi(x^0) - \frac{2}{A}\,\mathrm{Id},$$

we have $C_{\tau,\phi,1}(x^0,\xi) = C_{\tau,\psi,1}(x^0,\xi)$, i.e.

$$\frac{1}{i}\{\overline{p}_\psi, p_\psi\}(x^0,\xi) = C_{\tau,\phi,1}(x^0,\xi) + 4A\tau\left|\nabla_x\phi(x^0)\cdot\frac{\partial p}{\partial\xi}(x^0,\zeta)\right|^2 - \frac{4\tau}{A}\left|\frac{\partial p}{\partial\xi}(x^0,\zeta)\right|^2. \tag{3.6}$$

In identity (3.6), all terms are homogeneous of degree $2m-1$ in the variables (τ,ξ), so it is enough to prove the estimate for $(\tau,\xi)\in\mathbb{S}^n$, $\tau>0$. We now want this to be positive on the set $\{(\tau,\xi)\in\mathbb{S}^n : \tau>0,\ \xi_a=0,\ p_\phi(x^0,\xi)=0\} = \{(\tau,\xi)\in\mathbb{S}^n : \tau>0,\ \xi_a=0,\ p_\psi(x^0,\ \xi)=0\}$.

For this, notice first that $\frac{\partial}{\partial\tau}\frac{1}{i}\{\overline{p}_\phi, p_\phi\}\Big|_{\tau=0} = 2\,\mathrm{Re}\,\{\overline{p},\{p,\phi\}\}$. Hence, we can write

$$\frac{1}{i}\{\overline{p}_\phi, p_\phi\} = \frac{1}{i}\{\overline{p}, p\} + 2\tau\,\mathrm{Re}\,\{\overline{p},\{p,\phi\}\} + O(\tau^2), \qquad \tau\to 0^+, \tag{3.7}$$

with $O(\tau^2)$ uniform in $(\tau,\xi)\in\mathbb{S}^n$.

Moreover, by the Taylor formula, we have $p_\phi = p + i\tau\nabla\phi\cdot\frac{\partial p}{\partial\xi} + O(\tau^2) = p + i\tau\{p,\phi\} + O(\tau^2)$, with $O(\tau^2)$ uniform in $(\tau,\xi)\in\mathbb{S}^n$. Hence, on the compact set $\{(\tau,\xi)\in\mathbb{S}^n : \xi_a=0,\ p_\phi(x^0,\xi)=0\}$, we have $p=-i\tau\{p,\phi\}+O(\tau^2)$. But since P is analytically principally normal, (1.9) holds and we have $\{\overline{p},p\}=O(p)$ on the compact set $\{(\tau,\xi)\in\mathbb{S}^n : \xi_a=0\}$.

In particular, there is a constant C such that $\left|\frac{1}{i\tau}\{\overline{p},p\}\right| \le C(|\{p,\phi\}|+|\tau|)$ on $\{(\tau,\xi)\in\mathbb{S}^n : \xi_a=0,\ p_\phi(x^0,\xi)=0,\ \tau\neq 0\}$. Coming back to (3.7), on this set we have

$$\left|\frac{1}{i\tau}\{\overline{p}_\phi, p_\phi\} - 2\,\mathrm{Re}\,\{\overline{p},\{p,\phi\}\}\right| \le C(|\{p,\phi\}|+|\tau|). \tag{3.8}$$

Moreover, the first pseudoconvexity assumption (1.10) and Lemma A.1 below provide $C_1, C_2>0$ such that, on the set $\{\xi_a=0\}\cap\{|\xi|^2=1\}$, we have

$$2\,\mathrm{Re}\,\{\overline{p},\{p,\phi\}\} + C_1(|p|^2+|\{p,\phi\}|^2) \ge C_2.$$

This is also true by homogeneity for $|\xi|$ close to 1 with a different constant. Hence, on the set $\{(\tau,\xi)\in\mathbb{S}^n : \xi_a=0,\ p_\phi(x^0,\xi)=0,\ \tau\neq 0\}$, there exist constants $\tilde{C}, C>0$ such that $|\{p,\phi\}|\le\varepsilon$ and $|\tau|\le\varepsilon$ imply

$$\frac{1}{i\tau}\{\overline{p}_\phi, p_\phi\} \ge C_2 - \tilde{C}\big(|p|^2+|\{p,\phi\}|^2+|\{p,\phi\}|+|\tau|\big) \ge C_2 - C\varepsilon$$

where we have used $|p|\le C|\tau|\le C\varepsilon$ on this set.

Therefore, there exist $\varepsilon, C_3>0$ such that in $\{(\tau,\xi)\in\mathbb{S}^n : \xi_a=0,\ p_\phi(x^0,\xi)=0,\ \tau\neq 0\}$, we have

$$[|\{p,\phi\}|\le\varepsilon,\ |\tau|\le\varepsilon] \ \Rightarrow\ \frac{1}{i\tau}\{\overline{p}_\phi, p_\phi\} \ge C_3.$$

We now extend $\frac{1}{i\tau}\{\overline{p}_\phi, p_\phi\}$ to the compact set $K_\varepsilon = \{(\tau,\xi) \in \mathbb{S}^n : \xi_a = 0,\ p_\phi(x^0,\xi) = 0,\ 0 \le \tau \le \varepsilon\}$, by giving any positive value when $\tau = 0$. We are in a position to apply Lemma A.2 with $g = \frac{1}{i\tau}\{\overline{p}_\phi, p_\phi\}$ (its extension), $f = |\{p,\phi\}|^2$ and $h = \left|\frac{\partial p}{\partial \xi}(x^0,\zeta)\right|^2$. This yields $\frac{1}{i\tau}\{\overline{p}_\psi, p_\psi\}(x^0,\xi) > C$ on K_ε.

The case $\tau \ge \varepsilon$ is easier since $\frac{1}{i\tau}\{\overline{p}_\phi, p_\phi\}$ is continuous. We apply directly Lemma A.1 using the second pseudoconvexity assumption (1.11).

So, at this stage, we have proved that there exists C such that for A large enough, $\frac{1}{i\tau}\{\overline{p}_\psi, p_\psi\}(x^0,\xi) > C$ on $\{(\tau,\xi) \in \mathbb{S}^n : \xi_a = 0,\ p_\phi(x^0,\xi) = 0,\ \tau > 0\}$. Since $p_\psi(x^0,\xi) = p_\phi(x^0,\xi)$, this yields

$$\frac{1}{i}\{\overline{p}_\psi, p_\psi\}(x^0,\xi) > 0 \quad \text{if } p_\psi(x^0,\xi) = 0 \text{ and } \xi_a = 0,\ \tau > 0. \tag{3.9}$$

Combining (3.5) and (3.9) implies that ψ is a strongly pseudoconvex function in $\Omega \cap \{\xi_a = 0\}$ at x^0 for P.

Let us now prove the geometrical part of the lemma, i.e. (3). From now on, the parameter A is fixed. To simplify notation, we set $x^0 = 0$ and assume that $0 \le \rho \le \eta$. We also take a positive constant C_N such that $\frac{1}{C_N}N(x,0) \le |x| \le C_N N(x,0)$.

Let us first prove (3.2). We have

$$\frac{1}{A}|x|^2 = -\psi(x) + x\cdot\nabla\phi(0) + A(x\cdot\nabla\phi(0))^2 + \frac{1}{2}\phi''(0)(x,x),$$

which implies

$$\frac{1}{A}|x|^2 \le \eta + x\cdot\nabla\phi(0) + A(x\cdot\nabla\phi(0))^2 + \frac{1}{2}\phi''(0)(x,x)$$

on the set $\{\psi \ge -\eta\}$. Moreover, the Taylor expansion of ϕ yields $x\cdot\nabla\phi(0) + \frac{1}{2}\phi''(0)(x,x) = \phi(x) + f(x)$ with $|f(x)| \le \epsilon(|x|)|x|^2$, where $\epsilon : \mathbb{R}_+ \to \mathbb{R}_+$ is increasing and $\epsilon(s) \to 0^+$ as $s \to 0^+$. For $x \in \{\psi \ge -\eta\} \cap \{\phi \le \rho\}$, we thus obtain

$$\frac{1}{A}|x|^2 \le \eta + \rho + A(x\cdot\nabla\phi(0))^2 + \epsilon(|x|)|x|^2 \le 2\eta + A(x\cdot\nabla\phi(0))^2 + \epsilon(|x|)|x|^2. \tag{3.10}$$

Moreover, for $x \in \{\psi \ge -\eta\}$, the definition of ψ gives

$$\begin{aligned} x\cdot\nabla\phi(0) &= \psi(x) - A(x\cdot\nabla\phi(0))^2 - \frac{1}{2}\phi''(0)(x,x) + \frac{1}{A}|x|^2 \\ &\ge -\eta - (AC_0^2 + C_0/2)|x|^2 + \frac{1}{A}|x|^2 \\ &\ge -\eta - (AC_0^2 + C_0/2)|x|^2 \end{aligned}$$

for $C_0 = \max(|\nabla\phi(0)|, \max_{x\in B(0,R_0)}|\phi''(x)|)$. Also, for $x \in \{\phi \le \rho\}$, we have

$$x\cdot\nabla\phi(0) \le \phi(x) + C_0|x|^2/2 \le \rho + C_0|x|^2/2 \le \eta + C_0|x|^2/2.$$

Combining the last two inequalities, we obtain, for $x \in \{\phi \le \rho\} \cap \{\psi \ge -\eta\}$,

$$|x \cdot \nabla\phi(0)| \le \eta + (AC_0^2 + C_0/2)|x|^2,$$

and hence

$$|x \cdot \nabla\phi(0)|^2 \le \eta^2 + 2\eta(AC_0^2 + C_0/2)|x|^2 + (AC_0^2 + C_0/2)^2|x|^4.$$

Coming back to (3.10) yields, for $x \in \{\phi \le \rho\} \cap \{\psi > -\eta\}$,

$$\frac{1}{A}|x|^2 \le 2\eta + A\eta^2 + 2A\eta(AC_0^2 + C_0/2)|x|^2 + A(AC_0^2 + C_0/2)^2|x|^4 + \epsilon(|x|)|x|^2.$$

For $x \in \{\phi \le \rho\} \cap \{\psi \ge -\eta\} \cap B_N(0, R)$, this implies

$$\begin{aligned}\frac{1}{A}|x|^2 \le 2\eta + A\eta^2 + 2A\eta(AC_0^2 + C_0/2)|x|^2 + A(AC_0^2 + C_0/2)^2(C_N R)^2|x|^2 \\ + \epsilon(C_N R)|x|^2.\end{aligned}$$

Taking $R \le R_0$ with $R_0 = R_0(A, C_0)$ sufficiently small such that

$$A(AC_0^2 + C_0/2)^2(C_N R)^2 + \epsilon(C_N R) < \frac{1}{4A},$$

and $\eta < \eta_0$ sufficiently small such that

$$2A\eta(AC_0^2 + C_0/2) < \frac{1}{4A},$$

we have by absorption

$$|x|^2 \le 2A(2\eta + A\eta^2).$$

This gives $N(x, 0) < R/8$ as soon as $\eta < \eta_0$ for $\eta_0 = \eta_0(A, C_0, R)$ sufficiently small. This concludes the proof of (3.2) for the chosen constants as long as $0 \le \rho \le \eta$.

Let us now prove (3.3). Note that performing exactly the same computation as before with $\rho = \eta = 0$ and the same R, we obtain

$$\{\phi \le 0\} \cap \{\psi \ge 0\} \cap \overline{B}_N(0, R) = \{0\}. \tag{3.11}$$

Assume that the compact set $\{\psi \ge \eta_1\} \cap \overline{B}_N(0, R)$ is nonempty (otherwise (3.3) is trivial). The minimum of ϕ on that set is reached at some point x_m. We necessarily have $\phi(x_m) > 0$: otherwise, (3.11) implies $x_m = 0$, which is impossible since $\eta_1 > 0$ and $\psi(0) = 0$. So, in particular, $x \in \{\psi \ge \eta_1\} \cap \overline{B}_N(0, R)$ implies $\phi(x) \ge \phi(x_m) > 0$. This is (3.3) with some appropriate $0 < \rho < \min(\phi(x_m), \eta)$.

Finally, (3.4) is just a matter of continuity. Since $\psi(0) = 0$, there exists $r > 0$ such that $N(x, 0) \le r$ implies $|\psi(x)| \le \eta_2$. □

Remark 3.5. Note that the estimate (3.8) implies in particular that $2\,\mathrm{Re}\,\{\overline{p}, \{p, \phi\}\}$ is the limit as $\tau \to 0$ of $\frac{1}{i\tau}\{\overline{p}_\phi, p_\phi\}$ on the subset

$$\big\{(\tau, \xi) \in \mathbb{S}^n : \xi_a = 0,\ p_\phi(x^0, \xi) = \{p_\phi, \phi\}(x^0, \xi) = 0,\ \tau \neq 0\big\}.$$

However, this is not used directly in the above proof.

Now, thanks to Lemma 3.4 and the Carleman estimate of Theorem 2.2, we have the following result.

Corollary 3.6. *Let $x^0 \in \Omega = \Omega_a \times \Omega_b \subset \mathbb{R}^{n_a} \times \mathbb{R}^{n_b}$ and P be a partial differential operator on Ω of order m. Assume that*

- *P is analytically principally normal on $\{\xi_a = 0\}$ inside Ω (in the sense of Definition 1.6);*
- *there is a function ϕ defined in a neighborhood of x^0 such that $\phi(x^0) = 0$ and $\{\phi = 0\}$ is a C^2 strongly pseudoconvex oriented surface in the sense of Definition 1.7.*

Then there exists a quadratic polynomial $\psi : \Omega \to \mathbb{R}$ and $R_0 > 0$ such that $B(x^0, 4R_0) \subset \Omega$, and for any $R \in (0, R_0]$ there exist $\varepsilon, \delta, \rho, r, \mathsf{d}, \tau_0, C > 0$ such that $\delta \leq \mathsf{d}/8$ and

(1) *the Carleman estimate*

$$\tau \| Q^\psi_{\varepsilon,\tau} u\|^2_{m-1,\tau} \leq C\big(\|Q^\psi_{\varepsilon,\tau} P u\|^2_0 + \|e^{\tau(\psi-\mathsf{d})} P u\|^2_0 + \|e^{\tau(\psi-\mathsf{d})} u\|^2_{m-1,\tau}\big) \quad (3.12)$$

holds for all $\tau \geq \tau_0$ and all $u \in C^\infty_0(B(x^0, 4R))$;

(2) *we have*

$$\big(B(x^0, 5R/2) \setminus B(x^0, R/2)\big) \cap \{-9\delta \leq \psi \leq 2\delta\} \Subset \{\phi > 2\rho\} \cap B(x^0, 3R), \quad (3.13)$$

$$\{\delta/4 \leq \psi \leq 2\delta\} \cap B(x^0, 5R/2) \Subset \{\phi > 2\rho\} \cap B(x^0, 3R), \quad (3.14)$$

$$B(x^0, 2r) \Subset \{-\delta/2 \leq \psi \leq \delta/2\} \cap B(x^0, R). \quad (3.15)$$

Proof. First, Lemma 3.4 furnishes the function ψ for some A (large enough in its proof) and $R_0 > 0$. Once ψ is fixed, Theorem 2.2 yields the Carleman estimate (3.12) for some constants $\mathsf{R}, \mathsf{d}, \tau_0, \varepsilon, C$. Then, we take any $R < \min(\mathsf{R}/4, R_0/3)$ (with R_0 given by Lemma 3.4) and $\delta < \min(\mathsf{d}/8, \eta_0/9)$. Finally, the conclusion of Lemma 3.4 with $\eta = 9\delta$, $\eta_1 = \delta/4$, $\eta_2 = \delta/2$ implies (3.13)–(3.15), with possibly different constants, which concludes the proof. □

3.2. Step 2: Using the Carleman estimate

From now on, we let Ω, x^0, P and ϕ be as in Corollary 3.6. The function ψ and constants R_0, $R := R_0$ (that we fix now) and δ, ρ, r are provided by Corollary 3.6, as also are the constants d, τ_0, C of the Carleman estimate (3.12). We shall moreover assume that there exists $\mathsf{C} > 0$ such that

$$\frac{1}{\mathsf{C}}\mu \leq \lambda \leq \mathsf{C}\mu. \quad (3.16)$$

Actually, at the end of the proof, we will take $\lambda = c_1\mu$, but we believe that to keep the notation λ makes the presentation more readable by making a difference between μ which is the frequency and λ which is the regularization parameter. All the constants appearing in the following may depend upon the above ones.

Before going further, we need to introduce some cutoff functions that will be used all along the proof. We first let $\chi(s)$ be a smooth function supported in $(-8, 1)$ such that $\chi(s) = 1$ for $s \in [-7, 1/2]$, and set

$$\chi_\delta(s) := \chi(s/\delta). \tag{3.17}$$

Hence, $\chi_\delta(s)$ is a smooth function supported in $(-8\delta, \delta)$ such that $\chi_\delta(s) = 1$ for $s \in [-7\delta, \delta/2]$. We also define $\widetilde{\chi}$ so that $\widetilde{\chi} = 1$ on $(-\infty, 3/2)$ and supported in $s \leq 2$, and denote as well $\widetilde{\chi}_\delta(s) := \widetilde{\chi}(s/\delta)$. We finally recall that the functions σ_R and σ_{2R} are defined in (3.1).

In this part of the proof, we want to apply the Carleman estimate (3.12) (with weight ψ and constants d, τ_0, C given by Corollary 3.6) to the functions $\sigma_{2R}\sigma_{R,\lambda}\widetilde{\chi}_\delta(\psi)\chi_{\delta,\lambda}(\psi)u$ (for any $u \in C_0^\infty(\mathbb{R}^n)$), which is indeed compactly supported in $B(x^0, 4R)$ (according to the definition of σ_{2R} as in (3.1)). We first need to estimate the term

$$\|Q^{\psi}_{\varepsilon,\tau} P\sigma_{2R}\sigma_{R,\lambda}\widetilde{\chi}_\delta(\psi)\chi_{\delta,\lambda}(\psi)u\|_0,$$

which will appear on the right hand side of the inequality. Using $\operatorname{supp}\chi_\delta \subset (-\infty, \delta)$ with Lemma 2.13, together with (3.16), we first have

$$\begin{aligned}\|Q^{\psi}_{\varepsilon,\tau} P\sigma_{2R}\sigma_{R,\lambda}\widetilde{\chi}_\delta(\psi)\chi_{\delta,\lambda}(\psi)u\|_0 &\leq \|Q^{\psi}_{\varepsilon,\tau}\sigma_{2R}\sigma_{R,\lambda}\widetilde{\chi}_\delta(\psi)\chi_{\delta,\lambda}(\psi)Pu\|_0\\ &\quad + \|Q^{\psi}_{\varepsilon,\tau}[\sigma_{2R}\sigma_{R,\lambda}\widetilde{\chi}_\delta(\psi)\chi_{\delta,\lambda}(\psi), P]u\|_0\\ &\leq C\mu^{1/2}e^{C\tau^2/\mu}e^{\delta\tau}\|Pu\|_{B(x^0,4R)}\\ &\quad + \|Q^{\psi}_{\varepsilon,\tau}[\sigma_{2R}\sigma_{R,\lambda}\widetilde{\chi}_\delta(\psi)\chi_{\delta,\lambda}(\psi), P]u\|_0.\end{aligned} \tag{3.18}$$

The main task now consists in estimating the term containing the commutator, which we do in the following lemma.

Lemma 3.7. *With the above notations and assumptions, for any $\vartheta \in C_0^\infty(\mathbb{R}^n)$ such that $\vartheta(x) = 1$ on a neighborhood of $\{\phi \geq 2\rho\} \cap B(x^0, 3R)$, there exist $C, c > 0$ and $N > 0$ such that*

$$\begin{aligned}\|Q^{\psi}_{\varepsilon,\tau}[\sigma_{2R}\sigma_{R,\lambda}\widetilde{\chi}_\delta(\psi)\chi_{\delta,\lambda}(\psi), P]u\|_0 &\leq Ce^{2\delta\tau}\|M^{2\mu}_{\lambda}\vartheta_\lambda u\|_{m-1}\\ &\quad + C\mu^{1/2}\tau^N\big(e^{-8\delta\tau} + e^{-\frac{\varepsilon\mu^2}{8\tau}} + e^{-c\mu}e^{\delta\tau}\big)e^{C\tau^2/\mu}e^{\delta\tau}\|u\|_{m-1}\end{aligned} \tag{3.19}$$

for any $u \in C_0^\infty(\mathbb{R}^n)$, $\mu \geq 1$, λ such that (3.16) holds and $\tau \geq 1$.

We stress that all geometric constants are now fixed (see the beginning of Section 3.2). Hence, all constants appearing in the estimates may depend on them. In particular, the constant C in (3.19) depends on δ.

Proof of Lemma 3.7. The operator P can be written $P = \sum_{|\alpha|\leq m} p_\alpha(x)\partial^\alpha$, with p_α smooth and analytic in x_a in a neighborhood of $B(x^0, 4R) \subset \Omega$. By the Leibniz rule,

$$\begin{aligned} &p_\alpha(x)\partial^\alpha(\sigma_{2R}\sigma_{R,\lambda}\widetilde{\chi}_\delta(\psi)\chi_{\delta,\lambda}(\psi)u) \\ &\quad = p_\alpha(x) \sum_{\alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5=\alpha} C_{(\alpha_i)}\partial^{\alpha_1}(\chi_{\delta,\lambda}(\psi))\partial^{\alpha_2}(\sigma_{2R})\partial^{\alpha_3}(\sigma_{R,\lambda})\partial^{\alpha_4}(\widetilde{\chi}_\delta(\psi))\partial^{\alpha_5}u. \end{aligned}$$

The commutator $[\widetilde{\chi}_\delta(\psi)\chi_{\delta,\lambda}(\psi)\sigma_{2R}\sigma_{R,\lambda}, P]$ consists of all terms in the sum where at least one of the α_i is nonzero, for $i = 1, 2, 3$ or 4. Hence, we can split it into a sum of differential operators of order $m-1$ as

$$[P, \sigma_{2R}\sigma_{R,\lambda}\widetilde{\chi}_\delta(\psi)\chi_{\delta,\lambda}(\psi)] = B_1 + B_2 + B_3 + B_4,$$

where

1. B_1 contains the terms with $\alpha_1 \neq 0$ and $\alpha_2 = \alpha_4 = 0$;
2. B_2 contains some terms with $\alpha_2 \neq 0$;
3. B_3 contains the terms with $\alpha_3 \neq 0$ and $\alpha_1 = \alpha_2 = \alpha_4 = 0$;
4. B_4 contains some terms with $\alpha_4 \neq 0$.

Note that some terms could belong to several categories, and that all terms are supported in $\{\psi \leq 2\delta\} \cap B(x^0, 4R)$. More precisely:

1. B_1 consists of terms where there is at least one derivative on $\chi_{\delta,\lambda}(\psi)$ and none on σ_{2R} and $\widetilde{\chi}_\delta(\psi)$. According to the definition of χ and (3.17), there are only two possibilities for the localization of a derivative of χ_δ. Since $\chi'_{\delta,\lambda} = \frac{1}{\delta}(\chi')_{\delta,\lambda}$, $\partial^{\alpha_1}(\chi_{\delta,\lambda}(\psi))$ with $\alpha_1 \neq 0$ can be decomposed into two categories of terms: we shall use the notation $\chi^-_{\delta,\lambda}$ for those terms supported in $[-8\delta, -7\delta]$ and $\chi^+_{\delta,\lambda}$ for those supported in $[\delta/2, \delta]$. Hence, B_1 is a sum of generic terms of the form
$$B_\pm = b_\pm(x)\partial^\gamma = f\sigma_{2R}\partial^\beta(\sigma_{R,\lambda})\chi^\pm_{\delta,\lambda}(\psi)\widetilde{\chi}_\delta(\psi)\partial^\gamma,$$
where $|\beta|, |\gamma| \leq m-1$, $f \in C^\infty_0(\mathbb{R}^n)$ is analytic in x_a in $B(x^0, 4R)$, and $\chi^\pm_\delta$ is a derivative of χ_δ (with the above convention for the superscript $\pm$). The function f actually contains some terms coming from p_α and some derivatives of ψ. Notice that in the absence of regularization (i.e. the subscript λ), B_+ would be supported in
$$\big(\{\delta/2 \leq \psi \leq \delta\} \cap B(x^0, 2R)\big) \subset \big(\{\phi > 2\rho\} \cap \{\psi \leq \delta\} \cap B(x^0, 2R)\big),$$
and B_- in $\{-8\delta \leq \psi \leq -7\delta\} \cap B(x^0, 2R)$.
2. B_2 consists of terms where there is at least one derivative on σ_{2R}. Hence, B_2 is a sum of generic terms of the form
$$\check{B}_2 = b_2(x)\partial^\gamma = \widetilde{b}\partial^\beta(\sigma_{R,\lambda})(\chi^{(k)})_{\delta,\lambda}(\psi)\partial^\gamma,$$
where $k, |\beta|, |\gamma| \leq m-1$, the function $\widetilde{b}$ is smooth supported in $B(x^0, 4R)\setminus B(x^0, 2R)$ and $\widetilde{b}$ contains derivatives of σ_{2R}, some terms of $p_\alpha(x)$, and possibly some derivatives of ψ or $\widetilde{\chi}_\delta(\psi)$.

3. B_3 consists of terms where there is at least one derivative on $\sigma_{R,\lambda}$ and none on $\chi_{\delta,\lambda}(\psi)$, $\widetilde{\chi}_\delta(\psi)$ or σ_{2R}. Hence, B_3 is a sum of generic terms of the form
$$\check{B}_3 = b_3(x)\partial^\gamma = f\sigma_{2R}\partial^\beta(\sigma_{R,\lambda})\chi_{\delta,\lambda}(\psi)\widetilde{\chi}_\delta(\psi)\partial^\gamma,$$
where f is smooth in (x_a, x_b), analytic in x_a in a neighborhood of $B(x^0, 4R)$, $|\beta| \geq 1$ and $|\beta|, |\gamma| \leq m-1$. Notice also that in the absence of regularization (i.e. the subscript λ), $\check{B}_3$ would be supported in
$$\big(\{-8\delta \leq \psi \leq \delta\} \cap B(x^0, 2R) \setminus B(x^0, R)\big) \subset \big(\{\phi > 2\rho\} \cap \{\psi \leq \delta\} \cap B(x^0, 2R)\big).$$
4. B_4 consists of terms where there is at least one derivative on $\widetilde{\chi}_\delta(\psi)$. Hence, B_4 is a sum of generic terms of the form
$$\check{B}_4 = b_4(x)\partial^\gamma = \widetilde{b}\partial^\beta(\sigma_{R,\lambda})(\chi^{(k)})_{\delta,\lambda}(\psi)\partial^\gamma$$
where $k, |\beta|, |\gamma| \leq m-1$ and the function $\widetilde{b}$ is smooth supported in $B(x^0, 4R) \cap \{\psi \in [3\delta/2, 2\delta]\}$ and $\widetilde{b}$ contains derivatives of σ_{2R}, some terms from $p_\alpha(x)$, and some derivatives of ψ or $\widetilde{\chi}_\delta(\psi)$.

Now, proving an estimate of the last term in (3.18) consists in estimating successively the associated expressions with the generic terms $B_\pm$, $\check{B}_2$, $\check{B}_3$, $\check{B}_4$; the final estimate then follows as the LHS of (3.19) is bounded by a finite sum of such terms. Recall that δ is fixed, so that $C_\delta = C$ in the estimates below.

Estimating B_-. Using Lemma 2.13 applied to χ_δ^-, we have
$$\|Q^\psi_{\varepsilon,\tau}B_-u\|_0 \leq \|e^{\tau\psi}B_-u\|_0 \leq C_\delta\lambda^{1/2}e^{-7\delta\tau}e^{\tau^2/\lambda}\|u\|_{m-1} \leq C\mu^{1/2}e^{-7\delta\tau}e^{C\tau^2/\mu}\|u\|_{m-1}. \tag{3.20}$$

Estimating B_2. We use Lemma 2.13 applied to $\chi_\delta^{(k)}$ and Lemma 2.3 applied to $\widetilde{b}$ and $\partial^\beta(\sigma_R)$ where $\operatorname{supp}\widetilde{b} \cap \operatorname{supp}\sigma_R = \emptyset$. This yields
$$\begin{aligned}\|Q^\psi_{\varepsilon,\tau}B_2u\|_0 &\leq \|e^{\tau\psi}B_2u\|_0 \leq C_\delta\lambda^{1/2}e^{\delta\tau}e^{\tau^2/\lambda}e^{-c\lambda}\|u\|_{m-1}\\ &\leq C\mu^{1/2}e^{\delta\tau}e^{C\tau^2/\mu}e^{-c\mu}\|u\|_{m-1}.\end{aligned} \tag{3.21}$$

Estimating B_4. We use $e^{\tau\psi} \leq e^{2\delta\tau}$ and $|(\chi^{(k)})_{\delta,\lambda}(\psi)| \leq Ce^{-c\lambda}$ on $\{\psi \in [3\delta/2, 2\delta]\}$ thanks to Lemma 2.3 applied to $\chi^{(k)}$ and $\mathbb{1}_{[3\delta/2,2\delta]}$. This yields
$$\|Q^\psi_{\varepsilon,\tau}B_4u\|_0 \leq \|e^{\tau\psi}B_4u\|_0 \leq C_\delta e^{2\delta\tau}e^{-c\lambda}\|u\|_{m-1} \leq Ce^{2\delta\tau}e^{-c\mu}\|u\|_{m-1}. \tag{3.22}$$

First estimates on B_+ and B_3. With $\star = +$ or 3, we have
$$\begin{aligned}\big\|Q^\psi_{\varepsilon,\tau}B_\star u\big\|_0 &= \big\|e^{-\varepsilon\frac{|D_a|^2}{2\tau}}e^{\tau\psi}B_\star u\big\|_0\\ &\leq \big\|e^{-\varepsilon\frac{|D_a|^2}{2\tau}}M^\mu_\lambda e^{\tau\psi}B_\star u\big\|_0 + \big\|e^{-\varepsilon\frac{|D_a|^2}{2\tau}}(1-M^\mu_\lambda)e^{\tau\psi}B_\star u\big\|_0\\ &\leq \|M^\mu_\lambda e^{\tau\psi}B_\star u\|_0 + C\big(e^{-\frac{\varepsilon\mu^2}{8\tau}} + e^{-c\mu}\big)\|e^{\tau\psi}B_\star u\|_0\\ &\leq \|M^\mu_\lambda e^{\tau\psi}B_\star u\|_0 + C\lambda^{1/2}\big(e^{-\frac{\varepsilon\mu^2}{8\tau}} + e^{-c\mu}\big)e^{C\tau^2/\mu}e^{\delta\tau}\|u\|_{m-1},\end{aligned}$$

where the second inequality comes from the application of Lemma 2.14 and the third from Lemma 2.13.

Next, concerning the term with $\|M_\lambda^\mu e^{\tau\psi} B_\star u\|_0$, we have $B_\star = b_\star \partial^\gamma$ where $\star$ is either $+$ or 3. So, we can estimate

$$\|M_\lambda^\mu e^{\tau\psi} B_\star u\|_0 \leq \|M_\lambda^\mu e^{\tau\psi} b_\star (1 - M_\lambda^{2\mu})\partial^\gamma u\|_0 + \|M_\lambda^\mu e^{\tau\psi} b_\star M_\lambda^{2\mu} \partial^\gamma u\|_0,$$

where

$$\|M_\lambda^\mu e^{\tau\psi} b_\star (1 - M_\lambda^{2\mu})\partial^\gamma u\|_0 \leq C\tau^N e^{C\tau^2/\mu} e^{2\delta\tau - c\mu} \|u\|_{m-1},$$

according to Lemma 2.17 applied in the specific case of (2.34). Note that we use the fact that $f\sigma_{2R} = f$ in a neighborhood of $B(x^0, 2R) \supset \operatorname{supp} \sigma_R$, and $f\sigma_{2R}$ is therefore analytic in x_a on a neighborhood of this set. Next,

$$\|M_\lambda^\mu e^{\tau\psi} b_\star M_\lambda^{2\mu} \partial^\gamma u\|_0 \leq \|e^{\tau\psi} b_\star M_\lambda^{2\mu} \partial^\gamma u\|_0.$$

Combining the above four estimates, we now have

$$\|Q_{\varepsilon,\tau}^\psi B_\star u\|_0 \leq \|e^{\tau\psi} b_\star M_\lambda^{2\mu} \partial^\gamma u\|_0 + C\mu^{1/2}\tau^N \left(e^{-\frac{\varepsilon\mu^2}{8\tau}} + e^{\delta\tau} e^{-c\mu}\right) e^{C\tau^2/\mu} e^{\delta\tau} \|u\|_{m-1}. \tag{3.23}$$

Now, to estimate the first term of the RHS, we will distinguish whether $\star = +$ or 3, using the geometry of the "almost" location of each $b_\star$.

Estimating B_+. We have to treat terms of the form

$$B_+ = b_+ \partial^\gamma = f \tilde{\tilde{b}}_\lambda \chi_{\delta,\lambda}^+(\psi) \tilde{\chi}_\delta(\psi) \partial^\gamma,$$

where $\tilde{\tilde{b}} = \partial^\beta(\sigma_R)$, $|\beta| \leq m-1$, is supported in $B(x^0, 2R)$ and $f \in C_0^\infty(\mathbb{R}^n)$. We decompose $\mathbb{R}^n$ as $\mathbb{R}^n = O_1 \cup O_2 \cup O_3$ with

$$\begin{aligned} O_1 &= \{\psi \notin [\delta/4, 2\delta]\} \cap B(x^0, 5R/2), \\ O_2 &= B(x^0, 5R/2)^c, \\ O_3 &= \{\psi \in [\delta/4, 2\delta]\} \cap B(x^0, 5R/2). \end{aligned}$$

On O_1, since χ_δ^+ is supported in $[\delta/2, \delta]$ and using Lemma 2.3 with $f_2 = \mathbb{1}_{[\delta/4, 2\delta]^c}$, we have $|\chi_{\delta,\lambda}^+(\psi)| \leq e^{-c\lambda}$. Moreover, $e^{\tau\psi} \leq e^{2\delta\tau}$ on the support of $\tilde{\chi}_\delta$. Hence,

$$\|e^{\tau\psi} b_+ M_\lambda^{2\mu} \partial^\gamma u\|_{L^2(O_1)} \leq C e^{-c\lambda} e^{2\delta\tau} \|u\|_{m-1} \leq C e^{-c\mu} e^{2\delta\tau} \|u\|_{m-1}.$$

On O_2, using Lemma 2.3 with $f_2 = \mathbb{1}_{O_2}$ and $f_1 = \tilde{\tilde{b}}$ and then Lemma 2.13, we get

$$\|e^{\tau\psi} b_+ M_\lambda^{2\mu} \partial^\gamma u\|_{L^2(O_2)} \leq C\lambda^{1/2} e^{-c\lambda} e^{\delta\tau} e^{\tau^2/\lambda} \|u\|_{m-1} \leq C\mu^{1/2} e^{-c\mu} e^{\delta\tau} e^{C\tau^2/\mu} \|u\|_{m-1}.$$

Using (3.14), we can find a smooth cutoff function $\tilde{\vartheta}$ such that $\tilde{\vartheta} = 1$ on a neighborhood of O_3 and supported in $\{\phi > 2\rho\} \cap B(x^0, 3R)$. So, for λ large enough, we have $\tilde{\vartheta}_\lambda \geq 1/2$ on O_3. Moreover, $|e^{\tau\psi}| \leq e^{2\delta\tau}$ on O_3, and thus

$$\begin{aligned}\|e^{\tau\psi} b_+ M_\lambda^{2\mu} \partial^\gamma u\|_{L^2(O_3)} &\leq e^{2\delta\tau} \|b_+ M_\lambda^{2\mu} \partial^\gamma u\|_{L^2(O_3)} \leq C e^{2\delta\tau} \|M_\lambda^{2\mu} \partial^\gamma u\|_{L^2(O_3)} \\ &\leq C e^{2\delta\tau} \|\tilde{\vartheta}_\lambda M_\lambda^{2\mu} \partial^\gamma u\|_{L^2(O_3)} \leq C e^{2\delta\tau} \|\tilde{\vartheta}_\lambda M_\lambda^{2\mu} \partial^\gamma u\|_{L^2}.\end{aligned}$$

Let $\tilde{\tilde{\vartheta}} \in C_0^\infty$ be such that $\tilde{\tilde{\vartheta}} = 1$ on a neighborhood of $\operatorname{supp} \tilde{\vartheta}$ and supported in $\{\phi > 2\rho\} \cap B(x^0, 3R)$. This is possible since $\operatorname{supp} \tilde{\vartheta} \subset \{\phi > 2\rho\} \cap B(x^0, 3R)$. In particular, since $\vartheta = 1$ on $\{\phi > 2\rho\} \cap B(x^0, 3R)$ by assumption, we have $\vartheta = 1$ in a neighborhood of $\operatorname{supp} \tilde{\tilde{\vartheta}}$. Then, according to Lemma 2.6 and the properties of $\tilde{\tilde{\vartheta}}$, we have

$$\|\tilde{\vartheta}_\lambda M_\lambda^{2\mu} \partial^\gamma u\|_{L^2} \leq \|\tilde{\tilde{\vartheta}}_\lambda M_\lambda^{2\mu} u\|_{m-1} + e^{-c\lambda} \|u\|_{m-1},$$

and then

$$\|\tilde{\tilde{\vartheta}}_\lambda M_\lambda^{2\mu} u\|_{m-1} \leq \|M_\lambda^{2\mu} \vartheta_\lambda u\|_{m-1} + C e^{-c\mu} \|u\|_{m-1},$$

according to Lemma 2.11.

Combining the previous estimates with (3.23), we have obtained

$$\begin{aligned}\|Q_{\varepsilon,\tau}^\psi B_+ u\|_0 \leq{}& C e^{2\delta\tau} \|M_\lambda^{2\mu} \vartheta_\lambda u\|_{m-1} \\ &+ C \mu^{1/2} \tau^N \big(e^{-\frac{\varepsilon\mu^2}{8\tau}} + e^{-c\mu} e^{\delta\tau}\big) e^{C\tau^2/\mu} e^{\delta\tau} \|u\|_{m-1}. \end{aligned} \tag{3.24}$$

Estimating B_3. We now treat terms of the form

$$B_3 = b_3 \partial^\gamma = f \widetilde{\widetilde{b}}_\lambda \chi_{\delta,\lambda}(\psi) \widetilde{\chi}_\delta(\psi) \partial^\gamma,$$

where $\widetilde{\widetilde{b}} = \partial^\beta(\sigma_R)$, with $|\beta| \geq 1$, is supported in $B(x^0, 2R) \setminus B(x^0, R)$ and $f \in C_0^\infty(\mathbb{R}^n)$. We decompose $\mathbb{R}^n$ as $\mathbb{R}^n = O_1' \cup O_2' \cup O_3'$ with

$$\begin{aligned} O_1' &= \big\{\psi \notin [-9\delta, 2\delta] \cap \{|x - x^0| \in [R/2, 5R/2]\}\big\}, \\ O_2' &= \{|x - x^0| \notin [R/2, 5R/2]\}, \\ O_3' &= \big\{\psi \in [-9\delta, 2\delta] \cap \{|x - x^0| \in [R/2, 5R/2]\}\big\}. \end{aligned}$$

On $O_1' \cap \operatorname{supp} \widetilde{\chi}_\delta(\psi)$, we have $e^{\tau\psi} |\chi_{\delta,\lambda}(\psi)| \leq e^{-c\lambda} e^{2\delta\tau}$ as a consequence of Lemma 2.3 with $f_2 = \mathbb{1}_{[-9\delta,2\delta]^c}$, since χ_δ is supported in $[-8\delta, \delta]$. We thus obtain

$$\|e^{\tau\psi} b_3 M_\lambda^{2\mu} \partial^\gamma u\|_{L^2(O_1')} \leq C e^{-c\lambda} e^{2\delta\tau} \|u\|_{m-1} \leq C e^{-c\mu} e^{2\delta\tau} \|u\|_{m-1}.$$

On O_2', using Lemma 2.3 with $f_2 = \mathbb{1}_{O_2'}$ and $f_1 = \widetilde{\widetilde{b}}$ and using the support of $\widetilde{\chi}_\delta(\psi)$, we get

$$\|e^{\tau\psi} b_3 M_\lambda^{2\mu} \partial^\gamma u\|_{L^2(O_2')} \leq C e^{-c\lambda} e^{2\delta\tau} \|u\|_{m-1} \leq C e^{-c\mu} e^{2\delta\tau} \|u\|_{m-1}.$$

Using (3.13), we can find a function $\tilde{\vartheta}$ such that $\tilde{\vartheta} = 1$ on a neighborhood of O_3' and supported in $\{\phi > 2\rho\} \cap B(x^0, 3R)$. So, for λ large enough, we have $\tilde{\vartheta}_\lambda \geq 1/2$ on O_3'. Moreover, $|e^{\tau\psi}| \leq e^{2\delta\tau}$ on O_3'. This yields

$$\begin{aligned}\|e^{\tau\psi} b_3 M_\lambda^{2\mu} \partial^\gamma u\|_{L^2(O_3')} &\leq e^{2\delta\tau} \|b_3 M_\lambda^{2\mu} \partial^\gamma u\|_{L^2(O_3')} \leq Ce^{2\delta\tau} \|M_\lambda^{2\mu} \partial^\gamma u\|_{L^2(O_3')} \\ &\leq Ce^{2\delta\tau} \|\tilde{\vartheta}_\lambda M_\lambda^{2\mu} \partial^\gamma u\|_{L^2(O_3')}.\end{aligned}$$

We can then finish the estimates for B_3 just as for B_+ to obtain, combining the above estimates with (3.23),

$$\|Q_{\varepsilon,\tau}^\psi B_3 u\|_0 \leq Ce^{2\delta\tau} \|M_\lambda^{2\mu} \vartheta_\lambda u\|_{m-1} + C\mu^{1/2}\tau^N \big(e^{-\frac{\varepsilon\mu^2}{8\tau}} + e^{\delta\tau} e^{-c\mu}\big) e^{C\tau^2/\mu} e^{\delta\tau} \|u\|_{m-1}. \tag{3.25}$$

Combining (3.20), (3.21), (3.22), (3.24) and (3.25) concludes the estimate of the commutator (3.19) and the proof of Lemma 3.7. ◻

Remark 3.8. In the special case of terms of the form $p_\alpha(x_b)\partial^\alpha$, that is, with coefficients independent of x_a, we can obtain better estimates, uniform in the size of p_α, since

$$\begin{aligned}&\|Q_{\varepsilon,\tau}^\psi [\sigma_{2R}\sigma_{R,\lambda}\widetilde{\chi}_\delta(\psi)\chi_{\delta,\lambda}(\psi), p_\alpha(x_b)\partial^\alpha] u\|_0 \\ &\qquad = \|p_\alpha(x_b) Q_{\varepsilon,\tau}^\psi [\sigma_{2R}\sigma_{R,\lambda}\widetilde{\chi}_\delta(\psi)\chi_{\delta,\lambda}(\psi), \partial^\alpha] u\|_0 \\ &\qquad \leq \|p_\alpha\|_{L^\infty} \|Q_{\varepsilon,\tau}^\psi [\sigma_{2R}\sigma_{R,\lambda}\widetilde{\chi}_\delta(\psi)\chi_{\delta,\lambda}(\psi), \partial^\alpha] u\|_0.\end{aligned}$$

Also, for $\alpha = 0$, that is, for a potential $V(x_b)$, we have $[\sigma_{2R}\sigma_{R,\lambda}\chi_{\delta,\lambda}(\psi)\widetilde{\chi}_\delta(\psi), V] = 0$, so this term does not give any contribution.

This will be useful in Section 6 below, when we want estimates that are uniform with respect to lower order perturbations. We also refer to the paper [LL16], where these uniform estimates are used.

Moreover, if p_α is only analytic in x_a and bounded in x_b, all estimates of the commutator remain valid. Indeed, we only use Lemma 2.17 for $k = 0$, which remains true in that setting.

Now, we are ready to apply the Carleman estimate (3.12) to obtain the estimate of the following lemma.

Lemma 3.9. *With the previous notations and assumptions, for any $\vartheta \in C_0^\infty(\mathbb{R}^n)$ such that $\vartheta(x) = 1$ on a neighborhood of $\{\phi > 2\rho\} \cap B(x^0, 3R)$, there exist $\mu_0, C, c, N > 0$ such that*

$$\begin{aligned}&\tau \|Q_{\varepsilon,\tau}^\psi \sigma_{2R}\sigma_{R,\lambda}\chi_{\delta,\lambda}(\psi)\widetilde{\chi}_\delta(\psi) u\|_{m-1,\tau} \\ &\qquad \leq C\mu^{1/2} e^{C\tau^2/\lambda} e^{\delta\tau} \|Pu\|_{B(x^0,4R)} + Ce^{2\delta\tau} \|M_\lambda^{2\mu}\vartheta_\lambda u\|_{m-1} \\ &\qquad\quad + C\mu^{1/2}\tau^N \big(e^{-8\delta\tau} + e^{-\frac{\varepsilon\mu^2}{8\tau}} + e^{\delta\tau - c\mu}\big) e^{C\tau^2/\mu} e^{\delta\tau} \|u\|_{m-1}\end{aligned} \tag{3.26}$$

for any $u \in C_0^\infty(\mathbb{R}^n)$, $\mu \geq \mu_0$, λ such that (3.16) holds and $\tau \geq \tau_0$.

Proof. We only need to estimate the last two terms on the RHS of the Carleman estimate (3.12) (the first term being estimated in (3.18) and Lemma 3.7). Since we have chosen $\delta \leq \mathsf{d}/8$, we have $\delta \leq \mathsf{d} - 7\delta$, so that the support of χ_δ gives, using again Lemma 2.13 for $\tau \geq \tau_0$, $\frac{1}{\mathsf{C}}\mu \leq \lambda \leq \mathsf{C}\mu$,

$$\begin{aligned}\|e^{\tau(\psi-\mathsf{d})}\sigma_{2R}\sigma_{R,\lambda}\chi_{\delta,\lambda}(\psi)\widetilde{\chi}_\delta(\psi)u\|_{m-1,\tau} &\leq C\lambda^{1/2}\tau^{m-1}e^{-7\delta\tau}e^{\tau^2/\lambda}\|u\|_{m-1}\\ &\leq C\mu^{1/2}\tau^{m-1}e^{-7\delta\tau}e^{C\tau^2/\mu}\|u\|_{m-1}.\end{aligned} \tag{3.27}$$

We also need to estimate

$$\begin{aligned}&\|e^{\tau(\psi-\mathsf{d})}P\sigma_{2R}\sigma_{R,\lambda}\chi_{\delta,\lambda}(\psi)\widetilde{\chi}_\delta(\psi)u\|_0\\ &\qquad\leq \|e^{\tau(\psi-\mathsf{d})}\sigma_{2R}\sigma_{R,\lambda}\chi_{\delta,\lambda}(\psi)\widetilde{\chi}_\delta(\psi)Pu\|_0\\ &\qquad\quad+\|e^{\tau(\psi-\mathsf{d})}[\sigma_{2R}\sigma_{R,\lambda}\chi_{\delta,\lambda}(\psi)\widetilde{\chi}_\delta(\psi),P]u\|_0\\ &\qquad\leq Ce^{-\tau\mathsf{d}}\lambda^{1/2}e^{\delta\tau}e^{\tau^2/\lambda}(\|Pu\|_{L^2(B(x^0,4R))}+\|u\|_{m-1})\\ &\qquad\leq C\mu^{1/2}e^{-7\delta\tau}e^{C\tau^2/\mu}(\|Pu\|_{L^2(B(x^0,4R))}+\|u\|_{m-1})\end{aligned} \tag{3.28}$$

where we have applied several times Lemma 2.13 to $\chi_{\delta,\lambda}(\psi)$ or some of its derivatives of order less than $m-1$. So, the Carleman estimate (3.12) applied to $\sigma_{2R}\sigma_{R,\lambda}\chi_{\delta,\lambda}(\psi)\widetilde{\chi}_\delta(\psi)u$ together with (3.18), (3.19), (3.27) and (3.28) gives (3.26) for all $\tau_0 \leq \tau$, μ large enough, and λ such that (3.16) holds. □

3.3. *Step 3: A complex analysis argument*

The purpose of this part is to transfer the information given by the Carleman estimate to some estimates on the low frequencies of the function and conclude the proof of Theorem 3.1. The presence of the nonlocal regularizing term $e^{-\varepsilon|D_a|^2/(2\tau)}$ makes this task more intricate than in the usual case and imposes working by duality. As in [Tat95, Hör97, Tat99b, Tat99a], the idea is to proceed in the following three steps:

1. We make a kind of foliation along the level sets of ψ: if we want to measure u, we rather define the distribution $h_f = \psi_*(fu)$ by
$$\langle h_f, w\rangle_{\mathcal{E}'(\mathbb{R}),C^\infty(\mathbb{R})} = \langle fu, w(\psi)\rangle_{\mathcal{E}'(\mathbb{R}^n),C^\infty(\mathbb{R}^n)},$$
and estimate it for any test function f. Heuristically, $h_f(s)$ is the integral of fu on the level set $\{\psi(x) = s\}$.
2. We notice that the Fourier transform of h_f is $\hat{h}_f(\zeta) = \langle fu, e^{-i\zeta\psi}\rangle$ and can be extended to the complex domain if u is compactly supported. In particular, on the imaginary axis, $\hat{h}_f(i\tau) = \langle f, ue^{\tau\psi}\rangle$. Since the Carleman estimate gives information on the norm of $e^{\tau\psi}u$ for τ large, this can be translated into some information on $\hat{h}_f$ on the upper imaginary axis. A Phragmén–Lindelöf type argument allows one to transfer this estimate to the (almost) whole upper half-plane.
3. Finally, by using a change of contour, this information can be transferred to the real axis where we can estimate the real Fourier transform $\hat{h}_f$.

Note that in the problem of (qualitative) unique continuation, the third step is replaced by a Paley–Wiener type argument: a bound of exponential type for $|\hat{h}_f(\zeta)|$ on $\mathbb{C}$ implies some conditions on the support of h_f. Roughly speaking, if $\psi(x) = x_1$, the problem is to transfer information on the Laplace transform (with respect to the x_1 variable) $\int_{x_1 \geq C} e^{\tau x_1} f u$ (given by the Carleman estimate) to information on the Fourier transform using complex analysis. Moreover, since the Carleman estimate only gives some information on $e^{-\varepsilon |D_a|^2/(2\tau)} e^{\tau \psi} u$, we need to add some cutoff in frequency to this reasoning.

More precisely, let us define

$$\eta \in C_0^\infty((-4, 1)), \qquad \eta = 1 \text{ in } [-1/2, 1/2] \quad \text{and} \quad \eta_\delta(s) := \eta(s/\delta).$$

We first prove Lemma 3.10 below. We then complete the proof of Theorem 3.1, by estimating from below the left hand side of the inequality appearing in the lemma.

Lemma 3.10. *Under the above assumptions, there is* $\tilde{\tau}_0 = (\|\psi\|_{L^\infty(B(x^0,4R))} + 9\delta)^{1/2} \tau_0 > 0$ *such that for any* $\kappa, c_1 > 0$, *there exist* $\beta_0, C, c > 0$ (*depending on* δ, ψ, d, τ_0, κ, c_1, ε, R *and all the cutoff functions*) *such that for any* $0 < \beta < \beta_0$, *for all* $\mu \geq \tilde{\tau}_0/\beta$ *and* $u \in C_0^\infty(\mathbb{R}^n)$, *we have*

$$\|M^{\beta\mu} \sigma_{2R} \sigma_{R,\lambda} \chi_{\delta,\lambda}(\psi) \widetilde{\chi}_\delta(\psi) \eta_{\delta,\lambda}(\psi) u\|_{m-1} \leq C e^{-c\mu} (D + \|u\|_{m-1})$$

with

$$D = e^{\kappa\mu} (\|M_\lambda^{2\mu} \vartheta_\lambda u\|_{m-1} + \|Pu\|_{B(x^0,4R)}), \qquad \lambda = 2c_1\mu.$$

Proof. We now follow [Hör97, Proposition 2.1]. For any test function $f \in \mathcal{S}(\mathbb{R}^n)$, we define the following distribution (with $\beta > 0$ to be chosen later):

$$\langle h_f, w \rangle_{\mathcal{E}'(\mathbb{R}), C^\infty(\mathbb{R})} := \langle (M^{\beta\mu} f) \sigma_{2R} \sigma_{R,\lambda} \chi_{\delta,\lambda}(\psi) \widetilde{\chi}_\delta(\psi) u, w(\psi) \rangle_{\mathcal{E}'(\mathbb{R}^n), C^\infty(\mathbb{R}^n)}.$$

We choose the particular test functions $w = \eta_{\delta,\lambda}$, and want to estimate the quantity

$$\begin{aligned} \langle h_f, \eta_{\delta,\lambda} \rangle_{\mathcal{E}'(\mathbb{R}), C^\infty(\mathbb{R})} &= \langle (M^{\beta\mu} f) \sigma_{2R} \sigma_{R,\lambda} \chi_{\delta,\lambda}(\psi) \widetilde{\chi}_\delta(\psi) u, \eta_{\delta,\lambda}(\psi) \rangle_{\mathcal{E}'(\mathbb{R}^n), C^\infty(\mathbb{R}^n)} \\ &= \langle M^{\beta\mu} \sigma_{2R} \sigma_{R,\lambda} \chi_{\delta,\lambda}(\psi) \widetilde{\chi}_\delta(\psi) \eta_{\delta,\lambda}(\psi) u, f \rangle_{\mathcal{S}'(\mathbb{R}^n), \mathcal{S}(\mathbb{R}^n)}, \end{aligned}$$

uniformly with respect to f to finally obtain an estimate on

$$\|M^{\beta\mu} \sigma_{2R} \sigma_{R,\lambda} \chi_{\delta,\lambda}(\psi) \widetilde{\chi}_\delta(\psi) \eta_{\delta,\lambda}(\psi) u\|_{m-1}.$$

Being the Fourier transform of a compactly supported distribution, $\hat{h}_f$ is an entire function satisfying

$$\begin{aligned} \hat{h}_f(\zeta) &= \langle (M^{\beta\mu} f) \sigma_{2R} \sigma_{R,\lambda} \chi_{\delta,\lambda}(\psi) \widetilde{\chi}_\delta(\psi) u, e^{-i\zeta\psi} \rangle_{\mathcal{E}'(\mathbb{R}^n), C^\infty(\mathbb{R}^n)} \\ &= \langle \sigma_{2R} \sigma_{R,\lambda} \chi_{\delta,\lambda}(\psi) \widetilde{\chi}_\delta(\psi) u, e^{-i\zeta\psi} (M^{\beta\mu} f) \rangle_{\mathcal{E}'(\mathbb{R}^n), C^\infty(\mathbb{R}^n)} \\ &= \langle e^{-i\zeta\psi} \sigma_{2R} \sigma_{R,\lambda} \chi_{\delta,\lambda}(\psi) \widetilde{\chi}_\delta(\psi) u, (M^{\beta\mu} f) \rangle_{\mathcal{E}'(\mathbb{R}^n), C^\infty(\mathbb{R}^n)}, \qquad \zeta \in \mathbb{C}. \end{aligned}$$

Using $\operatorname{supp}\sigma_{2R} \subset B(x^0, 4R)$, we have the a priori estimate

$$\begin{aligned}|\hat{h}_f(\zeta)| &= |\langle e^{-i\zeta\psi}\sigma_{2R}\sigma_{R,\lambda}\chi_{\delta,\lambda}(\psi)\widetilde{\chi}_\delta(\psi)u, (M^{\beta\mu}f)\rangle_{\mathcal{E}'(\mathbb{R}^n),C^\infty(\mathbb{R}^n)}| \\ &\le \|e^{-i\zeta\psi}\sigma_{2R}\sigma_{R,\lambda}\chi_{\delta,\lambda}(\psi)\widetilde{\chi}_\delta(\psi)u\|_{m-1}\|(M^{\beta\mu}f)\|_{1-m} \\ &\le C\langle|\zeta|\rangle^{m-1}e^{|\operatorname{Im}\zeta|\,\|\psi\|_{L^\infty(B(x^0,4R))}}\|u\|_{m-1}\|f\|_{1-m}, \qquad \zeta\in\mathbb{C}.\end{aligned} \tag{3.29}$$

It will be in particular useful for $\zeta \in \mathbb{R}$, in which case the exponential vanishes.

Finally, for $\zeta = i\tau$ with $\tau > 0$, we have

$$\begin{aligned}|\hat{h}_f(i\tau)| &= |\langle (M^{\beta\mu}f), e^{\tau\psi}\sigma_{2R}\sigma_{R,\lambda}\chi_{\delta,\lambda}(\psi)\widetilde{\chi}_\delta(\psi)u\rangle_{C^\infty(\mathbb{R}^n),\mathcal{E}'(\mathbb{R}^n)}| \\ &= \left|\left\langle e^{\frac{\varepsilon}{2\tau}|D_a|^2}(M^{\beta\mu}f), e^{-\frac{\varepsilon}{2\tau}|D_a|^2}e^{\tau\psi}\sigma_{2R}\sigma_{R,\lambda}\chi_{\delta,\lambda}(\psi)\widetilde{\chi}_\delta(\psi)u\right\rangle_{\mathcal{S}(\mathbb{R}^n),\mathcal{S}'(\mathbb{R}^n)}\right| \\ &\le \left\|e^{\frac{\varepsilon}{2\tau}|D_a|^2}M^{\beta\mu}f\right\|_{1-m}\left\|e^{-\frac{\varepsilon}{2\tau}|D_a|^2}e^{\tau\psi}\sigma_{2R}\sigma_{R,\lambda}\chi_{\delta,\lambda}(\psi)\widetilde{\chi}_\delta(\psi)u\right\|_{m-1} \\ &\le e^{\frac{\varepsilon}{2\tau}\beta^2\mu^2}\|f\|_{1-m}\|Q^\psi_{\varepsilon,\tau}\sigma_{2R}\sigma_{R,\lambda}\chi_{\delta,\lambda}(\psi)\widetilde{\chi}_\delta(\psi)u\|_{m-1},\end{aligned}$$

as $M^{\beta\mu} = m\big(\frac{D_a}{\beta\mu}\big)$, with $|\xi_a| \le \beta\mu$ on $\operatorname{supp} m\big(\frac{\cdot}{\beta\mu}\big)$. Using (3.26) we obtain, for all $\tau \ge \tau_0$, $\mu \ge 1$, $\frac{1}{\mathsf{C}}\mu \le \lambda \le \mathsf{C}\mu$,

$$\begin{aligned}|\hat{h}_f(i\tau)| \le Ce^{\frac{\varepsilon}{2\tau}\beta^2\mu^2}\|f\|_{1-m}\big(&\mu^{1/2}e^{C\tau^2/\mu}e^{\delta\tau}\|Pu\|_{B(x^0,4R)} + e^{2\delta\tau}\|M_\lambda^{2\mu}\vartheta_\lambda u\|_{m-1} \\ &+ C\mu^{1/2}\tau^N\big(e^{-8\delta\tau} + e^{-\frac{\varepsilon\mu^2}{8\tau}} + e^{\delta\tau - c\mu}\big)e^{C\tau^2/\mu}e^{\delta\tau}\|u\|_{m-1}\big).\end{aligned}$$

Now, we choose

$$\lambda = 2c_1\mu,$$

and to simplify notation we write, for $\kappa > 0$,

$$D = e^{\kappa\mu}(\|M_\lambda^{2\mu}\vartheta_\lambda u\|_{m-1} + \|Pu\|_{B(x^0,4R)}).$$

With this notation, we have

$$\begin{aligned}|\hat{h}_f(i\tau)| &\le Ce^{\frac{\varepsilon}{2\tau}\beta^2\mu^2}\|f\|_{1-m}\big(\mu^{1/2}e^{C\tau^2/\mu}e^{\delta\tau}e^{-\kappa\mu}D + e^{2\delta\tau}e^{-\kappa\mu}D \\ &\qquad + \mu^{1/2}\tau^N\big(e^{-8\delta\tau} + e^{-\frac{\varepsilon\mu^2}{8\tau}} + e^{\delta\tau-c\mu}\big)e^{C\tau^2/\mu}e^{\delta\tau}\|u\|_{m-1}\big) \\ &\le C\mu^{1/2}\tau^N e^{\frac{\varepsilon}{2\tau}\beta^2\mu^2}e^{C\tau^2/\mu}e^{2\delta\tau}(D + \|u\|_{m-1})\|f\|_{1-m}\big(e^{-c\mu} + e^{-\frac{\varepsilon\mu^2}{8\tau}} + e^{-9\delta\tau}\big),\end{aligned} \tag{3.30}$$

where the new constant $c > 0$ may depend on κ.

We now come back to the quantity we want to estimate:

$$\begin{aligned}\langle M^{\beta\mu}\sigma_{2R}\sigma_{R,\lambda}\chi_{\delta,\lambda}(\psi)\widetilde{\chi}_\delta(\psi)\eta_{\delta,\lambda}(\psi)u, f\rangle_{\mathcal{S}'(\mathbb{R}^n),\mathcal{S}(\mathbb{R}^n)} &= \langle h_f, \eta_{\delta,\lambda}\rangle_{\mathcal{E}'(\mathbb{R}),C^\infty(\mathbb{R})} \\ &= \frac{1}{2\pi}\int_{\mathbb{R}}\hat{h}_f(\zeta)\hat{\eta}_{\delta,\lambda}(-\zeta)\,d\zeta.\end{aligned}$$

As $\eta_\delta \in C_0^\infty(-4\delta, \delta)$, the Paley–Wiener theorem implies that the function $\hat{\eta}_\delta$ is holomorphic in the lower complex half-plane together with the estimate

$$|\hat{\eta}_\delta(\zeta)| \leq Ce^{-4\delta \operatorname{Im} \zeta} \quad \text{for } \operatorname{Im} \zeta \leq 0,$$

that is, for $\operatorname{Im} \zeta \geq 0$,

$$|\hat{\eta}_\delta(-\zeta)| \leq Ce^{4\delta \operatorname{Im} \zeta}, \tag{3.31}$$

$$|\hat{\eta}_{\delta,\lambda}(-\zeta)| = |e^{-\zeta^2/\lambda}\hat{\eta}_\delta(-\zeta)| \leq Ce^{\frac{(\operatorname{Im}\zeta)^2-(\operatorname{Re}\zeta)^2}{\lambda}} e^{4\delta \operatorname{Im} \zeta}. \tag{3.32}$$

For a constant $0 < d \leq 1$ (beware that this d is not the d appearing in the Carleman estimate) to be chosen later, we split the integral into three parts:

$$\int_{\mathbb{R}} \hat{h}_f(\zeta)\hat{\eta}_{\delta,\lambda}(-\zeta)\, d\zeta =: I_- + I_0 + I_+$$

with $I_- := \int_{-\infty}^{-d\mu} \hat{h}_f(\zeta)\hat{\eta}_{\delta,\lambda}(-\zeta)\, d\zeta$, $I_0 := \int_{-d\mu}^{d\mu} \hat{h}_f(\zeta)\hat{\eta}_{\delta,\lambda}(-\zeta)\, d\zeta$, $I_+ := \int_{d\mu}^{+\infty} \hat{h}_f(\zeta)\hat{\eta}_{\delta,\lambda}(-\zeta)\, d\zeta$. According to (3.29) for $\zeta \in \mathbb{R}$ and (3.32), we have, for $\mu \geq 1$, $\lambda = 2c_1\mu$,

$$\begin{aligned}|I_\pm| &\leq C \int_{d\mu}^{+\infty} e^{-|\zeta|^2/\lambda}\langle\zeta\rangle^{m-1}\|u\|_{m-1}\|f\|_{1-m}\, d\zeta \leq C\mu^{2m}e^{-d^2\mu^2/\lambda}\|u\|_{m-1}\|f\|_{1-m} \\ &\leq C_d e^{-cd^2\mu}\|u\|_{m-1}\|f\|_{1-m}.\end{aligned} \tag{3.33}$$

So the main problem is to estimate I_0. For this, let us define

$$\mathcal{H}(\zeta) = \mu^{-1/2}(\zeta + i)^{-N}e^{i2\delta\zeta}\hat{h}_f(\zeta).$$

From (3.30), we have the estimate on the imaginary axis for all $\tau \geq \tau_0$, for $\mu \geq 1$, $\lambda = 2c_1\mu$,

$$|\mathcal{H}(i\tau)| \leq Ce^{\frac{\varepsilon}{2\tau}\beta^2\mu^2}e^{C\tau^2/\mu}(D + \|u\|_{m-1})\|f\|_{1-m}\left(e^{-c\mu} + e^{-\frac{\varepsilon\mu^2}{8\tau}} + e^{-9\delta\tau}\right).$$

Moreover, (3.29) implies (we can assume $N \geq m-1$ without loss of generality)

$$|\mathcal{H}(\zeta)| \leq Ce^{|\operatorname{Im}\zeta|\,\|\psi\|_{L^\infty(B(x^0,4R))}}\|u\|_{m-1}\|f\|_{1-m}, \qquad \zeta \in \mathbb{C},\ \operatorname{Im}\zeta \geq 0.$$

Next, we define $H := \mathcal{H}/c_0$ with

$$c_0 = C(D + \|u\|_{m-1})\|f\|_{1-m}, \tag{3.34}$$

and apply Lemma 3.11 below to the function H. This lemma implies the existence of $d_0 > 0$ (depending only on $\delta, \kappa, \|\psi\|_{L^\infty(B(x^0,4R))}, \varepsilon$ and the constants C, c appearing in the exponents of the estimates of $\mathcal{H}(i\tau)$) such that for any $d < d_0$, there exists $\beta_0 > 0$ (depending on the same parameters together with d) such that for any $0 < \beta < \beta_0$, for all $\mu \geq \tilde{\tau}_0/\beta := \tau_0(\|\psi\|_{L^\infty(B(x^0,4R))} + 9\delta)^{1/2}/\beta$, we have

$$|\mathcal{H}(\zeta)| \leq c_0 e^{-8\delta \operatorname{Im}\zeta} \quad \text{on } \overline{Q}_1 \cap \{d\mu/4 \leq |\zeta| \leq 2d\mu\}$$

where $Q_1 = \mathbb{R}_+^* + i\mathbb{R}_+^*$. The same procedure leads to the same estimate if Q_1 is replaced by $\mathbb{R}_-^* + i\mathbb{R}_+^*$, and hence, by the whole $\mathbb{C}_+ = \{\zeta \in \mathbb{C} : \operatorname{Im}\zeta \geq 0\}$. Coming back to $\hat{h}_f$, we obtain

$$|\hat{h}_f(\zeta)| \leq c_0 \mu^{1/2} \langle |\zeta| \rangle^N e^{-6\delta \operatorname{Im}\zeta} \leq c_0 \mu^{N+1/2} e^{-6\delta \operatorname{Im}\zeta} \quad \text{on } \mathbb{C}_+ \cap \{d\mu/4 \leq |\zeta| \leq 2d\mu\}. \tag{3.35}$$

where c_0 is defined in (3.34).

We now come back to I_0. The function $\hat{h}_f(\zeta)\hat{\eta}_{\delta,\lambda}(-\zeta)$ being holomorphic in $\mathbb{C}_+$, we make the following change of contour in the complex plane:

$$I_0 = \int_{\Gamma_+^V} \hat{h}_f(\zeta)\hat{\eta}_{\delta,\lambda}(-\zeta)\, d\zeta + \int_{\Gamma^H} \hat{h}_f(\zeta)\hat{\eta}_{\delta,\lambda}(-\zeta)\, d\zeta + \int_{\Gamma_-^V} \hat{h}_f(\zeta)\hat{\eta}_{\delta,\lambda}(-\zeta)\, d\zeta,$$

where the contours (oriented counterclockwise, see Figure 5) are defined by

$$\begin{aligned} \Gamma_\pm^V &= \{\operatorname{Re}\zeta = \pm d\mu, 0 \leq \operatorname{Im}\zeta \leq d\mu/2\}, \\ \Gamma^H &= \{-d\mu \leq \operatorname{Re}\zeta \leq d\mu, \operatorname{Im}\zeta = d\mu/2\}, \end{aligned}$$

with $d \in (0, d_0)$ still to be chosen later on.

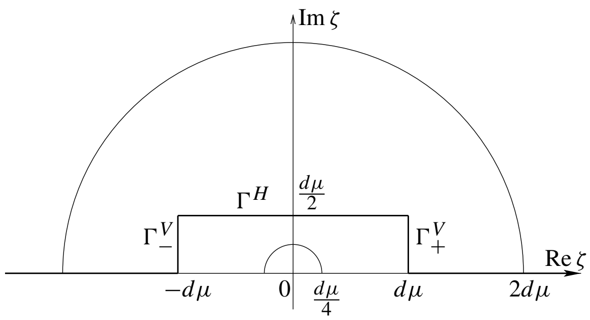

Fig. 5. Contours of integration.

Since $\Gamma_+^V \cup \Gamma^H \cup \Gamma_-^V \subset \mathbb{C}_+ \cap \{d\mu/4 \leq |\zeta| \leq 2d\mu\}$ and $\lambda = c_1\mu$, estimates (3.32) and (3.35) imply

$$\begin{aligned} |\hat{h}_f(\zeta)\hat{\eta}_{\delta,\lambda}(-\zeta)| &\leq c_0 \mu^{N+1/2} e^{-6\delta \operatorname{Im}\zeta} e^{\frac{(\operatorname{Im}\zeta)^2 - (\operatorname{Re}\zeta)^2}{2c_1\mu}} e^{4\delta \operatorname{Im}\zeta} \\ &\leq c_0 \mu^{N+1/2} e^{-2\delta \operatorname{Im}\zeta} e^{\frac{(\operatorname{Im}\zeta)^2 - (\operatorname{Re}\zeta)^2}{2c_1\mu}}, \qquad \zeta \in \Gamma_+^V \cup \Gamma^H \cup \Gamma_-^V. \end{aligned}$$

Using $3d^2\mu^2/4 \leq (\operatorname{Re}\zeta)^2 - (\operatorname{Im}\zeta)^2 \leq d^2\mu^2$ for $\zeta \in \Gamma_+^V \cup \Gamma_-^V$ we now obtain

$$|\hat{h}_f(\zeta)\hat{\eta}_{\delta,\lambda}(-\zeta)| \leq c_0 \mu^{N+1/2} e^{-2\delta \operatorname{Im}\zeta} e^{-\frac{3d^2\mu}{8c_1}}, \qquad \zeta \in \Gamma_+^V \cup \Gamma_-^V.$$

On Γ^H, we have $\operatorname{Im} \zeta = d\mu/2$, so that

$$|\hat{h}_f(\zeta)\hat{\eta}_{\delta,\lambda}(-\zeta)| \leq c_0 \mu^{N+1/2} e^{-\delta d\mu} e^{\frac{d^2}{8c_1}\mu}, \qquad \zeta \in \Gamma^H.$$

Now, we can fix $0 < d \leq \min(4c_1\delta, d_0)$ so that $e^{-\delta d\mu} e^{\frac{d^2}{8c_1}\mu} \leq Ce^{-c\mu}$ (for some $0 < c \leq 2c_1\delta^2$). As a consequence,

$$\begin{aligned}
|I_0| = \left| \int_{\Gamma_+^V \cup \Gamma^H \cup \Gamma_-^V} \hat{h}_f(\zeta)\hat{\eta}_{\delta,\lambda}(-\zeta)\, d\zeta \right| &\leq c_0 \mu^{N+1/2} |\Gamma_+^V \cup \Gamma^H \cup \Gamma_-^V| e^{-c\mu} \\
&\leq Ce^{-c\mu}(D + \|u\|_{m-1}) \|f\|_{1-m} \qquad (3.36)
\end{aligned}$$

for any $0 < \beta < \beta_0$ and all $\mu \geq \max(C, \tilde{\tau}_0/\beta)$ (as $|\Gamma_+^V \cup \Gamma^H \cup \Gamma_-^V| = Cd\mu$).

This together with (3.33) yields, for any $0 < \beta < \beta_0$ and all $\mu \geq \tilde{\tau}_0/\beta$,

$$\begin{aligned}
|\langle M^{\beta\mu}\sigma_{2R}\sigma_{R,\lambda}\chi_{\delta,\lambda}(\psi)\widetilde{\chi}_\delta(\psi)\eta_{\delta,\lambda}(\psi)u, f\rangle_{\mathcal{S}'(\mathbb{R}^n),\mathcal{S}(\mathbb{R}^n)}| &= (2\pi)^{-1} \left| \int_{\mathbb{R}} \hat{h}_f(\zeta)\hat{\eta}_{\delta,\lambda}(-\zeta)\, d\zeta \right| \\
&\leq Ce^{-c\mu}(D + \|u\|_{m-1}) \|f\|_{1-m}.
\end{aligned}$$

The constants being uniform with respect to $f \in \mathcal{S}(\mathbb{R}^n)$, this provides by duality

$$\|M^{\beta\mu}\sigma_{2R}\sigma_{R,\lambda}\chi_{\delta,\lambda}(\psi)\widetilde{\chi}_\delta(\psi)\eta_{\delta,\lambda}(\psi)u\|_{m-1} \leq Ce^{-c\mu}(D + \|u\|_{m-1}),$$

which concludes the proof of the lemma. □

With Lemma 3.10, we can now conclude the proof of the local estimate of Theorem 3.1. Lemma 3.11 and its proof are postponed to the end of the section.

End of the proof of Theorem 3.1. Using Lemma 2.3 with $m(2\,\cdot)$ and $1 - m(\cdot)$, we get

$$\|M_\lambda^{\beta\mu/2}(1 - M^{\beta\mu})\|_{H^{m-1}(\mathbb{R}^n) \to H^{m-1}(\mathbb{R}^n)} \leq Ce^{-c\lambda}.$$

Hence, Lemma 3.10 yields, for any $0 < \beta < \beta_0$ and all $\mu \geq \tilde{\tau}_0/\beta$ and $\lambda = 2c_1\mu$,

$$\begin{aligned}
&\|M_\lambda^{\beta\mu/2}\sigma_{2R}\sigma_{R,\lambda}\chi_{\delta,\lambda}(\psi)\widetilde{\chi}_\delta(\psi)\eta_{\delta,\lambda}(\psi)u\|_{m-1} \\
&\qquad \leq \|M_\lambda^{\beta\mu/2}(1 - M^{\beta\mu})\sigma_{2R}\sigma_{R,\lambda}\chi_{\delta,\lambda}(\psi)\widetilde{\chi}_\delta(\psi)\eta_{\delta,\lambda}(\psi)u\|_{m-1} \\
&\qquad\quad + \|M_\lambda^{\beta\mu/2} M^{\beta\mu}\sigma_{2R}\sigma_{R,\lambda}\chi_{\delta,\lambda}(\psi)\widetilde{\chi}_\delta(\psi)\eta_{\delta,\lambda}(\psi)u\|_{m-1} \\
&\qquad \leq Ce^{-c\mu}(D + \|u\|_{m-1}). \qquad (3.37)
\end{aligned}$$

Using Lemma 2.11, estimate (3.37) and the definition of r in Corollary 3.6, we get, for any $0 < \beta < \beta_0$ and all $\mu \geq \tilde{\tau}_0/\beta$ and $\lambda = 2c_1\mu$,

$$\begin{aligned}
&\|M_\lambda^{\beta\mu/4}\sigma_{r,\lambda}u\|_{m-1} \leq \|\sigma_{r,\lambda}M_\lambda^{\beta\mu/2}u\|_{m-1} + Ce^{-c\mu}\|u\|_{m-1} \\
&\leq \|\sigma_{r,\lambda}M_\lambda^{\beta\mu/2}\sigma_{2R}\sigma_{R,\lambda}\chi_{\delta,\lambda}(\psi)\widetilde{\chi}_\delta(\psi)\eta_{\delta,\lambda}(\psi)u\|_{m-1} \\
&\quad + \left\|\sigma_{r,\lambda}M_\lambda^{\beta\mu/2}\big(1 - \sigma_{2R}\sigma_{R,\lambda}\chi_{\delta,\lambda}(\psi)\widetilde{\chi}_\delta(\psi)\eta_{\delta,\lambda}(\psi)\big)u\right\|_{m-1} + Ce^{-c\mu}\|u\|_{m-1} \\
&\leq Ce^{-c\mu}(D + \|u\|_{m-1}) + \left\|\sigma_{r,\lambda}M_\lambda^{\beta\mu/2}\big(1 - \sigma_{2R}\sigma_{R,\lambda}\chi_{\delta,\lambda}(\psi)\eta_{\delta,\lambda}(\psi)\big)u\right\|_{m-1}. \qquad (3.38)
\end{aligned}$$

We know that $\sigma_R = \chi_\delta(\psi) = \widetilde{\chi}_\delta(\psi) = \eta_\delta(\psi) = 1$ on a neighborhood of $\operatorname{supp}\sigma_r$ according to (3.15) and the properties of χ, $\widetilde{\chi}_\delta$ and η. So, we can select $\Pi \in C_0^\infty(\mathbb{R}^n)$ such that $\Pi = 1$ on a neighborhood of $\operatorname{supp}\sigma_r$ and such that $\sigma_{2R} = \sigma_R = \chi_\delta(\psi) = \widetilde{\chi}_\delta(\psi) = \eta_\delta(\psi) = 1$ on a neighborhood of $\operatorname{supp}\Pi$. Now, we have

$$\begin{aligned}\big\|\sigma_{r,\lambda}M_\lambda^{\beta\mu/2}\big(1-\sigma_{2R}\sigma_{R,\lambda}\chi_{\delta,\lambda}(\psi)\widetilde{\chi}_\delta(\psi)\eta_{\delta,\lambda}(\psi)\big)u\big\|_{m-1}\\ \leq \big\|\sigma_{r,\lambda}M_\lambda^{\beta\mu/2}\big(1-\sigma_{2R}\sigma_{R,\lambda}\chi_{\delta,\lambda}(\psi)\widetilde{\chi}_\delta(\psi)\eta_{\delta,\lambda}(\psi)\big)(1-\Pi)u\big\|_{m-1}\\ +\big\|\sigma_{r,\lambda}M_\lambda^{\beta\mu/2}\big(1-\sigma_{2R}\sigma_{R,\lambda}\chi_{\delta,\lambda}(\psi)\widetilde{\chi}_\delta(\psi)\eta_{\delta,\lambda}(\psi)\big)\Pi u\big\|_{m-1}.\end{aligned} \tag{3.39}$$

To estimate the first term, we use Lemma 2.10 to obtain $\|\sigma_{r,\lambda}M_\lambda^{\beta\mu/2}(1-\Pi)\|_{H^{m-1}\to H^{m-1}} \leq Ce^{-c\mu}$. Concerning the second term, we have

$$\begin{aligned}\big\|\sigma_{r,\lambda}M_\lambda^{\beta\mu/2}\big(1-\sigma_{2R}\sigma_{R,\lambda}\chi_{\delta,\lambda}(\psi)\widetilde{\chi}_\delta(\psi)\eta_{\delta,\lambda}(\psi)\big)\Pi u\big\|_{m-1}\\ \leq C\big\|\big(1-\sigma_{2R}\sigma_{R,\lambda}\chi_{\delta,\lambda}(\psi)\widetilde{\chi}_\delta(\psi)\eta_{\delta,\lambda}(\psi)\big)\Pi u\big\|_{m-1} \leq Ce^{-c\mu}\|u\|_{m-1}\end{aligned} \tag{3.40}$$

where in the last inequality we have decomposed

$$\begin{aligned}1-\sigma_{2R}\sigma_{R,\lambda}\chi_{\delta,\lambda}(\psi)\widetilde{\chi}_\delta(\psi)\eta_{\delta,\lambda}(\psi) &= (1-\sigma_{2R})+\sigma_{2R}(1-\sigma_{R,\lambda})+\sigma_{2R}\sigma_{R,\lambda}(1-\chi_{\delta,\lambda}(\psi))\\ &\quad+\sigma_{2R}\sigma_{R,\lambda}\chi_{\delta,\lambda}(\psi)(1-\widetilde{\chi}_\delta(\psi))\\ &\quad+\sigma_{2R}\sigma_{R,\lambda}\chi_{\delta,\lambda}(\psi)\widetilde{\chi}_\delta(\psi)(1-\eta_{\delta,\lambda}(\psi))\end{aligned}$$

and used Lemmata 2.3 and 2.5; these can be applied thanks to the geometric fact that

$$\operatorname{dist}(\operatorname{supp}\Pi, \{x \in \mathbb{R}^n : \sigma_{2R}(x) \neq 1\}) > 0,$$

and the same is true with σ_{2R} replaced by σ_R, $\chi_\delta(\psi)$, $\widetilde{\chi}_\delta(\psi)$ or $\eta_\delta(\psi)$. We now have the existence of $\tilde{\tau}_0 > 0$ such that for any $\kappa, c_1 > 0$, there exist $\beta_0, C, c > 0$ such that for any $0 < \beta < \beta_0$, $\mu \geq \tilde{\tau}_0/\beta$ and $\lambda = 2c_1\mu$, the following estimate holds:

$$\|M_\lambda^{\beta\mu/4}\sigma_{r,\lambda}u\|_{m-1} \leq Ce^{-c\mu}(D+\|u\|_{m-1}), \quad D = e^{\kappa\mu}(\|M_\lambda^{2\mu}\vartheta_\lambda u\|_{m-1}+\|Pu\|_{B(x^0,4R)}).$$

This concludes the proof of Theorem 3.1 with $\kappa' = c$, after replacing μ and μ_0 by $\mu/2$ and $\mu_0/2$ respectively. □

It only remains to prove Lemma 3.11 below.

Lemma 3.11. *Let $\delta, \kappa, R_0, C_1, \varepsilon, \tau_0 > 0$. Then there exists $d_0 = d_0(\delta, \kappa, R_0, C_1, \varepsilon)$ such that for any $d < d_0$, there exists $\beta_0(\delta, \kappa, R_0, c_1, \varepsilon, d)$ such that for any $0 < \beta < \beta_0$ and all $\mu \geq \tau_0(R_0+9\delta)^{1/2}/\beta$, the following holds: for every holomorphic function H in $Q_1 = \mathbb{R}_+^* + i\mathbb{R}_+^*$, continuous on $\overline{Q}_1$ and satisfying*

$$|H(i\tau)| \leq e^{\varepsilon\frac{\beta^2}{2\tau}\mu^2}e^{C_1\tau^2/\mu}\max\big(e^{-\kappa\mu}, e^{-\frac{\varepsilon\mu^2}{8\tau}}, e^{-9\delta\tau}\big) \quad \textit{for } \tau \in [\tau_0, +\infty), \tag{3.41}$$

$$|H(\zeta)| \leq e^{R_0\operatorname{Im}\zeta} \quad \textit{on } \overline{Q}_1, \tag{3.42}$$

we have

$$|H(\zeta)| \leq e^{-8\delta\operatorname{Im}\zeta} \quad \textit{on } \overline{Q}_1 \cap \{d\mu/4 \leq |\zeta| \leq 2d\mu\}. \tag{3.43}$$

The proof of Lemma 3.11 essentially consists in performing a scaling argument to get rid of the parameter μ, and then using complex analysis arguments: construction of an appropriate harmonic function and application of a quantitative maximum principle. These technical arguments are given in Lemmata B.1, B.2 (construction of the harmonic function, and associated estimates) and B.4 (maximum principle), and are postponed to Appendix B for the sake of readability.

Proof of Lemma 3.11. The function H is holomorphic in Q_1 and $z \mapsto \log|z|$ is subharmonic on $\mathbb{C}^*$. As a consequence, the function

$$g^\mu : \zeta \mapsto \mu^{-1} \log|H(\mu\zeta)|$$

is subharmonic on Q_1 (which is invariant by dilations). Assumption (3.41) (used for $\tau\mu \in [\tau_0, +\infty)$) yields

$$g^\mu(i\tau) \le C_1\tau^2 + \frac{\varepsilon\beta^2}{\tau} + \max\left(-\kappa, -9\delta\tau, -\frac{\varepsilon}{8\tau}\right) \quad \text{for } \tau \in [\tau_0/\mu, +\infty), \tag{3.44}$$

and assumption (3.42) yields

$$g^\mu(\zeta) \le R_0 \operatorname{Im}\zeta \quad \text{on } \overline{Q}_1. \tag{3.45}$$

Now, we set, for $y \in \mathbb{R}_+$,

$$\begin{aligned} f_1^\mu(y) &= R_0 y \mathbb{1}_{[0,\tau_0/\mu)}(y) \\ &\quad + \mathbb{1}_{[\tau_0/\mu,+\infty)}(y) \min\left\{R_0 y, \max\left(-\kappa, -9\delta y, -\frac{\varepsilon}{8y}\right) + C_1 y^2 + \frac{\varepsilon\beta^2}{y}\right\}. \end{aligned} \tag{3.46}$$

According to Lemma B.2, there exists $d_0 = d_0(\delta, \kappa, R_0, \varepsilon, C_1)$ such that for every $d < d_0$, there exists $\beta_0(\delta, \kappa, R_0, d, \varepsilon, C_1)$ such that for any $0 < \beta < \beta_0$ and any $\mu \ge \tau_0(R_0 + 9\delta)^{1/2}/\beta$, the function f_1^μ is continuous and the associated function f^μ given by Lemma B.1 with $f_0 = 0$ and $f_1 = f_1^\mu$ satisfies

$$\begin{aligned} &f^\mu \in C^0(\overline{Q}_1), \quad \Delta f^\mu = 0 \text{ in } Q_1, \quad |f^\mu(x,y)| \le C_\mu(1 + |(x,y)|) \text{ in } Q_1, \\ &f^\mu = f_1^\mu \text{ on } i\mathbb{R}_+, \quad f^\mu = 0 \text{ on } \mathbb{R}_+, \end{aligned}$$

together with

$$f^\mu(\zeta) \le -8\delta \operatorname{Im}\zeta \quad \text{on } \overline{Q}_1 \cap \{d/4 \le |\zeta| \le 2d\}.$$

This yields

$$f^\mu(\zeta/\mu) \le -8\delta(\operatorname{Im}\zeta)/\mu \quad \text{on } \overline{Q}_1 \cap \{d\mu/4 \le |\zeta| \le 2d\mu\}. \tag{3.47}$$

Now, as g^μ is subharmonic and f^μ is harmonic, the function

$$h^\mu(\zeta) := g^\mu(\zeta) - f^\mu(\zeta)$$

is subharmonic too. As a consequence of (3.44)–(3.46), we have

$$h^\mu(\zeta) \le 0 \quad \text{on } \mathbb{R}_+ \cup i\mathbb{R}_+.$$

Moreover, (3.45) and $|f^\mu(\zeta)| \le C(1+|\zeta|)$ also yield

$$h^\mu(\zeta) \le C_\mu + (C_\mu + R_0)|\zeta|.$$

According to Lemma B.4, this implies

$$h^\mu(\zeta) \le 0 \quad \text{on } \overline{Q}_1,$$

and hence

$$|H(\mu\zeta)| = e^{\mu g^\mu(\zeta)} \le e^{\mu f^\mu(\zeta)} \quad \text{on } \overline{Q}_1.$$

Finally, coming back to (3.47), we obtain

$$|H(\zeta)| \le e^{-8\delta \operatorname{Im}\zeta} \quad \text{on } \overline{Q}_1 \cap \{d\mu/4 \le |\zeta| \le 2d\mu\},$$

which concludes the proof of the lemma. □

4. Semiglobal estimates

4.1. Some tools for propagating information

The local estimate of Theorem 3.1 only provides information on the low frequency part of the function. Iterating this result allows us to propagate the low frequency information. In this section, we define some tools that will be useful for this iterative procedure. They are aimed at describing how information on the low frequency part of the solution can be deduced from one subregion to another one.

Definition 4.1. Fix an open subset Ω of $\mathbb{R}^n = \mathbb{R}^{n_a} \times \mathbb{R}^{n_b}$, a differential operator P of order m defined in Ω, and two finite collections $(V_j)_{j\in J}$ and $(U_i)_{i\in I}$ of bounded open sets in $\mathbb{R}^n$. We say that $(V_j)_{j\in J}$ is *under the dependence of* $(U_i)_{i\in I}$, denoted

$$(V_j)_{j\in J} \trianglelefteq (U_i)_{i\in I},$$

if for any $\vartheta_i \in C_0^\infty(\mathbb{R}^n)$ such that $\vartheta_i(x) = 1$ on a neighborhood of $\overline{U_i}$, for any $\widetilde{\vartheta}_j \in C_0^\infty(V_j)$ and for all $\kappa, \alpha > 0$, there exist $C, \kappa', \beta, \mu_0 > 0$ such that for all $(\mu, u) \in [\mu_0, +\infty) \times C_0^\infty(\mathbb{R}^n)$, we have

$$\sum_{j\in J} \|M_\mu^{\beta\mu} \widetilde{\vartheta}_{j,\mu} u\|_{m-1} \le Ce^{\kappa\mu}\Big(\sum_{i\in I} \|M_\mu^{\alpha\mu} \vartheta_{i,\mu} u\|_{m-1} + \|Pu\|_{L^2(\Omega)}\Big) + Ce^{-\kappa'\mu}\|u\|_{m-1}.$$

If the cardinality of I is 1, and U is the only set of the family $(U_i)_{i\in I}$, we simply write $(V_j)_{j\in J} \trianglelefteq U$. We use the same convention if the cardinality of J is 1.

Recall that the norm $\|\cdot\|_{m-1}$ is always taken in the whole $\mathbb{R}^n$.

Remark 4.2. The relation $\trianglelefteq$ actually depends on the splitting $\mathbb{R}^n = \mathbb{R}^{n_a} \times \mathbb{R}^{n_b}$, the set Ω and the operator P. However, in the main part of this work, $\mathbb{R}^n = \mathbb{R}^{n_a} \times \mathbb{R}^{n_b}$, Ω and P will be fixed, so it should not lead to confusion (in particular in applications). The dependence of $\trianglelefteq$ upon these objects will be mentioned when needed.

For applications, it is important that the functions u are not necessarily supported in Ω.

In the following, we will only need to use the relation $\trianglelefteq$ in appropriate coordinate charts. However, it will not be a problem for what we want to prove, even on a compact manifold. Indeed, we will fix some coordinate chart on an open set $\Omega \subset \mathbb{R}^n$ close to a point or close to a trajectory. Then, we will use the relation $\trianglelefteq$ relative to Ω to finally obtain estimates which will be invariant by changes of coordinates.

Now, we list some general properties of the relation $\trianglelefteq$, which actually hold without using any assumption on the set Ω or the operator P.

Proposition 4.3. (1) *If $(V_j)_{j\in J} \trianglelefteq (U_i)_{i\in I}$ with $U_i = U$ for all $i \in I$, then $(V_j)_{j\in J} \trianglelefteq U$.*
(2) *If $(V_j)_{j\in J} \trianglelefteq (U_i)_{i\in I}$ with $U_i \subset W_i$ for all $i \in I$, then $(V_j)_{j\in J} \trianglelefteq (W_i)_{i\in I}$.*
(3) *If $V \subset U$, then $V \trianglelefteq U$. In particular, $U \trianglelefteq U$.*
(4) *$\bigcup_{i\in I} U_i \trianglelefteq (U_i)_{i\in I}$.*
(5) *If $V_i \trianglelefteq U_i$ for any $i \in I$, then $(V_i)_{i\in I} \trianglelefteq (U_i)_{i\in I}$. In particular, $(U_i)_{i\in I} \trianglelefteq (U_i)_{i\in I}$.*

Proof. Property (1) is obvious from the definition, and (2) is also immediate since $\vartheta_i(x) = 1$ on a neighborhood of W_i implies $\vartheta_i(x) = 1$ on a neighborhood of $U_i \subset W_i$.

Property (3) is a consequence of Lemma 2.11 applied with $\alpha\mu/2$ instead of μ, $\lambda = \mu$, $f_1 = \vartheta$ and $f = \tilde{\vartheta}$. The assumptions on ϑ and $\tilde{\vartheta}$ ensure that $f_1 = 1$ on a uniform neighborhood of $\operatorname{supp} f$. This gives the result with $\beta = \alpha/2$.

Property (4) is a consequence of Lemma 2.12 with the same parameters as for (3), but with $b_i = \vartheta_i$.

Property (5) is almost a consequence of the definition. Actually, the only difference is that a priori, we have one β_i for each $i \in I$. Taking the worst of the constants C, κ', μ_0 given by the application of the definition for any i gives

$$\sum_{i\in I} \|M_\mu^{\beta_i\mu}\tilde{\vartheta}_{i,\mu}u\|_{m-1} \le Ce^{\kappa\mu}\Big(\sum_{i\in I} \|M_\mu^{\alpha\mu}\vartheta_{i,\mu}u\|_{m-1} + \|Pu\|_{L^2(\Omega)}\Big) + Ce^{-\kappa'\mu}\|u\|_{m-1}$$

with $\vartheta_i = 1$ on $\overline{U_i}$ and $\tilde{\vartheta}_i \in C_0^\infty(V_i)$. But taking $2\beta = \min\{\beta_i : i \in I\}$, we have

$$\begin{aligned}\|M_\mu^{\beta\mu}\tilde{\vartheta}_{i,\mu}u\|_{m-1} &\le \|M_\mu^{\beta_i\mu}M_\mu^{\beta\mu}\tilde{\vartheta}_{i,\mu}u\|_{m-1} + \|M_\mu^{\beta\mu}(1 - M_\mu^{\beta_i\mu})\tilde{\vartheta}_{i,\mu}u\|_{m-1}\\ &\le \|M_\mu^{\beta_i\mu}\tilde{\vartheta}_{i,\mu}u\|_{m-1} + Ce^{-c\mu}\|u\|_{m-1},\end{aligned}$$

where we have used Lemma 2.3 and the properties of the support of $m(\cdot/\beta)$ and $1 - m(\cdot/\beta_i)$ for the last estimate. The second part comes from the first, together with $U_i \trianglelefteq U_i$ for all $i \in I$. $\square$

The relation is not transitive but we have the following weaker but sufficient property: if $(V_j)_{j\in J} \trianglelefteq (\tilde{U}_i)_{i\in I}$ and $\tilde{U}_i \Subset U_i$ (that is, $\overline{\tilde{U}_i} \subset U_i$) and $(U_i)_{i\in I} \trianglelefteq (W_k)_{k\in K}$, then

$(V_j)_{j\in J} \trianglelefteq (W_k)_{k\in K}$ (this is proved by introducing functions $f_i \in C_0^\infty(U_i)$ equal to 1 on $\widetilde{U}_i$: see the proof of Proposition 4.5(6) below).

For this reason, it is convenient to introduce the following stronger property.

Definition 4.4. Given an open set Ω in $\mathbb{R}^n = \mathbb{R}^{n_a} \times \mathbb{R}^{n_b}$, a differential operator P of order m defined in Ω, and two finite collections $(V_j)_{j\in J}$ and $(U_i)_{i\in I}$ of bounded open sets in $\mathbb{R}^n$, we say that $(V_j)_{j\in J}$ is *under the strong dependence* of $(U_i)_{i\in I}$ if there exist $\widetilde{U}_i \Subset U_i$ such that $(V_j)_{j\in J} \trianglelefteq (\widetilde{U}_i)_{i\in I}$. In that case, we write

$$(V_j)_{j\in J} \lhd (U_i)_{i\in I}.$$

This makes the relation transitive, but it becomes more strict in the sense that we do not always have $U \lhd U$. We again summarize the properties of this relation.

Proposition 4.5. (1) *$(V_j)_{j\in J} \lhd (U_i)_{i\in I}$ implies $(V_j)_{j\in J} \trianglelefteq (U_i)_{i\in I}$.*
(2) *If $(V_j)_{j\in J} \lhd (U_i)_{i\in I}$ with $U_i = U$ for all $i \in I$, then $(V_j)_{j\in J} \lhd U$.*
(3) *If $V_i \Subset U_i$ for any $i \in I$, then $(V_i)_{i\in I} \lhd (U_i)_{i\in I}$.*
(4) *If $V_i \Subset U_i$ for any $i \in I$, then $\bigcup_{i\in I} V_i \lhd (U_i)_{i\in I}$.*
(5) *If $V_i \lhd U_i$ for any $i \in I$, then $(V_i)_{i\in I} \lhd (U_i)_{i\in I}$. In particular, if $U_i \lhd U$ for any $i \in I$, then $(U_i)_{i\in I} \lhd U$.*
(6) *The relation is transitive, that is,*

$$[(V_j)_{j\in J} \lhd (U_i)_{i\in I} \text{ and } (U_i)_{i\in I} \lhd (W_k)_{k\in K}] \Rightarrow (V_j)_{j\in J} \lhd (W_k)_{k\in K}.$$

Proof. Property (1) is obvious. For (2), the assumption gives some $(\widetilde{U}_i)_{i\in I}$ with $(V_j)_{j\in J} \trianglelefteq (\widetilde{U}_i)_{i\in I}$ and $\widetilde{U}_i \Subset U$ for all $i \in I$. Since $\overline{\widetilde{U}_i} \subset U$ for all $i \in I$ and I is finite, we have $\overline{\bigcup_{i\in I} \widetilde{U}_i} = \bigcup_{i\in I} \overline{\widetilde{U}_i} \subset U$. Denote $W = \bigcup_{i\in I} \widetilde{U}_i$. We have $\widetilde{U}_i \subset W$ for all $i \in I$, so property (2) and then Proposition 4.3(1) give $(V_j)_{j\in J} \trianglelefteq W$, which implies $(V_j)_{j\in J} \lhd U$ since $W \Subset U$.

For (3), we use $(V_i)_{i\in I} \trianglelefteq (V_i)_{i\in I}$ from Proposition 4.3(5) and $V_i \Subset U_i$.

For (4), we use Proposition 4.3(4), which gives $\bigcup_{i\leq I} V_i \trianglelefteq (V_i)_{i\in I}$. This means $\bigcup_{i\leq I} V_i \lhd (U_i)_{i\in I}$ by the definition of $\lhd$.

For (5), assume $V_i \trianglelefteq \widetilde{U}_i$ with $\widetilde{U}_i \Subset U_i$. Then Proposition 4.3(5) gives $(V_i)_{i\in I} \trianglelefteq (\widetilde{U}_i)_{i\in I}$, which yields $(V_i)_{i\in I} \lhd (U_i)_{i\in I}$ by definition. The second part is direct by combining with (2).

For (6), the assumptions give the existence of $\widetilde{U}_i \Subset U_i$ and $\widetilde{W}_k \Subset W_k$ such that

$$(V_j)_{j\in J} \trianglelefteq (\widetilde{U}_i)_{i\in I} \quad \text{and} \quad (U_i)_{i\in I} \trianglelefteq (\widetilde{W}_k)_{k\in K}.$$

Since $\widetilde{U}_i \Subset U_i$, we can pick $\chi_i \in C_0^\infty(U_i)$ such that $\chi_i = 1$ in a neighborhood of $\overline{\widetilde{U}_i}$. Let $\alpha, \kappa > 0$, and take $\vartheta_k \in C_0^\infty(\mathbb{R}^n)$ (for all $k \in K$) such that $\vartheta_k = 1$ on a neighborhood of $\overline{\widetilde{W}_k}$, and $\widetilde{\vartheta}_j \in C_0^\infty(V_j)$ (for all $j \in J$). Since $(U_i)_{i\in I} \trianglelefteq (\widetilde{W}_k)_{k\in K}$ and $\chi_i \in C_0^\infty(U_i)$, there exist $C, \kappa', \beta, \mu_0 > 0$ such that

$$\sum_{i\in I} \|M_\mu^{\beta\mu} \chi_{i,\mu} u\|_{m-1} \leq Ce^{\kappa\mu/2}\Big(\sum_{k\in K} \|M_\mu^{\alpha\mu} \vartheta_{k,\mu} u\|_{m-1} + \|Pu\|_{L^2(\Omega)}\Big) + Ce^{-\kappa'\mu}\|u\|_{m-1}.$$

Now, we apply the relation given by $(V_j)_{j\in J} \trianglelefteq (\widetilde{U}_i)_{i\in I}$ with α replaced by the above β and κ replaced by $\kappa_1 = \min(\kappa', \kappa)/2 > 0$. Since $\chi_i = 1$ in a neighborhood of $\widetilde{U}_i$ and $\widetilde{\vartheta}_j \in C_0^\infty(V_j)$, there exist $C', \kappa'', \beta', \mu_0' > 0$ such that

$$\sum_{j\in J} \|M_\mu^{\beta'\mu}\widetilde{\vartheta}_{j,\mu}u\|_{m-1} \le C'e^{\kappa_1\mu}\Big(\sum_{i\in I}\|M_\mu^{\beta\mu}\chi_{i,\mu}u\|_{m-1} + \|Pu\|_{L^2(\Omega)}\Big) + C'e^{-\kappa''\mu}\|u\|_{m-1}.$$

Combining the above two estimates now yields

$$\begin{aligned}
&\sum_{j\in J}\|M_\mu^{\beta'\mu}\widetilde{\vartheta}_{j,\mu}u\|_{m-1}\\
&\quad\le CC'e^{(\kappa/2+\kappa_1)\mu}\sum_{k\in K}\|M_\mu^{\alpha\mu}\vartheta_{k,\mu}u\|_{m-1} + C'e^{\kappa_1\mu}(1+Ce^{\kappa\mu/2})\|Pu\|_{L^2(\Omega)}\\
&\qquad + (C'e^{-\kappa''\mu} + CC'e^{(\kappa_1-\kappa')\mu})\|u\|_{m-1}.
\end{aligned}$$

Since $\kappa/2+\kappa_1 \le \kappa$ and $\kappa_1-\kappa' < \kappa'/2-\kappa' = -\kappa'/2 < 0$, this gives $(V_j)_{j\in J} \trianglelefteq (\widetilde{W}_k)_{k\in K}$, which implies the result since $\widetilde{W}_k \Subset W_k$.

Note that in the proofs above, we have not mentioned the restriction $\mu \ge \mu_0$ each time. Yet, all the estimates have to be taken with that restriction, taking the worst constant μ_0 when several restrictions are involved. □

Corollary 4.6. *Under the assumptions of Theorem 3.1, there exists $R_0 > 0$ such that for any $R \in (0, R_0)$, there exist $r, \rho > 0$ such that*

$$B(x^0, r) \trianglelefteq \{\phi > 2\rho\} \cap B(x^0, 3R), \qquad B(x^0, r) \triangleleft \{\phi > \rho\} \cap B(x^0, 4R).$$

Proof. First, we restrict R_0 so that $B(x^0, 4R_0) \subset \Omega$. Theorem 3.1 gives the existence of constants $R, r, \rho, \tilde{\tau}_0 > 0$.

Let $\kappa, \alpha > 0$. We apply the result with $\mu = \alpha\mu'$, $c_1 = 1/\alpha$ and κ replaced by κ/α to obtain, uniformly for $\mu' \ge \tilde{\tau}_0/(\alpha\beta)$,

$$\|M_{\mu'}^{\beta\alpha\mu'}\sigma_{r,\mu'}u\|_{m-1} \le Ce^{\kappa\mu'}(\|M_{\mu'}^{\alpha\mu'}\vartheta_{\mu'}u\|_{m-1} + \|Pu\|_{L^2(B(x^0,4R))}) + Ce^{-\alpha\kappa'\mu'}\|u\|_{m-1}.$$

Now, let $\widetilde{\vartheta} \in C_0^\infty(B(x^0, r))$. Since $\sigma_r = 1$ on $B(x^0, r)$, Lemma 2.11 gives

$$\|M_{\mu'}^{\beta\alpha\mu'/2}\widetilde{\vartheta}_{\mu'}u\|_{m-1} \le \|M_{\mu'}^{\beta\alpha\mu'}\sigma_{r,\mu'}u\|_{m-1} + Ce^{-c\mu'}\|u\|_{m-1},$$

which implies the first statement. The second one comes directly from the compact inclusion of $\{\phi > 2\rho\} \cap B(x^0, 3R)$ into $\{\phi > \rho\} \cap B(x^0, 4R)$. □

4.2. Semiglobal estimates along foliation by graphs

This section is devoted to the proof of Theorem 1.11. Actually, this result is a corollary of the following stronger theorem, stated here in the context of zones of dependence.

Theorem 4.7. *Under the assumptions of Theorem 1.11, for any open neighborhood $\hat{\omega}$ of S_0, there exists an open neighborhood U of K such that $U \lhd \hat{\omega}$.*

In the present section, we first prove that Theorem 4.7 implies Theorem 1.11, and then prove Theorem 4.7.

Proof that Theorem 4.7 implies Theorem 1.11. We first apply Theorem 4.7 for a neighborhood $\hat{\omega}$ of S_0 such that $\hat{\omega} \Subset \tilde{\omega}$, where $\tilde{\omega}$ is as in the statement of Theorem 1.11. We obtain $U \lhd \hat{\omega}$. Take $\chi \in C_0^\infty(U)$ such that $\chi = 1$ on a neighborhood U_χ of K, and $\varphi \in C_0^\infty(\tilde{\omega})$ such that $\varphi = 1$ on a neighborhood of $\hat{\omega}$. We find that for any $\kappa > 0$, there exist $C, \beta, \kappa', \mu_0 > 0$ such that for $\mu \geq \mu_0$,

$$\|M_\mu^{\beta\mu} \chi_\mu u\|_{m-1} \leq Ce^{\kappa\mu}\left(\|M_\mu^\mu \varphi_\mu u\|_{m-1} + \|Pu\|_{L^2(\Omega)}\right) + Ce^{-\kappa'\mu}\|u\|_{m-1}. \tag{4.1}$$

But since $\varphi \in C_0^\infty(\tilde{\omega})$, taking again $\widetilde{\varphi} \in C_0^\infty(\tilde{\omega})$ with $\widetilde{\varphi} = 1$ on a neighborhood of $\operatorname{supp}\varphi$, we get, thanks to Lemma 2.3,

$$\begin{aligned}
\|M_\mu^\mu \varphi_\mu u\|_{m-1} &\leq \|M_\mu^\mu \widetilde{\varphi}\varphi_\mu u\|_{m-1} + \|(1-\widetilde{\varphi})\varphi_\mu u\|_{m-1} \\
&\leq \sum_{|\alpha|+|\beta|\leq m-1} \|D_a^\alpha M_\mu^\mu (D_b^\beta \widetilde{\varphi}\varphi_\mu u)\|_0 + Ce^{-c\mu}\|u\|_{m-1}.
\end{aligned}$$

Next,

$$\begin{aligned}
\|D_a^\alpha M_\mu^\mu f\|_0 &\leq \|\xi_a^\alpha m_\mu(\xi_a/\mu)\|_{L^\infty(\mathbb{R}^{n_a})}\|f\|_0 \\
&\leq \mu^{|\alpha|}\|\xi_a^\alpha m_\mu(\xi_a)\|_{L^\infty(\mathbb{R}^{n_a})}\|f\|_0 \leq C\mu^{|\alpha|}\|f\|_0,
\end{aligned}$$

since $\xi_a \mapsto \xi_a^\alpha m_\mu(\xi_a)$ is uniformly bounded on $\mathbb{R}^{n_a}$ for $\mu \geq 1$. As a consequence,

$$\begin{aligned}
\|M_\mu^\mu \varphi_\mu u\|_{m-1} &\leq C \sum_{|\alpha|+|\beta|\leq m-1} \mu^{|\alpha|}\|D_b^\beta(\widetilde{\varphi}\varphi_\mu u)\|_0 + Ce^{-c\mu}\|u\|_{m-1} \\
&\leq C\mu^{m-1}\sum_{|\beta|\leq m-1}\|D_b^\beta u\|_{L^2(\tilde{\omega})} + Ce^{-c\mu}\|u\|_{m-1} \\
&\leq C\mu^{m-1}\|u\|_{H_b^{m-1}(\tilde{\omega})} + Ce^{-c\mu}\|u\|_{m-1}.
\end{aligned}$$

In the particular case where $n_a = n$, we slightly change the estimate:

$$\begin{aligned}
\|M_\mu^\mu \varphi_\mu u\|_{m-1} &\leq \|M^{2\mu} M_\mu^\mu \varphi_\mu u\|_{m-1} + \|(1-M^{2\mu})M_\mu^\mu \varphi_\mu u\|_{m-1} \\
&\leq C\mu^{s+m-1}\|\varphi_\mu u\|_{-s} + Ce^{-c\mu}\|u\|_{m-1} \\
&\leq C\mu^{s+m-1}\|\widetilde{\varphi}\varphi_\mu u\|_{-s} + C\mu^{s+m-1}\|(1-\widetilde{\varphi})\varphi_\mu u\|_{-s} + Ce^{-c\mu}\|u\|_{m-1} \\
&\leq C\mu^{s+m-1}\|\widetilde{\varphi} u\|_{H^{-s}} + Ce^{-c\mu}\|u\|_{m-1}.
\end{aligned}$$

In (4.1), the constant $\kappa > 0$ is arbitrary (all other constants in that estimate depending on it): imposing $\kappa < c/2$ and noticing that $\mu^{m-1} \leq C_m e^{\kappa\mu}$, we obtain, with $c' := \min(c/2, \kappa')$,

$$\|M_\mu^{\beta\mu}\chi_\mu u\|_{m-1} \leq Ce^{2\kappa\mu}\left(\|u\|_{H_b^{m-1}(\tilde{\omega})} + \|Pu\|_{L^2(\Omega)}\right) + Ce^{-c'\mu}\|u\|_{m-1}. \tag{4.2}$$

In the analytic case, $n_a = n$, using $\mu^{s+m-1} \le C_s e^{\kappa\mu}$ we have similarly

$$\|M_\mu^{\beta\mu}\chi_\mu u\|_{m-1} \le Ce^{2\kappa\mu}\big(\|\widetilde{\varphi}u\|_{H^{-s}} + \|Pu\|_{L^2(\Omega)}\big) + Ce^{-c'\mu}\|u\|_{m-1}.$$

Now, let $\widetilde{\chi} \in C_0^\infty(U_\chi)$ be such that $\widetilde{\chi} = 1$ in a neighborhood of K. We have, using again Lemma 2.3,

$$\begin{aligned}\|\widetilde{\chi}u\|_0 &\le \|\widetilde{\chi}\chi_\mu u\|_0 + \|(1-\chi_\mu)\widetilde{\chi}u\|_0 \le C\|\chi_\mu u\|_0 + Ce^{-c\mu}\|u\|_{m-1}\\ &\le C\|M_\mu^{\beta\mu}\chi_\mu u\|_0 + C\|(1-M_\mu^{\beta\mu})\chi_\mu u\|_0 + Ce^{-c\mu}\|u\|_{m-1}.\end{aligned} \tag{4.3}$$

For the second term on the right hand side, we write

$$\|(1-M_\mu^{\beta\mu})\chi_\mu u\|_0 \le C \sup_{(\xi_a,\xi_b)\in\mathbb{R}^{n_a+n_b}} \left|\frac{(1-m_\mu)\big(\frac{\xi_a}{\beta\mu}\big)}{|\xi_a|^{m-1} + \langle\xi_b\rangle^{m-1}}\right| \|\chi_\mu u\|_{m-1}.$$

In the range $|\xi_a| \ge \beta\mu/2$ with $\mu \ge \mu_0$, we have the loose estimate

$$\left|\frac{(1-m_\mu)\big(\frac{\xi_a}{\beta\mu}\big)}{|\xi_a|^{m-1} + \langle\xi_b\rangle^{m-1}}\right| \le \frac{C}{\mu^{m-1}}. \tag{4.4}$$

In the range $|\xi_a| \le \beta\mu/2$, using $\operatorname{dist}\big(\operatorname{supp}(1-m(\cdot/\beta)), \{|\xi_a| \le \beta/2\}\big) > 0$, we have

$$\left|(1-m_\mu)\left(\frac{\xi_a}{\beta\mu}\right)\right| \le Ce^{-c\mu},$$

in view of Lemma 2.3. In this range of ξ_a, this yields

$$\left|\frac{(1-m_\mu)\big(\frac{\xi_a}{\beta\mu}\big)}{|\xi_a|^{m-1} + \langle\xi_b\rangle^{m-1}}\right| \le Ce^{-c\mu},$$

so that (4.4) holds for all $\xi_a \in \mathbb{R}^{n_a}$ and $\mu \ge \mu_0$. This yields

$$\|(1-M_\mu^{\beta\mu})\chi_\mu u\|_0 \le \frac{C}{\mu^{m-1}}\|\chi_\mu u\|_{m-1},$$

which, combined with (4.2) and (4.3), gives, for $\mu \ge \mu_0$,

$$\|\widetilde{\chi}u\|_0 \le Ce^{2\kappa\mu}\big(\|u\|_{H_b^{m-1}(\tilde{\omega})} + \|Pu\|_{L^2(\Omega)}\big) + \frac{C}{\mu^{m-1}}\|u\|_{m-1}.$$

Similarly, in the analytic case $n_a = n$, we have

$$\|\widetilde{\chi}u\|_0 \le Ce^{2\kappa\mu}\big(\|\widetilde{\varphi}u\|_{H^{-s}} + \|Pu\|_{L^2(\Omega)}\big) + \frac{C}{\mu^{m-1}}\|u\|_{m-1}.$$

Note also that in order to prove the precise statement in this case (for all $\widetilde{\varphi}$ such that...), we first fix $\widetilde{\varphi}$, and then $\hat{\omega}$ and φ, having the above support properties. The rest of the proof remains unchanged.

Finally, the case $n_a = 0$ is a direct consequence of (4.1) since there is no regularization.

Now, we notice that the previous estimates are true for any neighborhood Ω of K (with constants and open subsets depending on Ω). Denoting now by Ω the neighborhood of K given by the assumptions of the theorem, we can apply the previous estimates to an open neighborhood $\widetilde{\Omega}$ of K with $\widetilde{\Omega} \Subset \Omega$. This shows that for any neighborhood $\widetilde{\omega} \subset \widetilde{\Omega}$ of S_0, there exists an open neighborhood $\widetilde{U}$ of K (that we can require to be included in $\widetilde{\Omega}$) so that

$$\|u\|_{L^2(\widetilde{U})} \leq Ce^{2\kappa\mu}\big(\|u\|_{H_b^{m-1}(\widetilde{\omega})} + \|Pu\|_{L^2(\widetilde{\Omega})}\big) + \frac{C}{\mu^{m-1}}\|u\|_{m-1}. \tag{4.5}$$

Take χ_0 supported in Ω and such that $\chi_0 = 1$ in $\widetilde{\Omega}$. In particular, $\|P(\chi_0 u)\|_{L^2(\widetilde{\Omega})} = \|Pu\|_{L^2(\widetilde{\Omega})} \leq \|Pu\|_{L^2(\Omega)}$, $\|\chi_0 u\|_{L^2(\widetilde{U})} = \|u\|_{L^2(\widetilde{U})}$, $\|\chi_0 u\|_{H_b^{m-1}(\widetilde{\omega})} = \|u\|_{H_b^{m-1}(\widetilde{\omega})}$ and $\|\chi_0 u\|_{m-1} \leq C\|u\|_{H^{m-1}(\Omega)}$. Applying inequality (4.5) to $\chi_0 u$ gives

$$\|u\|_{L^2(\widetilde{U})} \leq Ce^{2\kappa\mu}\big(\|u\|_{H_b^{m-1}(\widetilde{\omega})} + \|Pu\|_{L^2(\Omega)}\big) + \frac{C}{\mu^{m-1}}\|u\|_{H^{m-1}(\Omega)}.$$

This concludes the proof of Theorem 1.11 in the general case. The end of the proof in the cases $n_a = n$ and $n_a = 0$ is similar. □

Now, we come to the proof of the main result of this section, namely Theorem 4.7. This proof consists in two main steps: first, to define the adapted geometrical context, and second, to iterate the local result in this geometric context, using an induction argument.

Proof of Theorem 4.7. To begin with, we choose $\omega_1 \Subset \omega_2 \Subset \hat{\omega}$ where ω_1 is another open neighborhood of S_0 (see Figure 1). We fix R small enough such that

$$2R < \min\big(\mathrm{dist}(K, \Omega^c), \mathrm{dist}(\omega_1^c, S_0)\big), \tag{4.6}$$

define the set

$$K^R = \bigcup_{x \in K} B(x, 2R),$$

and pick a cutoff function

$$\chi_K \in C_0^\infty(\Omega), \qquad \chi_K = 1 \text{ on } K^R, \qquad \operatorname{supp} \chi_K \cap \{x_n \leq 0\} \subset \omega_1. \tag{4.7}$$

Given any point $x \in K$, there exists $\varepsilon > 0$ such that $x \in S_\varepsilon$. We denote by $R_0 > 0$ the constant given by Theorem 3.1 associated to the point x and the function ϕ_ε. Next, we set

$$R_x := \min(R_0/2, R/4), \tag{4.8}$$

and then

$$r_x := \min(r/2, 3R_x), \qquad \rho_x = \rho,$$

where $r, \rho > 0$ are the constants given by Theorem 3.1 (and Corollary 4.6) associated to x, ϕ_ε and R_x.

For any $\varepsilon \in (0, 1]$ and $x \in S_\varepsilon$, we have $\phi_\varepsilon(x) = 0$. So, we can write

$$S_\varepsilon \subset \bigcup_{x \in S_\varepsilon} B(x, r_x),$$

and, since S_ε is compact, we can extract a finite covering, i.e. there is a finite set I_ε of indices and a finite family $(x_i^\varepsilon)_{i \in I_\varepsilon}$ of points such that

$$S_\varepsilon \subset \bigcup_{i \in I_\varepsilon} B(x_i^\varepsilon, r_{x_i^\varepsilon}), \qquad x_i^\varepsilon \in S_\varepsilon.$$

For $x_i^\varepsilon \in S_\varepsilon$, we rename the associated radii, setting

$$R_i^\varepsilon := R_{x_i^\varepsilon}, \qquad r_i^\varepsilon := r_{x_i^\varepsilon}, \qquad \rho_i^\varepsilon := \rho_{x_i^\varepsilon},$$

and define

$$\rho_\varepsilon := \min_{i \in I_\varepsilon} \rho_i^\varepsilon > 0.$$

Since $\phi_\varepsilon = 0$ on S_ε, we still have

$$S_\varepsilon \subset \Big(\bigcup_{i \in I_\varepsilon} B(x_i^\varepsilon, r_i^\varepsilon) \Big) \cap \{\phi_\varepsilon < \rho_\varepsilon\} =: \mathcal{U}_\varepsilon.$$

The definition of $\mathcal{U}_\varepsilon$ is illustrated in Figure 6. Therefore, for $\varepsilon \in (0, 1]$, $\mathcal{U}_\varepsilon$ is an open neighborhood of the compact surface S_ε. Since G is C^1, we claim that we can find $g(\varepsilon) > 0$ such that

$$\mathcal{V}_\varepsilon := \bigcup_{\varepsilon' \in (\varepsilon - g(\varepsilon), \varepsilon + g(\varepsilon))} S_{\varepsilon'} \subset \mathcal{U}_\varepsilon \tag{4.9}$$

(the definition of $\mathcal{V}_\varepsilon$ is illustrated in Figure 7). Indeed, since $G \in C^1(\bar{D} \times (0, 1])$, we can find $C > 0$ such that

$$|G(x', \varepsilon) - G(x', \varepsilon')| \leq C|\varepsilon - \varepsilon'|,$$

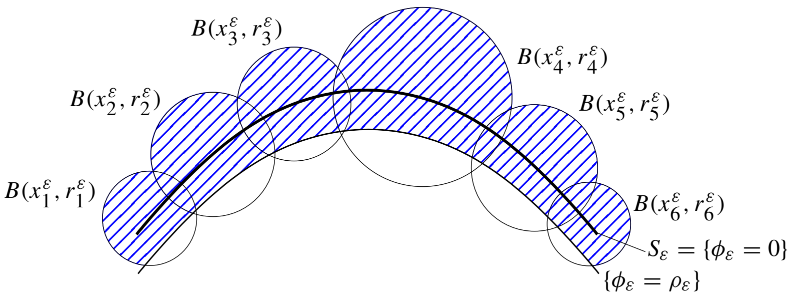

Fig. 6. Definition of the set $\mathcal{U}_\varepsilon$, striped in blue.

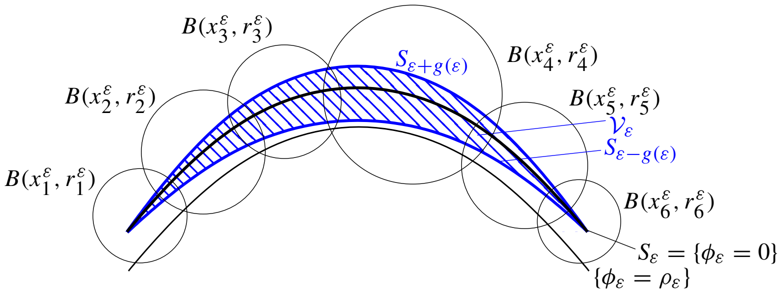

Fig. 7. Definition of the set $\mathcal{V}_\varepsilon$, striped in blue.

uniformly for $x' \in \overline{D}$. In particular, if $|\varepsilon - \varepsilon'| \le \frac{1}{2C} \operatorname{dist}(S_\varepsilon, \mathcal{U}_\varepsilon^c)$ with $\operatorname{dist}(S_\varepsilon, \mathcal{U}_\varepsilon^c) > 0$, we have

$$\begin{aligned}\operatorname{dist}[(x', G(x', \varepsilon')), S_\varepsilon] &\le \operatorname{dist}[(x', G(x', \varepsilon)), (x', G(x', \varepsilon'))] \le |G(x', \varepsilon) - G(x', \varepsilon')| \\ &\le \operatorname{dist}(S_\varepsilon, \mathcal{U}_\varepsilon^c)/2.\end{aligned}$$

This holds for any $x' \in \overline{D}$, so that $S_{\varepsilon'}$ is contained in a neighborhood of S_ε of size $\operatorname{dist}(S_\varepsilon, \mathcal{U}_\varepsilon^c)/2$, and hence contained in $\mathcal{U}_\varepsilon$. This proves (4.9) with

$$g(\varepsilon) = \operatorname{dist}(S_\varepsilon, \mathcal{U}_\varepsilon^c)/(2C) > 0.$$

As a consequence of (4.9), we have in particular, for any $\varepsilon \in (0, 1]$,

$$\mathcal{V}_\varepsilon \subset \mathcal{U}_\varepsilon \subset \{\phi_\varepsilon < \rho_\varepsilon\}. \tag{4.10}$$

Now, we also have

$$K \subset \Big(S_0 \cup \bigcup_{\varepsilon \in (0,1]} \mathcal{V}_\varepsilon\Big) \subset \Big(\omega_1 \cup \bigcup_{\varepsilon \in (0,1]} \mathcal{V}_\varepsilon\Big).$$

The same argument as above using the fact that ω_1 is a neighborhood of S_0 shows that there exists ε_0 such that

$$\mathcal{V}_0 := \bigcup_{\varepsilon \in [0,\varepsilon_0)} S_\varepsilon \subset \omega_1.$$

As a consequence,

$$K \subset \Big(\mathcal{V}_0 \cup \bigcup_{\varepsilon \in [\varepsilon_0,1]} \mathcal{V}_\varepsilon\Big), \qquad \mathcal{V}_0 \subset \omega_1.$$

From the covering $[\varepsilon_0, 1] \subset \bigcup_{\varepsilon \in [\varepsilon_0,1]} (\varepsilon - g(\varepsilon), \varepsilon + g(\varepsilon))$, we now extract a finite covering $[\varepsilon_0, 1] \subset \bigcup_{j \in J} (\varepsilon_j - g(\varepsilon_j), \varepsilon_j + g(\varepsilon_j))$, where J is a finite set of indices. In particular, this yields a finite covering

$$[0, 1] \subset [0, \varepsilon_0) \cup \bigcup_{i \in J} (\varepsilon_j - g(\varepsilon_j), \varepsilon_j + g(\varepsilon_j)). \tag{4.11}$$

As a consequence (with $\mathcal{V}_{\varepsilon_j}$ defined in (4.9)),

$$K \subset \omega_1 \cup \bigcup_{j\in J} \mathcal{V}_{\varepsilon_j} \quad \Big(\subset \omega_1 \cup \bigcup_{j\in J} \bigcup_{i\in I_{\varepsilon_j}} (B(x_i^{\varepsilon_j}, r_i^{\varepsilon_j}) \cap \{\phi_{\varepsilon_j} < \rho_{\varepsilon_j}\})\Big). \tag{4.12}$$

Now, we reorder the set J by increasing order of $\varepsilon_j - g(\varepsilon_j)$, that is,

$$J = [\![1, N]\!] \quad \text{with} \quad \varepsilon_j - g(\varepsilon_j) \leq \varepsilon_{j+1} - g(\varepsilon_{j+1}) \text{ for all } j \in [\![1, N-1]\!]. \tag{4.13}$$

Note that if $\varepsilon_j - g(\varepsilon_j) = \varepsilon_{j+1} - g(\varepsilon_{j+1})$, we can suppress the $\mathcal{V}_{\varepsilon_j}$ associated to the smaller $\varepsilon_j + g(\varepsilon_j)$, and the covering property remains true. We will also need to check that

$$\varepsilon_{k+1} - g(\varepsilon_{k+1}) < \max_{1\leq j\leq k} (\varepsilon_j + g(\varepsilon_j)). \tag{4.14}$$

Indeed, if this is not the case, then $\varepsilon_{k+1} - g(\varepsilon_{k+1}) \geq \max_{0\leq j\leq k}(\varepsilon_j + g(\varepsilon_j))$. In particular, for $j \leq k$, we have $\varepsilon_j + g(\varepsilon_j) \leq \varepsilon_{k+1} - g(\varepsilon_{k+1})$ and $\varepsilon_{k+1} - g(\varepsilon_{k+1}) \notin (\varepsilon_j - g(\varepsilon_j), \varepsilon_j + g(\varepsilon_j))$. But for $j \geq k+1$, by increasing choice (4.13), we have $\varepsilon_{k+1} - g(\varepsilon_{k+1}) \leq \varepsilon_j - g(\varepsilon_j)$, and in particular $\varepsilon_{k+1} - g(\varepsilon_{k+1}) \notin (\varepsilon_j - g(\varepsilon_j), \varepsilon_j + g(\varepsilon_j))$. Hence $\varepsilon_{k+1} - g(\varepsilon_{k+1}) \notin \bigcup_{j\in J}(\varepsilon_j - g(\varepsilon_j), \varepsilon_j + g(\varepsilon_j))$. Moreover, $\varepsilon_{k+1} - g(\varepsilon_{k+1}) \geq \max_{1\leq j\leq k}(\varepsilon_j + g(\varepsilon_j)) \geq \varepsilon_0$ as $\varepsilon_j \geq \varepsilon_0$ for $j \geq 1$ and hence $\varepsilon_{k+1} - g(\varepsilon_{k+1}) \notin [0, \varepsilon_0)$. This contradicts (4.11) and proves (4.14).

The preparatory definitions were made to state the following geometrical lemma to be proved later.

Lemma 4.8. *With the notation of the proof of Theorem 4.7, for any $k \in [\![0, N-1]\!]$ and $i \in I_{\varepsilon_{k+1}}$ we have*

$$\{\phi_{\varepsilon_{k+1}} > \rho_{\varepsilon_{k+1}}\} \cap B(x_i^{\varepsilon_{k+1}}, 4R_i^{\varepsilon_{k+1}}) \Subset \Big[\omega_1 \cup \bigcup_{j\in[\![1,k]\!]} \bigcup_{\ell\in I_{\varepsilon_j}} B(x_\ell^{\varepsilon_j}, r_\ell^{\varepsilon_j})\Big],$$

where we consider the union $\bigcup_{j\in[\![1,k]\!]}$ to be empty if $k = 0$.

Now, we are going to use an abstract iteration argument, so we set the following notations for $j \in [\![1, N]\!]$ and $i \in I_{\varepsilon_j}$:

$$\begin{aligned} I_j &= I_{\varepsilon_j}, \quad U_{i,j} = B(x_i^{\varepsilon_j}, 2r_i^{\varepsilon_j}), \quad \omega_{i,j} = B(x_i^{\varepsilon_j}, r_i^{\varepsilon_j}), \\ V_{i,j} &= \{\phi_{\varepsilon_j} > \rho_{\varepsilon_j}\} \cap B(x_i^{\varepsilon_j}, 4R_i^{\varepsilon_j}), \quad V_0 = \hat{\omega}, \quad U_0 = \omega_1. \end{aligned}$$

The choice of the $r_i^{\varepsilon_j}$ and $\rho_i^{\varepsilon_j} \leq \rho_{\varepsilon_j}$ according to Corollary 4.6 implies

$$U_{i,j} \lhd V_{i,j}.$$

Moreover, we have $\omega_{i,j} \Subset U_{i,j}$, and Lemma 4.8 can be written as

$$V_{i,k+1} \Subset \Big[U_0 \cup \bigcup_{j\in[\![1,k]\!]} \bigcup_{i\in I_j} \omega_{i,j}\Big].$$

Now, we are in a position to apply the following iteration proposition, to be proved later.

Proposition 4.9. *Assume that there exist open sets U_0, $U_{i,j}$, $\omega_{i,j} \Subset U_{i,j}$, with $j \in [\![1, N]\!]$ and $i \in I_j$ (I_j finite), such that*

$$U_{i,j} \lhd V_{i,j} \quad \text{and} \quad \omega_{i,j} \Subset U_{i,j} \quad \text{for all } j \in [\![1, N]\!] \text{ and } i \in I_j;$$
$$V_{i,k+1} \Subset \Big[U_0 \cup \bigcup_{j\in[\![1,k]\!]} \bigcup_{\ell\in I_j} \omega_{\ell,j}\Big] \quad \text{for } k \in [\![0, N-1]\!] \text{ and } i \in I_{k+1},$$

where we consider the union $\bigcup_{j\in[\![1,k]\!]}$ to be empty if $k = 0$. Then

$$\Big[U_0 \cup \bigcup_{j\in[\![1,N]\!]} \bigcup_{\ell\in I_j} \omega_{\ell,j}\Big] \lhd V_0$$

for any open set V_0 such that $U_0 \Subset V_0$.

Now, we always have $\omega_2 \lhd \hat{\omega}$, as a consequence of properties (5) (second part) and (6) of Proposition 4.5. Hence, with $U := \omega_1 \cup \bigcup_{j\in[\![1,N]\!]} \bigcup_{\ell\in I_{\varepsilon_j}} B(x_\ell^{\varepsilon_j}, r_\ell^{\varepsilon_j})$, Proposition 4.9 yields $U \lhd \hat{\omega}$. Since U is a neighborhood of K by the covering property (4.12), this concludes the proof of Theorem 4.7, up to the proofs of Lemma 4.8 and Proposition 4.9. ◻

4.2.1. Proof of Lemma 4.8. We first prove, for later use, that for any $x' \in \overline{D}$ and any $\varepsilon \in (0, 1]$, we have

$$G(x', \varepsilon - g(\varepsilon)) \geq G(x', \varepsilon) - \rho_\varepsilon. \tag{4.15}$$

Indeed, let $x \in \mathcal{V}_\varepsilon$, so $x \in S_{\varepsilon'}$ for some $\varepsilon' \in (\varepsilon - g(\varepsilon), \varepsilon + g(\varepsilon))$. That is, $x_n = G(x', \varepsilon')$. Using (4.10), we have $\phi_\varepsilon(x) < \rho_\varepsilon$, that is, $G(x', \varepsilon) - x_n < \rho_\varepsilon$ and so $G(x', \varepsilon) - G(x', \varepsilon') < \rho_\varepsilon$. This is true for any point $x = (x', G(x', \varepsilon'))$ for $\varepsilon' \in (\varepsilon - g(\varepsilon), \varepsilon + g(\varepsilon))$. Letting $\varepsilon' \to \varepsilon - g(\varepsilon)$ and using the continuity of G, we get $G(x', \varepsilon) - G(x', \varepsilon - g(\varepsilon)) \leq \rho_\varepsilon$, which is (4.15).

We now come back to the proof of the lemma. As a consequence of the definitions of $\mathcal{U}_\varepsilon$ and $\mathcal{V}_\varepsilon \subset \mathcal{U}_\varepsilon$ and of (4.12), for all $k \in [\![0, N]\!]$ we have

$$\Big[\mathcal{V}_0 \cup \bigcup_{j\in[\![1,k]\!]} \mathcal{V}_{\varepsilon_j}\Big] \Subset \Big[\omega_1 \cup \bigcup_{j\in[\![1,k]\!]} \bigcup_{\ell\in I_{\varepsilon_j}} B(x_\ell^{\varepsilon_j}, r_\ell^{\varepsilon_j})\Big]. \tag{4.16}$$

By (4.16), it is sufficient to prove, for any $k \in [\![0, N-1]\!]$ and all $i \in I_{\varepsilon_{k+1}}$,

$$\big(\{\phi_{\varepsilon_{k+1}} \geq \rho_{\varepsilon_{k+1}}\} \cap B(x_i^{\varepsilon_{k+1}}, 4R_i^{\varepsilon_{k+1}})\big) \subset \Big(\omega_1 \cup \bigcup_{j\in[\![1,k]\!]} \mathcal{V}_{\varepsilon_j}\Big),$$

which will follow from the following two inclusions:

$$\big(\{\phi_{\varepsilon_{k+1}} \geq \rho_{\varepsilon_{k+1}}\} \cap K\big) \subset \Big(\omega_1 \cup \bigcup_{j\in[\![1,k]\!]} \mathcal{V}_{\varepsilon_j}\Big), \tag{4.17}$$

$$\big(\{\phi_{\varepsilon_{k+1}} \geq \rho_{\varepsilon_{k+1}}\} \cap K^c\big) \cap B(x_i^{\varepsilon_{k+1}}, 4R_i^{\varepsilon_{k+1}}) \subset \omega_1. \tag{4.18}$$

Let us first prove (4.17). Since $K \subset \omega_1 \cup \bigcup_{j\in[\![1,N]\!]} \mathcal{V}_{\varepsilon_j}$ by (4.12), we have

$$\left(\{\phi_{\varepsilon_{k+1}} \geq \rho_{\varepsilon_{k+1}}\} \cap K\right) \subset \Big(\omega_1 \cup \bigcup_{j\in[\![1,N]\!]} (\mathcal{V}_{\varepsilon_j} \cap \{\phi_{\varepsilon_{k+1}} \geq \rho_{\varepsilon_{k+1}}\})\Big). \tag{4.19}$$

Moreover, using (4.10), we get

$$\mathcal{V}_{\varepsilon_{k+1}} \subset \{\phi_{\varepsilon_{k+1}} < \rho_{\varepsilon_{k+1}}\}.$$

Now, we will use the fact that G is increasing in ε to prove that also

$$\mathcal{V}_{\varepsilon_j} \subset \{\phi_{\varepsilon_{k+1}} < \rho_{\varepsilon_{k+1}}\} \qquad \text{for } j \geq k+1. \tag{4.20}$$

Actually, for $x \in \mathcal{V}_{\varepsilon_j}$ with $j \geq k+1$, we have $x_n = G(x', \varepsilon)$ for some $\varepsilon > \varepsilon_j - g(\varepsilon_j) \geq \varepsilon_{k+1} - g(\varepsilon_{k+1})$ (it is here that we use the fact that the ε_j's are ordered as in (4.13)). But since G is strictly increasing in ε, this implies $x_n > G(x', \varepsilon_{k+1} - g(\varepsilon_{k+1}))$. Using the inequality (4.15), true for any $\varepsilon \in (0, 1]$, we obtain $x_n > G(x', \varepsilon_{k+1}) - \rho_{\varepsilon_{k+1}}$. This gives $\phi_{\varepsilon_{k+1}}(x', x_n) < \rho_{\varepsilon_{k+1}}$ and therefore (4.20). As a consequence, on the right hand side of (4.19) only the terms for $j \leq k$ are nonempty, which implies (4.17).

We now prove (4.18). Since $x_i^{\varepsilon_{k+1}} \in K$ and $4R_i^{\varepsilon_{k+1}} \leq R$, it is sufficient to prove

$$\{\phi_{\varepsilon_{k+1}} \geq 0\} \cap K^c \cap K^R \subset \omega_1.$$

We first notice that, according to the definition of K, we have

$$K^c = \{x_n < 0\} \cup \{x_n > G(x', 1)\}.$$

In addition, since for $x' \in \overline{D}$, G is increasing in ε, we have

$$\{\phi_{\varepsilon_{k+1}} \geq 0\} \cap \{x' \in \overline{D}\} = \{x_n \leq G(x', \varepsilon_{k+1}),\ x' \in \overline{D}\} \subset \{x_n \leq G(x', 1)\}.$$

But for $x' \notin \overline{D}$, we have $G(x', \varepsilon_{k+1}) < 0$ and hence

$$\{\phi_{\varepsilon_{k+1}} \geq 0\} \cap \{x' \notin \overline{D}\} = \{x_n \leq G(x', \varepsilon_{k+1}),\ x' \notin \overline{D}\} \subset \{x_n < 0\}.$$

As a consequence, $\{\phi_{\varepsilon_{k+1}} \geq 0\} \cap K^c \subset \{x_n < 0\}$. We are thus left with proving

$$\left(\{x_n < 0\} \cap K^R\right) \subset \omega_1.$$

This is direct thanks to (4.6) since $\operatorname{dist}(x, K) = \operatorname{dist}(x, S_0)$ for $x_n < 0$. This concludes the proof of (4.18).

We finally check that the proof works the same way for the degenerate case $k = 0$, which corresponds to the same proof with $\emptyset$ instead of $\bigcup_{j\in[\![1,k]\!]}$. This concludes the proof of Lemma 4.8. $\square$

Remark 4.10. In this process, we can also require that the points $x_i^{\varepsilon_j}$ be far from $\{x_n = 0\}$, by forcing $B(x_i^{\varepsilon_j}, 4R_i^{\varepsilon_j}) \cap \{x_n = 0\} = \emptyset$.

Indeed, if $B(x_i^{\varepsilon_j}, 4R_i^{\varepsilon_j}) \cap \{x_n = 0\} \neq \emptyset$, we have necessarily $\operatorname{dist}(x_i^{\varepsilon_j}, S_0) < 4R_i^{\varepsilon_j}$ because $\operatorname{dist}(x_i^{\varepsilon_j}, \{x_n = 0\})$ is necessarily reached at a point in $S_0 = \overline{D} \times \{0_{x_n}\}$, since $x_i^{\varepsilon_j} \in S_{\varepsilon_j} \subset \overline{D} \times \mathbb{R}_{x_n}$. But in the process (see (4.6) and (4.8)) we have chosen $R_i^{\varepsilon_j} \leq \operatorname{dist}(\omega_1^c, S_0)/8$. This implies $\operatorname{dist}(x_i^{\varepsilon_j}, \omega_1^c) \geq \operatorname{dist}(\omega_1^c, S_0) - \operatorname{dist}(x_i^{\varepsilon_j}, S_0) > 8R_i^{\varepsilon_j} - 4R_i^{\varepsilon_j}$ and so $B(x_i^{\varepsilon_j}, 4R_i^{\varepsilon_j}) \subset \omega_1$. In particular, these points $x_i^{\varepsilon_j}$ can be removed without affecting the set

$$\omega_1 \cup \bigcup_{j \in [\![1,k]\!]} \bigcup_{i \in I_{\varepsilon_j}} B(x_i^{\varepsilon_j}, r_i^{\varepsilon_j}),$$

for any k.

This fact was not used here but it will be useful later in the presence of boundary.

4.2.2. Semiglobal estimates by iteration: proof of Proposition 4.9. To prove Proposition 4.9, we use induction on $k \in [\![1, N]\!]$. We make the following induction assumption at step k:

$$\text{For any } j \in [\![1, k]\!] \text{ and } i \in I_j, \text{ we have } U_{i,j} \lhd V_0. \tag{IA$_k$}$$

We first explain why this proves Proposition 4.9, and then perform the induction argument. Note that using Proposition 4.5(4) and since we can select W_0 with $U_0 \Subset W_0 \Subset V_0$ and $\omega_{i,j} \Subset U_{i,j}$, we have

$$\Big[U_0 \cup \bigcup_{j \in [\![1,k]\!]} \bigcup_{\ell \in I_j} \omega_{\ell,j}\Big] \lhd (W_0, U_{i,j})_{j \in [\![1,k]\!],\, i \in I_j}.$$

So, since $W_0 \lhd V_0$, with the use of properties (5) (second part) and (6) of Proposition 4.5, (IA$_k$) directly implies

$$\Big[U_0 \cup \bigcup_{j \in [\![1,k]\!]} \bigcup_{\ell \in I_j} \omega_{\ell,j}\Big] \lhd V_0. \tag{4.21}$$

In particular, (IA$_N$) implies (4.21) for $k = N$, which is the result of the proposition, namely

$$U := \Big[U_0 \cup \bigcup_{j \in [\![1,N]\!]} \bigcup_{\ell \in I_j} \omega_{\ell,j}\Big] \lhd V_0. \tag{4.22}$$

We now come to the proof of (IA$_k$) by induction.

For $k = 1$, we need to prove $U_{i,1} \lhd V_0$ for $i \in I_1$. But the assumption with $k = 0$ gives $V_{i,1} \Subset U_0$, which implies $V_{i,1} \lhd U_0$. Since $U_{i,1} \lhd V_{i,1}$ by assumption, we get $U_{i,1} \lhd U_0$ by transitivity. Since also $U_0 \lhd V_0$, we obtain $U_{i,1} \lhd V_0$ for all $i \in I_1$ as desired.

We now prove (IA$_k$) $\Rightarrow$ (IA$_{k+1}$) for $k \in [\![1, N-1]\!]$. The assumption of the proposition gives

$$V_{i,k+1} \Subset \Big[U_0 \cup \bigcup_{j \in [\![1,k]\!]} \bigcup_{\ell \in I_j} \omega_{\ell,j}\Big] \quad \text{for all } i \in I_{k+1}.$$

Combined with Proposition 4.5(3), this yields

$$V_{i,k+1} \lhd \Big[U_0 \cup \bigcup_{j\in[\![1,k]\!]} \bigcup_{\ell\in I_j} \omega_{\ell,j} \Big] \quad \text{for all } i \in I_{k+1}.$$

Using (4.21) for k since $(\mathrm{IA})_k$ is true, together with the transitivity of $\lhd$, we get

$$V_{i,k+1} \lhd V_0 \quad \text{for all } i \in I_{k+1}.$$

Since $U_{i,j} \lhd V_{i,j}$, the transitivity property gives again $U_{i,k+1} \lhd V_0$ for all $i \in I_{k+1}$. This implies (IA_{k+1}) and thus proves the induction property for $k \in [\![1, N-1]\!]$. This concludes the proof of Proposition 4.9. □

4.3. Semiglobal estimates along foliation by hypersurfaces

The previous framework, where we define hypersurfaces by graphs, may look a bit rigid for applications. Having defined hypersurfaces by graphs was mainly convenient to make the foliation more effective and order the hypersurfaces more easily.

Now, we give a slight variant of Theorem 4.7, more adapted to some possible changes of variables.

Theorem 4.11. *Let $\Omega \subset \mathbb{R}^n = \mathbb{R}^{n_a} \times \mathbb{R}^{n_b}$ and let P be a smooth differential operator of order m on Ω, analytically principally normal in $\{\xi_a = 0\}$. Let Φ be a diffeomorphism of class C^2 from Ω to $\widetilde{\Omega} = \Phi(\Omega)$. Assume that the geometric setting of Theorem 1.11 is satisfied for some D, G, K, ϕ_ε on $\widetilde{\Omega}$ (and not on Ω). Assume further that for any $\varepsilon \in [0, 1+\eta)$, the oriented surface $\{\phi_\varepsilon \circ \Phi = 0\} = \Phi^{-1}(S_\varepsilon)$ (well defined on Ω) is strongly pseudoconvex with respect to P on $\Phi^{-1}(S_\varepsilon)$.*

Then, for any neighborhood ω of $\Phi^{-1}(S_0)$, there exists an open neighborhood $U \subset \Omega$ of $\Phi^{-1}(K)$ such that $U \lhd \omega$, where $\lhd = \lhd_{\Omega,P}$ is related to the operator P defined on Ω (see Remark 4.2).

Proof. The proof is exactly the same as that of Theorem 1.11/4.7 except that the local uniqueness estimates are performed in Ω. So, for any $x \in \Phi^{-1}(S_\varepsilon)$, it furnishes some r_x, R_x and ρ_x such that

$$B_\Omega(x, r_x) \lhd_{\Omega,P} \big(\{\phi_\varepsilon \circ \Phi > \rho_x\} \cap B_\Omega(x, 4R_x)\big).$$

But since Φ is a homeomorphism, this implies the existence of $\widetilde{r}_x$ and $\widetilde{R}_x$ (that can still be chosen small enough) such that

$$\Phi^{-1}[B_{\widetilde{\Omega}}(\Phi(x), \widetilde{r}_x)] \Subset B_\Omega(x, r_x) \quad \text{and} \quad B_\Omega(x, 4R_x) \Subset \Phi^{-1}[B_{\widetilde{\Omega}}(\Phi(x), 4\widetilde{R}_x)],$$

and hence

$$\Phi^{-1}[B_{\widetilde{\Omega}}(\Phi(x), \widetilde{r}_x)] \lhd_{\Omega,P} \big(\{\phi_\varepsilon \circ \Phi > \rho_x\} \cap \Phi^{-1}[B_{\widetilde{\Omega}}(\Phi(x), 4\widetilde{R}_x)]\big).$$

where B_Ω (resp. $B_{\widetilde{\Omega}}$) denote balls in Ω (resp. $\widetilde{\Omega}$).

The geometric part of the proof of Theorem 1.11/4.7 is then exactly the same, performed in $\widetilde{\Omega}$, i.e. replacing r_x, R_x by $\widetilde{r}_x$, $\widetilde{R}_x$. Once the geometric part is done, the iteration process, performed in Ω, is exactly the same by replacing each geometric term by the preimage in Ω (for instance $\Phi^{-1}[B_{\widetilde{\Omega}}(\Phi(x_i^{\varepsilon_k}), 4\widetilde{R}_{x_i^{\varepsilon_k}})]$ replaces $B(x_i^{\varepsilon_k}, 4R_{x_i^{\varepsilon_k}})$ etc.). □

5. The Dirichlet problem for some second order operators

In this section, we shall consider a particular class of operators as described in Remark 1.10, that is, with symbols of the form $p_2(x,\xi) = Q_x(\xi)$ where Q_x is a smooth family of real quadratic forms. Assuming that the variables x_a are tangent to the boundary, and that the functions satisfy Dirichlet boundary conditions, we prove a counterpart of the local estimate of Theorem 3.1 for this boundary value problem. For this, the main goal to achieve is to prove a Carleman estimate adapted to this boundary value problem. All local, semiglobal and global results will then follow.

This situation is of particular interest for the wave equation for which x_a is the time variable, which is always tangent to the boundary of cylindrical domains.

For simplicity, we shall further assume that the principal symbol of the operator P is independent of the x_a variable. More precisely, in Theorem 5.2 below, we first assume that *no* coefficient of P depends on x_a. This is then relaxed in Corollary 5.4, where we explain how to include lower order terms that are analytic in x_a. It would be in principle possible to allow the principal part of P to depend analytically on x_a, but it would require some additional technicalities in the (already rather technical) proofs.

5.1. Some notation

Here, we shall always assume that the analytic variables are tangential to the boundary, that is,

$$x = (x_a, x_b) \in \mathbb{R}^{n_a} \times \mathbb{R}^{n_b}_+ \quad \text{with} \quad \mathbb{R}^{n_b}_+ = \mathbb{R}^{n_b-1} \times \mathbb{R}_+, \quad x_b = (x_b', x_b^n).$$

When the distinction between analytic and nonanalytic variables is not essential, we shall split the variables according to

$$x = (x', x_n) \in \mathbb{R}^n_+ = \mathbb{R}^{n-1} \times \mathbb{R}_+ \quad \text{with} \quad x' = (x_a, x_b') \in \mathbb{R}^{n_a+n_b-1}, \quad x_n = x_b^n \in \mathbb{R}_+.$$

We also denote by $\xi' = (\xi_a, \xi_b') \in \mathbb{R}^{n-1}$ the cotangential variables and $\xi_n = \xi_b^n$ the conormal variable, by $D' = (D_a, D_{x_b'}) = \frac{1}{i}(\partial_{x_a}, \partial_{x_b'})$ the associated tangential derivations and by $D_n = D_{x_b^n} = \frac{1}{i}\partial_{x_n}$ the normal derivation.

For any $r_0 > 0$, we define

$$K_{r_0} = \{x \in \mathbb{R}^n_+ : |x_a| \le r_0,\ |x_b| \le r_0\} = \overline{B}_{\mathbb{R}^{n_a}}(0, r_0) \times \overline{B}_{\mathbb{R}^{n_b}}(0, r_0) \cap \{x_n \ge 0\}. \tag{5.1}$$

We denote by $C_0^\infty(\mathbb{R}^n_+)$ the space of restrictions to $\mathbb{R}^n_+$ of functions in $C_0^\infty(\mathbb{R}^n)$, and by $C_0^\infty(K_{r_0})$ the space of $C_0^\infty(\mathbb{R}^n_+)$ functions supported in K_{r_0}. The trace of a function $f \in C_0^\infty(\mathbb{R}^n_+)$ at $x_n = 0$ is denoted by $f_{|x_n=0}$.

We denote by $(f, g) = \int_{\mathbb{R}^n_+} f\overline{g}$ and $\|f\|^2_{0,+} = (f, f)$ the $L^2(\mathbb{R}^n_+)$ inner product and norm. For $k \in \mathbb{N}$, $\|\cdot\|_{k,+}$ will denote the classical Sobolev norm on $\mathbb{R}^n_+$ and $\|\cdot\|_{k,+,\tau}$ the associated weighted norms, that is,

$$\|f\|^2_{k,+,\tau} = \sum_{j+|\alpha|\le k} \tau^{2j} \|\partial^\alpha f\|^2_{0,+}, \qquad \tau \ge 1. \tag{5.2}$$

We also define the tangential Sobolev norms by

$$|f|_{k,\tau}^2 = \|(|D'| + \tau)^k f\|_{0,+}^2 \sim \sum_{j+|\alpha|\le k} \tau^{2j} \|\partial_{x'}^\alpha f\|_{0,+}^2, \qquad \tau \ge 1.$$

For $f, g \in C_0^\infty(\mathbb{R}_+^n)$, we shall also use $(f, g)_0 = \int_{\mathbb{R}^{n-1}} f_{|x_n=0}(x') g_{|x_n=0}(x')\, dx'$.

Finally, for $j \in \mathbb{N}$, we denote by $\mathcal{D}_\tau^k$ the space of *tangential* differential operators, i.e. operators of the form

$$P(x, D', \tau) = \sum_{j+|\alpha|\le k} a_{j,\alpha}(x) \tau^j D'^\alpha,$$

and by

$$\sigma(P) = p(x, \xi', \tau) = \sum_{j+|\alpha|=k} a_{j,\alpha}(x) \tau^j \xi'^\alpha$$

their principal symbols.

Remark 5.1. Denote by T the restriction operator from $\mathcal{D}'(\mathbb{R}^n)$ to $\mathcal{D}'(\mathbb{R}_+^n)$. We write $H^k(\mathbb{R}_+^n) = T(H^k(\mathbb{R}^n))$ with the restriction Sobolev norms

$$\begin{aligned} \|u\|_{k,+} &:= \inf\{\|v\|_k : v \in H^k(\mathbb{R}^n),\ Tv = u \text{ in } \mathcal{D}'(\mathbb{R}_+^n)\} \\ &= \inf\{\|v\|_k : v \in H^k(\mathbb{R}^n),\ v = u \text{ a.e. on } \mathbb{R}_+^n\}. \end{aligned}$$

We have the property

$$\|u\|_{k,+} \approx \sup_{|\alpha|\le k} \|\partial^\alpha u\|_{L^2(\mathbb{R}_+^n)}$$

(see [Hör85, Corollary B.2.5] with different notations $\overline{H}_{(k,0)}(\mathbb{R}_+^n)$). Moreover, the set $C_0^\infty(\mathbb{R}_+^n) = T(C_0^\infty(\mathbb{R}^n))$ of restrictions of smooth functions is dense in $H^k(\mathbb{R}_+^n)$ (see [Hör85, Theorem B.2.1]). As a conclusion, if L is a linear operator from $H^k(\mathbb{R}^n)$ to $H^l(\mathbb{R}^n)$ of norm C that sends $\ker(T) \cap H^k$ into $\ker(T) \cap H^l$, then L extends to a linear operator from $H^k(\mathbb{R}_+^n)$ to $H^l(\mathbb{R}_+^n)$ and we have

$$\|Lu\|_{l,+} \le C\|u\|_{k,+}.$$

In particular, this will be the case for all "tangential" operators.

5.2. The Carleman estimate

In this section, we state and prove the counterpart of the Carleman estimate (2.4) associated to the Dirichlet problem for some second order operators (including the wave operator). Recall that the operator $Q_{\varepsilon,\tau}^\psi$ is defined in (2.3) and acts in the variable x_a only, and hence is tangential to the boundary.

Theorem 5.2 (Local Carleman estimate). *Let $r_0 > 0$ and $P = D^2_{x_b^n} + r(x_b, D_{x_a}, D_{x_b'})$ be a differential operator of order 2 on a neighborhood of K_{r_0}, with real principal part, where $r(x_b, D_{x_a}, D_{x_b'})$ does not depend on x_a and is a smooth x_b^n family of second order operators in the (tangential) variable (x_a, x_b').*

Let ψ be a quadratic polynomial such that $\psi'_{x_b^n} \neq 0$ on K_{r_0} and

$$\{p, \{p, \psi\}\}(x, \xi) > 0 \quad \text{if } p(x, \xi) = 0,\ x \in K_{r_0} \text{ and } \xi_a = 0,\ \xi \neq 0, \tag{5.3}$$

$$\frac{1}{i\tau}\{\overline{p}_\psi, p_\psi\}(x, \xi) > 0 \quad \text{if } p_\psi(x, \xi) = 0,\ x \in K_{r_0} \text{ and } \xi_a = 0,\ \tau > 0, \tag{5.4}$$

where $p_\psi(x, \xi) = p(x, \xi + i\tau\nabla\psi)$.

Then there exist $\varepsilon, \mathsf{d}, C, \tau_0 > 0$ such that for any $\tau > \tau_0$ all $u \in C_0^\infty(K_{r_0/4})$ we have

$$\begin{aligned}\tau\|Q^\psi_{\varepsilon,\tau}u\|^2_{1,+,\tau} \le C\big(&\|Q^\psi_{\varepsilon,\tau}Pu\|^2_{0,+} + e^{-\mathsf{d}\tau}\|e^{\tau\psi}u\|^2_{1,+,\tau} + \tau^3|(Q^\psi_{\varepsilon,\tau}u)_{|x_n=0}|^2_0 \\ &+ e^{-\mathsf{d}\tau}|e^{\tau\psi}u_{|x_n=0}|^2_0 + \tau|(D(Q^\psi_{\varepsilon,\tau}u))_{|x_n=0}|^2_0 + e^{-\mathsf{d}\tau}|e^{\tau\psi}Du_{|x_n=0}|^2_0\big).\end{aligned} \tag{5.5}$$

If moreover $\psi'_{x_n} > 0$ for $(x', x_n = 0) \in K_{r_0}$, then for all $u \in C_0^\infty(K_{r_0/4})$ such that $u_{|x_n=0} = 0$, we have

$$\tau\|Q^\psi_{\varepsilon,\tau}u\|^2_{1,+,\tau} \le C\big(\|Q^\psi_{\varepsilon,\tau}Pu\|^2_{0,+} + e^{-\mathsf{d}\tau}\|e^{\tau\psi}u\|^2_{1,+,\tau}\big). \tag{5.6}$$

Note that the operators P considered here satisfy in particular assumption (H) (i.e. have a real valued principal symbol independent of x_a).

The proof of this theorem relies on a Carleman estimate interpolating between the "boundary elliptic Carleman estimates" of Lebeau and Robbiano [LR95] and the "partially analytic Carleman estimates" of Tataru [Tat95] (see also [Hör97]).

Let us first state two corollaries that explain how to deal with lower order terms, and then prove Theorem 5.2.

Corollary 5.3. *Under the assumptions of Theorem 5.2, there exist $\varepsilon, \mathsf{d}, C, \tau_0 > 0$ such that for any $V \in L^\infty(K_{r_0})$, $W \in L^\infty(K_{r_0}; \mathbb{R}^n)$, independent of x_a and any $\tau > \tau_0 \max\{1, \|V\|^{2/3}_{L^\infty}, \|W\|^2_{L^\infty}\}$, the Carleman estimates (5.5) or (5.6) are satisfied with P replaced by $P_{V,W} = P + W \cdot \nabla + V$.*

Here the constant C does not depend on the lower order terms V or $W \cdot \nabla$ (that are independent of x_a).

Proof of Corollary 5.3. Applying the Carleman estimates (5.5) or (5.6) for $P = P_{V,W} - iW \cdot D - V$, we need to estimate the term

$$Q^\psi_{\varepsilon,\tau}Pu = Q^\psi_{\varepsilon,\tau}P_{V,W}u - iW \cdot Q^\psi_{\varepsilon,\tau}(Du) - VQ^\psi_{\varepsilon,\tau}u$$

where $V = V(x_b)$, $W = W(x_b)$. Notice first that

$$C\|VQ^\psi_{\varepsilon,\tau}u\|^2_{0,+} \le C\|V\|^2_{L^\infty}\|Q^\psi_{\varepsilon,\tau}u\|^2_{0,+} \le \tfrac{1}{4}\tau^3\|Q^\psi_{\varepsilon,\tau}u\|^2_{0,+} \le \tfrac{1}{4}\tau\|Q^\psi_{\varepsilon,\tau}u\|^2_{1,+,\tau}$$

as soon as $\tau^3/(4C) \geq \|V\|^2_{L^\infty}$ (recall the definition of τ-depending norms in (5.2)). Next, we recall how $Q^\psi_{\varepsilon,\tau}$ commutes with differentiation (using the fact that ψ is a quadratic polynomial):

$$Q^\psi_{\varepsilon,\tau}(Du) = (D - \varepsilon\psi''_{x,x_a} D_a + i\tau\psi')Q^\psi_{\varepsilon,\tau} u$$

(see e.g. (5.7)–(5.8) below), and consequently

$$C\|iW \cdot Q^\psi_{\varepsilon,\tau}(Du)\|^2_{0,+} \leq C'\|W\|^2_{L^\infty}\|Q^\psi_{\varepsilon,\tau}u\|^2_{1,+,\tau} \leq \tfrac{1}{4}\tau\|Q^\psi_{\varepsilon,\tau}u\|^2_{1,+,\tau}$$

as soon as $\tau/(4C') \geq \|W\|^2_{L^\infty}$. For such τ, these two terms may hence be absorbed in the left hand side of the inequality. This concludes the proof. □

Corollary 5.4. *Under the assumptions of Theorem 5.2, let $R(x, D)$ be a differential operator of order 1 with coefficients which can be extended to bounded functions in $\{(z_a, x_b) \in \mathbb{C}^{n_a} \times \mathbb{R}^{n_b} : |z_a| < 5r_0, |x_b| < 5r_0\}$ and are analytic with respect to z_a for fixed x_b. Then there exist $\varepsilon, \mathsf{d}, C, \tau_0 > 0$ such that for any any $\tau > \tau_0$, the Carleman estimates (5.5) or (5.6) are satisfied with P replaced by $P_R = P + R$.*

Proof. Lemma 4.8 of [Hör97] yields

$$\|Q^\psi_{\varepsilon,\tau}R(x, D)u\|_{0,+} \leq C\|Q^\psi_{\varepsilon,\tau}u\|_{1,+,\tau} + Ce^{-\tau\mathsf{d}}\|e^{\tau\psi}u\|_{1,+,\tau}$$

for all $u \in C^\infty_0(K_{r_0/4})$. Actually, it is stated for the interior case, with the norm $\|\cdot\|_{1,+,\tau}$ replaced by $\|\cdot\|_{1,\tau}$. Yet, the estimates used for the proof, [Hör97, (3.13), (3.14)], are actually made first in the variable x_a and then integrated in x_b. Since the variable x_a is tangential, the same proof gives the expected result.

As in Corollary 5.3, we can absorb the term $C\|Q^\psi_{\varepsilon,\tau}u\|_{1,+,\tau}$ for τ large enough. The second term has the same form as the right hand side of the Carleman estimate, up to changing d. □

Remark 5.5. This theorem, as well as its consequences, may be extended with some modification to the Neumann case following Lebeau–Robbiano [LR97]. It could also be generalized to a larger class of operators and boundary conditions (satisfying a Lopatinskii condition) following Tataru [Tat96] and Bellassoued–Le Rousseau [BLR15].

We now turn to the proof of Theorem 5.2. For this, we define the conjugated operator $P_\psi = e^{\tau\psi}Pe^{-\tau\psi} = P(x, D + i\tau\psi')$, and also let $P_{\psi,\varepsilon}$ be the conjugate of P_ψ with respect to $e^{-\frac{\varepsilon}{2\tau}|D_a|^2}$, that is,

$$e^{-\frac{\varepsilon}{2\tau}|D_a|^2}P_\psi w = P_{\psi,\varepsilon}e^{-\frac{\varepsilon}{2\tau}|D_a|^2}w \quad \text{for all } w, \tag{5.7}$$

or equivalently

$$Q^\psi_{\varepsilon,\tau}Pu = P_{\psi,\varepsilon}Q^\psi_{\varepsilon,\tau}u \quad \text{for all } u,$$

with, as usual, $Q^\psi_{\varepsilon,\tau} = e^{-\frac{\varepsilon}{2\tau}}e^{\tau\psi}$. Since P is independent of x_a, we have

$$P_{\psi,\varepsilon} = P(x, D - \varepsilon\psi''_{x,x_a}D_a + i\tau\psi'), \tag{5.8}$$

where $\psi''_{x,x_a} D_a = \psi''_{xx}((D_a, 0))$ (with the notation of [Hör97]); recall that ψ is a quadratic polynomial such that ψ''_{xx} is a constant symmetric matrix with real coefficients.

When proving the theorem, we shall drop the index $+$ in the norms to lighten notation; of course, all inner norms and integrals are meant on $\mathbb{R}^n_+$. We first need the following proposition.

Proposition 5.6. *Under the assumptions of Theorem 5.2, there exist $C, \tau_0 > 0$ such that for any $\tau > \tau_0$ and $f \in C_0^\infty(K_{r_0})$, we have*

$$\tau \|f\|^2_{1,\tau} \leq C\big(\|P_{\psi,\varepsilon} f\|_0^2 + \tau \|D_a f\|_0^2 + \tau^3 |f_{|x_n=0}|_0^2 + \tau |Df_{|x_n=0}|_0^2\big). \tag{5.9}$$

If moreover $\psi'_{x_n} > 0$ for $(x', x_n = 0) \in K_{r_0}$, then

$$\tau \|f\|^2_{1,\tau} \leq C\big(\|P_{\psi,\varepsilon} f\|_0^2 + \tau \|D_a f\|_0^2\big) \quad \textit{for all } f \in C_0^\infty(K_{r_0}) \textit{ such that } f_{|x_n=0} = 0. \tag{5.10}$$

Remark 5.7. As stated, ε is fixed and all constants may depend on it. It is likely that one could perform uniform estimates in the limit $\varepsilon \to 0^+$, so as to recover the estimate in the case $\varepsilon = 0$, i.e. get rid of the term $\tau \|D_a f\|_0^2$ on the right hand sides. This would require some additional work (in particular, the introduction of a uniform metric on the phase space, see e.g. [Hör97, (5.7)]), and is not needed in the applications we have in mind here.

Proof of Proposition 5.6. Defining $\tilde{Q}_2^\varepsilon = \frac{1}{2}(P_{\psi,\varepsilon} + P^*_{\psi,\varepsilon})$ and $\tilde{Q}_1^\varepsilon = \frac{1}{2i\tau}(P_{\psi,\varepsilon} - P^*_{\psi,\varepsilon})$, we have

$$P_{\psi,\varepsilon} = \tilde{Q}_2^\varepsilon + i\tau \tilde{Q}_1^\varepsilon.$$

We also denote by p_ψ^ε the principal symbol of $P_{\psi,\varepsilon}$ and by $\tilde{q}_j^\varepsilon$ that of $\tilde{Q}_j^\varepsilon$, $j = 1, 2$ (which is real valued), so that

$$p_\psi^\varepsilon = \tilde{q}_2^\varepsilon + i\tau \tilde{q}_1^\varepsilon.$$

We have

$$\begin{cases} \tilde{Q}_2^\varepsilon = D_n^2 - 2\varepsilon \psi''_{x_n,x_a}(D_n; D_a) + Q_2^\varepsilon, \\ \tilde{Q}_1^\varepsilon = D_n \psi'_{x_n} + \psi'_{x_n} D_n + 2Q_1^\varepsilon, \end{cases} \tag{5.11}$$

and hence

$$\begin{cases} \tilde{q}_2^\varepsilon = \xi_n^2 - 2\varepsilon \psi''_{x_n,x_a}(\xi_n; \xi_a) + q_2^\varepsilon, \\ \tilde{q}_1^\varepsilon = 2\xi_n \psi'_{x_n} + 2q_1^\varepsilon. \end{cases} \tag{5.12}$$

In these expressions, the operators $Q_2^\varepsilon \in \mathcal{D}_\tau^2$ and $Q_1^\varepsilon \in \mathcal{D}_\tau^1$ have principal symbols

$$q_2^\varepsilon = \varepsilon^2 (\psi''_{x_n,x_a}\xi_a)^2 - \tau^2 (\psi'_{x_n})^2 + r(x, \xi' - \varepsilon \psi''_{x',x_a}\xi_a) - \tau^2 r(x, \psi'_{x'}),$$
$$q_1^\varepsilon = -\varepsilon \psi''_{x_n,x_a}(\psi'_{x_n}; \xi_a) + \tilde{r}(x_b, \xi' - \varepsilon \psi''_{x',x_a}\xi_a, \psi'_{x'}),$$

where $\tilde{r}$ is the bilinear form associated with the quadratic form r. Note that even if this does not appear in notation, all these operators depend upon the parameter τ.

With this notation, we hence have $p_\psi = \tilde{q}_2^0 + i\tau\tilde{q}_1^0$, so that $\frac{1}{i\tau}\{\overline{p}_\psi, p_\psi\} = 2\{\tilde{q}_2^0, \tilde{q}_1^0\}$. Assumptions (5.3) and (5.4) then translate respectively into

$$\{\tilde{q}_2^0, \tilde{q}_1^0\}(x,\xi) > 0 \quad \text{if } p(x,\xi) = 0,\ x \in K_{r_0} \text{ and } \xi_a = 0,\ \tau = 0, \tag{5.13}$$

$$\{\tilde{q}_2^0, \tilde{q}_1^0\}(x,\xi) > 0 \quad \text{if } p_\psi(x,\xi) = 0,\ x \in K_{r_0} \text{ and } \xi_a = 0,\ \tau > 0, \tag{5.14}$$

where the second assertion is a direct consequence of (5.4), and the first one follows from (5.3) together with the fact that, as p is real, we have

$$\lim_{\tau\to 0^+} \frac{1}{i\tau}\{\overline{p}_\psi, p_\psi\} = \frac{\partial}{\partial\tau}\frac{1}{i}\{\overline{p}_\psi, p_\psi\}\bigg|_{\tau=0} = 2\{p,\{p,\psi\}\}.$$

Next, we have the integration by parts formulæ:

$$\begin{cases} (g, \tilde{Q}_2^\varepsilon f) = (\tilde{Q}_2^\varepsilon g, f) - i[(g, D_n f)_0 + (D_n g, f)_0 - 2\varepsilon(g, \psi''_{x_n,x_a} D_a f)_0], \\ (g, \tilde{Q}_1^\varepsilon f) = (\tilde{Q}_1^\varepsilon g, f) - 2i(\psi'_{x_n} g, f)_0. \end{cases} \tag{5.15}$$

So, for $f \in C_0^\infty(K_{r_0})$ we have

$$\|P_{\psi,\varepsilon} f\|_0^2 = \|\tilde{Q}_2^\varepsilon f\|_0^2 + \tau^2\|\tilde{Q}_1^\varepsilon f\|_0^2 + i\tau[(\tilde{Q}_1^\varepsilon f, \tilde{Q}_2^\varepsilon f) - (\tilde{Q}_2^\varepsilon f, \tilde{Q}_1^\varepsilon f)]. \tag{5.16}$$

Hence, by the integration by parts formulæ (5.15),

$$\|P_{\psi,\varepsilon} f\|_0^2 = \|\tilde{Q}_2^\varepsilon f\|_0^2 + \tau^2\|\tilde{Q}_1^\varepsilon f\|_0^2 + i\tau([\tilde{Q}_2^\varepsilon, \tilde{Q}_1^\varepsilon] f, f) + \tau\mathcal{B}^\varepsilon(f) \tag{5.17}$$

with the boundary term

$$\begin{aligned} \mathcal{B}^\varepsilon(f) &= [(\tilde{Q}_1^\varepsilon f, D_n f)_0 + (D_n \tilde{Q}_1^\varepsilon f, f)_0 - 2\varepsilon(\tilde{Q}_1^\varepsilon f, \psi''_{x_n,x_a} D_a f)_0] - 2(\psi'_{x_n}\tilde{Q}_2^\varepsilon f, f)_0 \\ &= 2(\psi'_{x_n} D_n f, D_n f)_0 + (M_1^\varepsilon f, D_n f)_0 + (M_1'^\varepsilon D_n f, f)_0 + (M_2^\varepsilon f, f)_0 \end{aligned} \tag{5.18}$$

for some tangential operator M_1^ε of order 1 (in (ξ',τ)) (note that terms of order 2 in D_n cancel).

Now that we have made the exact computations, we will make some estimates on the symbols of the interior part of the commutator. The idea is to transfer the positivity assumption on the full symbol to some positivity of a tangential symbol, which will then allow one to apply the tangential Gårding inequality.

The first step is to perform a factorization of $[\tilde{Q}_2^\varepsilon, \tilde{Q}_1^\varepsilon]$ with respect to $\tilde{Q}_1^\varepsilon$ and $\tilde{Q}_2^\varepsilon$ to obtain a tangential remainder. Since $[\tilde{Q}_2^\varepsilon, \tilde{Q}_1^\varepsilon]$ is of order 2, it can be written $i[\tilde{Q}_2^\varepsilon, \tilde{Q}_1^\varepsilon] = C_2 + C_1 D_n + C_0 D_n^2$ where $C_i \in \mathcal{D}_\tau^i$. But using (5.11), and $\psi'_{x_n} \neq 0$ on K_{r_0}, we can replace $D_n = \frac{1}{2\psi'_{x_n}}\tilde{Q}_1^\varepsilon + \mathcal{D}_\tau^1$ and $D_n^2 = \tilde{Q}_2^\varepsilon + 2\varepsilon\psi''_{x_n,x_a}(D_n; D_a) - Q_2^\varepsilon$. So, in particular, we can write

$$i[\tilde{Q}_2^\varepsilon, \tilde{Q}_1^\varepsilon] = B_0^\varepsilon \tilde{Q}_2^\varepsilon + B_1^\varepsilon \tilde{Q}_1^\varepsilon + B_2^\varepsilon, \tag{5.19}$$

where $B_i^\varepsilon \in \mathcal{D}_\tau^i$ with real symbol b_i^ε. Now, we need to

- use the assumption to get some positivity of the symbol $\{\overline{p}_\psi, p_\psi\}$—this is Lemma 5.8;

- transfer this positivity to $\{\overline{p}^\varepsilon_\psi, p^\varepsilon_\psi\}$ for ε small enough by approximation—this is Lemma 5.9;
- transfer this information to tangential information on the symbol—this is Lemma 5.10.

Lemma 5.8. *There exist $C_1, C_2 > 0$ such that for all $(x,\xi) \in K_{r_0} \times \mathbb{R}^n$ and $\tau > 0$, we have*

$$|\xi|^2 + \tau^2 \leq C_1\{\tilde{q}_2^0, \tilde{q}_1^0\}(x,\xi) + C_2\left[\frac{|p_\psi(x,\xi)|^2}{|\xi|^2+\tau^2} + |\xi_a|^2\right].$$

Proof. All the terms are homogeneous of degree 2 in (ξ,τ) and continuous on the compact set $K_{r_0} \times \{(\xi,\tau) \in \mathbb{R}^n \times \mathbb{R}_+ : |\xi|^2 + \tau^2 = 1\}$. Thus, on this set, the result is a consequence of (5.13), (5.14) and Lemma A.1 applied to $f = |p_\psi(x,\xi)|^2/(|\xi|^2+\tau^2) + |\xi_a|^2 \geq 0$, $g = \{\tilde{q}_2^0, \tilde{q}_1^0\}$ and $h = 0$. The result on the whole $K_{r_0} \times \mathbb{R}^n \times \mathbb{R}_+$ follows by homogeneity. ◻

Lemma 5.9. *There exists ε_0 such that for all $\varepsilon \in (0,\varepsilon_0)$, there exist $C_1, C_2 > 0$ such that for all $(x,\xi) \in K_{r_0} \times \mathbb{R}^n$ and $\tau > 0$, we have*

$$|\xi|^2 + \tau^2 \leq C_1\{\tilde{q}_2^\varepsilon, \tilde{q}_1^\varepsilon\}(x,\xi) + C_2\left[\frac{|p^\varepsilon_\psi(x,\xi)|^2}{|\xi|^2+\tau^2} + |\xi_a|^2\right].$$

Proof. By the same argument, we may restrict to the compact set $K_{r_0} \times \{(\xi,\tau) \in \mathbb{R}^n \times \mathbb{R}_+ : |\xi|^2 + \tau^2 = 1\}$. There, the inequality follows from Lemma 5.8 and the continuity of the maps $\varepsilon \mapsto q_j^\varepsilon$, $j = 1, 2$, from $\mathbb{R}$ to $C^1(V)$, where V is a neighborhood of $K_{r_0} \times \{(\xi,\tau) \in \mathbb{R}^n \times \mathbb{R}_+ : |\xi|^2 + \tau^2 = 1\}$ in $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_+$. ◻

Now, we set

$$\mu^\varepsilon(x,\xi') = (q_1^\varepsilon)^2 + 2\varepsilon q_1^\varepsilon \psi''_{x_n,x_a}(\psi'_{x_n};\xi_a) + (\psi'_{x_n})^2 q_2^\varepsilon.$$

For $\tau > 0$, the symbol $\mu^\varepsilon(x,\xi')$ has the property that $\mu^\varepsilon(x,\xi') = 0$ if and only if there exists ξ_n real such that $p^\varepsilon_\psi(x,\xi',\xi_n) = 0$. This can be seen by noticing that $\tau \operatorname{Im} p^\varepsilon_\psi = \tilde{q}_1^\varepsilon = 0$ if and only if $\xi_n = -q_1^\varepsilon/\psi'_{x_n}$ (see (5.12)), as a function of τ, x, ξ'. Note that given τ, x, ξ', the formula

$$p^\varepsilon_\psi(x,\xi',\xi_n) = \operatorname{Re} p^\varepsilon_\psi(x,\xi',\xi_n) = \tilde{q}_2^\varepsilon(x,\xi',\xi_n) = (\psi'_{x_n})^{-2}\mu^\varepsilon(x,\xi') \quad \text{for } \xi_n = -\frac{q_1^\varepsilon}{\psi'_{x_n}}$$

always holds (even if $\tau = 0$). Notice also that $\mu^\varepsilon(x,\xi')$ is a tangential symbol of order 2.

Lemma 5.10. *There exists ε_0 such that for all $\varepsilon \in (0,\varepsilon_0)$, there exist $C_1, C_2 > 0$ such that for all $(x,\xi') \in K_{r_0} \times \mathbb{R}^{n-1}$ and $\tau > 0$, we have*

$$|\xi'|^2 + \tau^2 \leq C_1 b_2^\varepsilon + C_2\left[\frac{[\mu^\varepsilon(x,\xi')]^2}{|\xi'|^2+\tau^2} + |\xi_a|^2\right]. \tag{5.20}$$

Proof. Note first that for any (x, ξ', ξ_n) with $\xi_n = -q_1(x, \xi')/\psi'_{x_n}$, we have $\tilde{q}_1^\varepsilon(x, \xi', \xi_n) = 0$ and

$$p_\psi^\varepsilon(x, \xi', \xi_n) = \tilde{q}_2^\varepsilon(x, \xi', \xi_n) = (\psi'_{x_n})^{-2}\mu^\varepsilon(x, \xi').$$

Now, assume $\mu^\varepsilon(x, \xi') = 0$ and $\xi_a = 0$. Setting $\xi_n = -q_1(x, \xi')/\psi'_{x_n}$, we have $p_\psi^\varepsilon(x, \xi', \xi_n) = 0$. Using Lemma 5.9, we have $\{\tilde{q}_2^\varepsilon, \tilde{q}_1^\varepsilon\}(x, \xi', \xi_n) > 0$. According to the definition of B_2^ε in (5.19), we have $b_2^\varepsilon(x, \xi') > 0$. As a consequence,

$$[\mu^\varepsilon(x, \xi') = 0 \text{ and } \xi_a = 0] \Rightarrow b_2^\varepsilon(x, \xi') > 0.$$

Moreover, all terms in (5.20) are homogeneous of degree 2 in (ξ', τ) and continuous on $(\xi', \tau) \neq (0, 0)$. Hence, applying Lemma A.1 below on the compact set $K_{r_0} \times \{(\xi', \tau) \in \mathbb{R}^{n-1} \times \mathbb{R}_+ : |\xi'|^2 + \tau^2 = 1,\ \xi_a = 0\}$ yields (5.20) on that set. The conclusion follows by homogeneity. □

Let us now come back to the proof of Proposition 5.6. Taking the real part of (5.17) and using (5.19), we obtain

$$\begin{aligned}\|P_{\psi,\varepsilon} f\|_0^2 - \tau \operatorname{Re} \mathcal{B}^\varepsilon(f) = \|\tilde{Q}_2^\varepsilon f\|_0^2 + \tau^2 \|\tilde{Q}_1^\varepsilon f\|_0^2 + \tau \operatorname{Re}(B_2^\varepsilon f, f) \\ + \tau \operatorname{Re}\big((B_0^\varepsilon \tilde{Q}_2^\varepsilon + B_1^\varepsilon \tilde{Q}_1^\varepsilon) f, f\big).\end{aligned} \tag{5.21}$$

Concerning the remainder term, we have

$$\begin{aligned}\tau\left|\operatorname{Re}\big((B_0^\varepsilon \tilde{Q}_2^\varepsilon + B_1^\varepsilon \tilde{Q}_1^\varepsilon) f, f\big)\right| &\leq \tau \|f\|_0 \|\tilde{Q}_2^\varepsilon f\|_0 + \tau |f|_1 \|\tilde{Q}_1^\varepsilon f\|_0 \\ &\leq \tau^{-1/2}(\tau |f|_{1,\tau}^2 + \|\tilde{Q}_2^\varepsilon f\|_0^2 + \tau^2 \|\tilde{Q}_1^\varepsilon f\|_0^2).\end{aligned} \tag{5.22}$$

We now define the tangential differential operator

$$\Sigma = (Q_1^\varepsilon)^2 + 2\varepsilon Q_1^\varepsilon \psi''_{x_n, x_a}(\psi'_{x_n}; D_a) + (\psi'_{x_n})^2 Q_2^\varepsilon,$$

having principal symbol μ^ε. We also let G be a tangential pseudodifferential operator with principal symbol $\frac{\mu^\varepsilon(x,\xi')}{|\xi'|^2+\tau^2}$. The operator

$$C_1 B_2^\varepsilon + C_2(G^*\Sigma + D_a^2)$$

lies in the tangential class $S\big((|\xi'| + \tau)^2, |dx'|^2 + \frac{|d\xi'|^2}{(|\xi'|+\tau)^2}\big)$ (see [Hör85, Chapter XVIII] or [Ler10]), in which symbols are allowed to depend smoothly upon the variable x_n. According to Lemma 5.10, it is elliptic in this class, so that the tangential Gårding inequality yields, for τ sufficiently large,

$$|f|_{1,\tau}^2 \leq C \operatorname{Re}\big((B_2^\varepsilon f, f) + \operatorname{Re}(\Sigma f, G f) + \|D_a f\|_0^2\big). \tag{5.23}$$

Writing $\psi'_{x_n} D_n = \frac{1}{2}(\tilde{Q}_1^\varepsilon - [D_n, \psi'_{x_n}]) - Q_1^\varepsilon$ (where ψ'_{x_n} does not vanish) allows us to estimate the full norm $\|f\|_{1,\tau}$ according to

$$\|f\|_{1,\tau} \leq C(\|\tilde{Q}_1^\varepsilon f\|_0 + |f|_{1,\tau}). \tag{5.24}$$

Recalling the definitions of $\tilde{Q}_i^\varepsilon$ in terms of Q_i^ε in (5.11), $i = 1, 2$, we also have

$$\begin{aligned}
\Sigma &= \big(\psi'_{x_n} D_n - \tfrac{1}{2}(\tilde{Q}_1^\varepsilon - [D_n, \psi'_{x_n}])\big)^2 + 2\varepsilon Q_1^\varepsilon \psi''_{x_n,x_a}(\psi'_{x_n}; D_a) \\
&\quad + (\psi'_{x_n})^2\big(\tilde{Q}_2^\varepsilon - D_n^2 + 2\varepsilon\psi''_{x_n,x_a}(D_n; D_a)\big) \\
&= \big(\psi'_{x_n} D_n - \tfrac{1}{2}(\tilde{Q}_1^\varepsilon - [D_n, \psi'_{x_n}])\big)\psi'_{x_n} D_n + \tfrac{1}{2} Q_1(\tilde{Q}_1^\varepsilon - [D_n, \psi'_{x_n}]) \\
&\quad + 2\varepsilon Q_1^\varepsilon \psi''_{x_n,x_a}(\psi'_{x_n}; D_a) + (\psi'_{x_n})^2\big(\tilde{Q}_2^\varepsilon - D_n^2 + 2\varepsilon\psi''_{x_n,x_a}(D_n; D_a)\big).
\end{aligned} \tag{5.25}$$

In this expression, notice that second order derivatives in x_n, namely the terms $(\psi'_{x_n})^2 D_n^2$, cancel. Hence, we obtain

$$\Sigma \in (\psi'_{x_n})^2\tilde{Q}_2^\varepsilon - \tfrac{1}{2}\psi'_{x_n} D_n \tilde{Q}_1^\varepsilon + 2\varepsilon\psi''_{x_n,x_a}((\psi'_{x_n})^2 D_n + Q_1^\varepsilon \psi'_{x_n}; D_a) + \mathcal{D}_\tau^1 \tilde{Q}_1^\varepsilon + \mathcal{D}_\tau^1 + \mathcal{D}_\tau^0 D_n.$$

We now want to estimate the term $\operatorname{Re}(\Sigma f, Gf)$ in (5.23). For this, integrating by parts in the tangential direction x_a, we have

$$|(\psi''_{x_n,x_a}((\psi'_{x_n})^2 D_n + Q_1^\varepsilon \psi'_{x_n}; D_a) f, Gf)| \le C\|\langle D_a\rangle f\|\,\|f\|_{1,\tau}.$$

This yields

$$\begin{aligned}
|(\Sigma f, Gf)| &\le C\|\tilde{Q}_2^\varepsilon f\|_0\|f\|_0 + \left|\left(\frac{1}{2i}\psi'_{x_n}\tilde{Q}_1^\varepsilon f, Gf\right)_0\right| \\
&\qquad + \|\tilde{Q}_1^\varepsilon f\|_0\|f\|_{1,\tau} + \|f\|_0\|f\|_{1,\tau} + C\|\langle D_a\rangle f\|\,\|f\|_{1,\tau} \\
&\le \left|\left(\frac{1}{2i}\psi'_{x_n}\tilde{Q}_1^\varepsilon f, Gf\right)_0\right| \\
&\qquad + C\|f\|_{1,\tau}\big(\tau^{-1}\|\tilde{Q}_2^\varepsilon f\|_0 + \|\tilde{Q}_1^\varepsilon f\|_0 + \tau^{-1}\|f\|_{1,\tau} + \|D_a f\|_0\big).
\end{aligned} \tag{5.26}$$

According to (5.23), (5.24) and (5.26), this now implies

$$\|f\|_{1,\tau}^2 \lesssim \operatorname{Re}(B_2^\varepsilon f, f) + \|\tilde{Q}_1^\varepsilon f\|_0^2 + \left|\left(\frac{1}{2i}\psi'_{x_n}\tilde{Q}_1^\varepsilon f, Gf\right)_0\right| + \tau^{-2}\|\tilde{Q}_2^\varepsilon f\|_0^2 + \|D_a f\|_0^2.$$

Coming back to (5.21), we obtain, for τ large enough,

$$\begin{aligned}
\tau\|f\|_{1,\tau}^2 &\lesssim \|P_{\psi,\varepsilon} f\|_0^2 - \tau \operatorname{Re}\mathcal{B}^\varepsilon(f) - \|\tilde{Q}_2^\varepsilon f\|_0^2 - \tau^2\|\tilde{Q}_1^\varepsilon f\|_0^2 + \tau\|D_a f\|_0^2 \\
&\qquad + \tau\left|\left(\frac{1}{2i}\psi'_{x_n}\tilde{Q}_1^\varepsilon f, Gf\right)_0\right| \\
&\lesssim \|P_{\psi,\varepsilon} f\|_0^2 - \tau \operatorname{Re}\mathcal{B}^\varepsilon(f) + \tau\|D_a f\|_0^2 + \tau\left|\left(\frac{1}{2i}\psi'_{x_n}\tilde{Q}_1^\varepsilon f, Gf\right)_0\right|.
\end{aligned}$$

Recalling the definition of $\tilde{Q}_1^\varepsilon$ in (5.11), we have $\psi'_{x_n}\tilde{Q}_1^\varepsilon = 2(\psi'_{x_n})^2 D_n + G_1$, where $G_1 \in \mathcal{D}_\tau^1$ is a differential operator of order 1 (in (τ, D')), we finally have

$$\tau\|f\|_{1,\tau}^2 \lesssim \|P_{\psi,\varepsilon} f\|_0^2 - \tau \operatorname{Re}\mathcal{B}^\varepsilon(f) + \tau\|D_a f\|_0^2 + \tau|(D_n f + G_1 f, Gf)_0|, \tag{5.27}$$

where G a tangential pseudodifferential operator of order zero. Recalling the form of $\mathcal{B}^{\varepsilon}(f)$ in (5.18) gives the bound $|\mathcal{B}^{\varepsilon}(f)| \lesssim \tau^2 |f_{|x_n=0}|_0^2 + |Df_{|x_n=0}|_0^2$, which concludes the proof of (5.9).

Now if $f_{|x_n=0} = 0$, all tangential derivatives vanish. With (5.27) and the form of $\mathcal{B}^{\varepsilon}(f)$ in (5.18), this yields

$$\tau \|f\|_{1,\tau}^2 \lesssim \|P_{\psi,\varepsilon} f\|_0^2 - 2\tau(\psi'_{x_n} D_n f, D_n f)_0 + \tau \|D_a f\|_0^2,$$

which proves (5.10) since $\psi'_{x_n} > 0$ for $(x', x_n = 0) \in K$. This concludes the proof of Proposition 5.6. $\square$

We now turn to the proof of Theorem 5.2.

Proof of Theorem 5.2. In the proof, we consider functions $u \in C_0^\infty(K_{r_0/4})$ where K_r is defined in (5.1). Let $\chi \in C_0^\infty(B_{\mathbb{R}^{n_a}}(0, r_0))$ be such that $\chi = 1$ on $B_{\mathbb{R}^{n_a}}(0, r_0/2)$. Setting

$$v = Q_{\varepsilon,\tau}^{\psi} u = e^{-\frac{\varepsilon}{2\tau}|D_a|^2}(e^{\tau\psi} u) \quad \text{and} \quad f = \chi(x_a) v(x),$$

we have $\operatorname{supp} f \subset K_{r_0}$ so that we may apply Proposition 5.6 to f. We have

$$v - f = (1-\chi) Q_{\varepsilon,\tau}^{\psi} u = (1-\chi) e^{-\frac{\varepsilon}{2\tau}|D_a|^2}(\check{\chi} e^{\tau\psi} u)$$

for some $\check{\chi} \in C_0^\infty(B_{\mathbb{R}^{n_a}}(0, r_0/3))$ with $\check{\chi} = 1$ in a neighborhood of $\overline{B}_{\mathbb{R}^{n_a}}(0, r_0/4)$. As a consequence of Lemma 2.4, we have, for $\tau \geq \tau_0$,

$$\|v\|_{1,\tau} \leq \|f\|_{1,\tau} + C e^{-C\tau/\varepsilon} \|e^{\tau\psi} u\|_{1,\tau}. \tag{5.28}$$

Now, it remains to estimate the terms on the RHS of Proposition 5.6 in terms of v. Notice first that the same reasoning as for Lemma 2.4 (using that D_a is tangential) allows us to estimate the boundary terms as

$$|f_{|x_n=0}|_0 \leq |v_{|x_n=0}|_0 + C e^{-C\tau/\varepsilon} |e^{\tau\psi} u_{|x_n=0}|_0, \tag{5.29}$$

and, with $Dv - Df = D((1-\chi) e^{-\frac{\varepsilon}{2\tau}|D_a|^2}(\check{\chi} e^{\tau\psi} u))$,

$$\begin{aligned}
|Df_{|x_n=0}|_0 &\leq |Dv_{|x_n=0}|_0 + C e^{-C\tau/\varepsilon} |e^{\tau\psi} u_{|x_n=0}|_0 + C e^{-C\tau/\varepsilon} |e^{\tau\psi}(\tau\psi' + D) u_{|x_n=0}|_0 \\
&\quad + C e^{-C\tau/\varepsilon} |e^{\tau\psi} D u_{|x_n=0}|_0 \\
&\leq |Dv_{|x_n=0}|_0 + C\tau e^{-C\tau/\varepsilon} |e^{\tau\psi} u_{|x_n=0}|_0 + C e^{-C\tau/\varepsilon} |e^{\tau\psi} D u_{|x_n=0}|_0.
\end{aligned} \tag{5.30}$$

Second, we estimate $\|P_{\psi,\varepsilon} f\|_0 = \|P_{\psi,\varepsilon} \chi v\|_0 = \|\chi P_{\psi,\varepsilon} v\|_0 + \|[P_{\psi,\varepsilon}, \chi] v\|_0$. For the commutator, we write

$$[P_{\psi,\varepsilon}, \chi] v = [P_{\psi,\varepsilon}, \chi] e^{-\frac{\varepsilon}{2\tau}|D_a|^2} \check{\chi} e^{\tau\psi} u.$$

We notice that $[P_{\psi,\varepsilon}, \chi]$ is a differential operator of order 1 in (D, τ) with coefficients supported in $\operatorname{supp} \chi'_{x_a}$, that is, away from $\operatorname{supp} \check{\chi}$. In particular, Lemma 2.4 implies $\|[P_{\psi,\varepsilon}, \chi] v\|_0 \leq C e^{-c\tau/\varepsilon} \|e^{\tau\psi} u\|_{1,\tau}$. This yields

$$\|P_{\psi,\varepsilon} f\|_0 \leq \|P_{\psi,\varepsilon} v\|_0 + C e^{-c\tau/\varepsilon} \|e^{\tau\psi} u\|_{1,\tau}. \tag{5.31}$$

Now, it remains to treat the term $\|D_a f\|_0$. We find similarly that

$$\begin{aligned}\|D_a f\|_0 = \|D_a(\chi v)\|_0 &\le \|\chi D_a v\|_0 + \left\|\chi'_{x_a} e^{-\frac{\varepsilon}{2\tau}|D_a|^2} \check{\chi} e^{\tau\psi} u\right\|_0 \\ &\le \|D_a v\|_0 + Ce^{-c\tau/\varepsilon}\|e^{\tau\psi}u\|_0, \end{aligned} \tag{5.32}$$

where we have again used Lemma 2.4.

Let ς be a small constant to be fixed later. We distinguish between frequencies of size smaller and larger than $\varsigma\tau$. For $\tau \ge 1/(\varsigma^2\varepsilon)$ large enough (so that $\sqrt{\tau/\varepsilon} \ge \varsigma\tau$ and the function $s \mapsto se^{-\frac{\varepsilon}{2\tau}s^2}$ is decreasing on $s \ge \sqrt{\tau/\varepsilon}$) we get

$$\begin{aligned}\|D_a v\|_0 = \|D_a e^{-\frac{\varepsilon}{2\tau}|D_a|^2} e^{\tau\psi} u\|_0 &\le \|D_a \mathbb{1}_{|D_a|\le\varsigma\tau} v\|_0 + \left\|D_a \mathbb{1}_{|D_a|\ge\varsigma\tau} e^{-\frac{\varepsilon}{2\tau}|D_a|^2} e^{\tau\psi} u\right\|_0 \\ &\le \varsigma\tau\|v\|_0 + \varsigma\tau e^{-\tau\varsigma^2\varepsilon/2}\|e^{\tau\psi}u\|_0. \end{aligned} \tag{5.33}$$

We may now apply Proposition 5.6 to f. Combining the Carleman estimate (5.9) with (5.29)–(5.33), we obtain, for some $C_1 > 0$ and $\tau \ge \tau_0$ with τ_0 (depending also on ς, ε) sufficiently large,

$$\begin{aligned} C_1\tau\|v\|_{1,\tau}^2 &\le \|P_{\psi,\varepsilon}v\|_0^2 + e^{-2c\tau/\varepsilon}\|e^{\tau\psi}u\|_{1,\tau}^2 + \varsigma^2\tau^3\|v\|_0^2 + \varsigma^2\tau^3 e^{-\tau\varsigma^2\varepsilon}\|e^{\tau\psi}u\|_0^2 \\ &\quad + \tau^3|v_{|x_n=0}|_0^2 + \tau^3 e^{-2c\tau/\varepsilon}|e^{\tau\psi}u_{|x_n=0}|_0^2 + \tau|Dv_{|x_n=0}|_0^2 + \tau e^{-2c\tau/\varepsilon}|e^{\tau\psi}Du_{|x_n=0}|_0^2. \end{aligned}$$

For fixed $\varsigma \le \sqrt{C_1/2}$ this yields, for some $\mathsf{d} > 0$ (ε is already fixed) and $\tau \ge \tau_0$,

$$\begin{aligned} \tfrac{1}{2}C_1\tau\|v\|_{1,\tau}^2 &\le \|P_{\psi,\varepsilon}v\|_0^2 + e^{-\mathsf{d}\tau}\|e^{\tau\psi}u\|_{1,\tau}^2 + \tau^3|v_{|x_n=0}|_0^2 + e^{-\mathsf{d}\tau}|e^{\tau\psi}u_{|x_n=0}|_0^2 \\ &\quad + \tau|Dv_{|x_n=0}|_0^2 + e^{-\mathsf{d}\tau}|e^{\tau\psi}Du_{|x_n=0}|_0^2. \end{aligned} \tag{5.34}$$

Similarly, if moreover $\psi'_{x_n} > 0$ for $(x', x_n = 0) \in K_{r_0}$, then (5.10) yields, for all $u \in C_0^\infty(K_{r_0/4})$ such that $u_{|x_n=0} = 0$,

$$C_1\tau\|v\|_{1,\tau}^2 \le \|P_{\psi,\varepsilon}v\|_0^2 + e^{-2c\tau/\varepsilon}\|e^{\tau\psi}u\|_{1,\tau}^2 + \varsigma^2\tau^3\|v\|_0^2 + \varsigma^2\tau^3 e^{-\tau\varsigma^2\varepsilon}\|e^{\tau\psi}u\|_0^2,$$

and hence

$$\tfrac{1}{2}C_1\tau\|v\|_{1,\tau}^2 \le \|P_{\psi,\varepsilon}v\|_0^2 + e^{-\mathsf{d}\tau}\|e^{\tau\psi}u\|_{1,\tau}^2. \tag{5.35}$$

Rewriting (5.34)–(5.35) in terms of u concludes the proof of Theorem 5.2. □

5.3. *The local quantitative uniqueness result*

The Carleman estimates of the previous section have been proved when P has a very specific form. Before proving the local quantitative uniqueness result, we first state them in a more invariant way that can be obtained by change of coordinates in x_b. When doing so, we strengthen the assumptions made on the operator P, still encompassing the cases of wave and Schrödinger operators (or more generally of the form of Remark 1.10).

From now on, and until the end of the section, P will have the following property:

Assumption 5.1. P is a differential operator on $\mathbb{R}^{n_a} \times \mathbb{R}^{n_b}_+$, of order 2, with coefficients analytic in x_a. Moreover, P has principal symbol independent of x_a of the form $p(x, \xi) = q_{x_b}(\xi_a) + \tilde{q}_{x_b}(\xi_b)$, where $q_{x_b}, \tilde{q}_{x_b}$ are smooth x_b-families of real quadratic forms on $\mathbb{R}^{n_a}$ and $\mathbb{R}^{n_b}$ respectively.

Moreover, given $V \in L^\infty(\mathbb{R}^{n_b}_+)$ and $W \in L^\infty(\mathbb{R}^{n_b}_+; \mathbb{R}^n)$, independent of x_a, we set $P_{V,W} = P + W \cdot \nabla + V$.

Note that operators P satisfying Assumption 5.1 also satisfy assumption (H).

The proof of the local quantitative uniqueness result will then be essentially the same as in the boundaryless case. The following proposition is the counterpart, in the boundary case, of the end of the first step in Section 3 (hence containing the geometrical part of the proof of the local uniqueness result).

Proposition 5.11. *Let $x^0 \in \{x_n = 0\}$ and let P satisfy Assumption 5.1. Assume that $\{x_n = 0\}$ is noncharacteristic with respect to P. Let ϕ be a function defined in a neighborhood of x^0 in $\mathbb{R}^n$ such that $\phi(x^0) = 0$, and $\{\phi = 0\}$ is a C^2 strongly pseudoconvex oriented surface at x^0 in the sense of Definition 1.7.*

Then there exists $R_0 > 0$ and a smooth function $\psi : B(x^0, 4R_0) \to \mathbb{R}$ which is a quadratic polynomial with respect to $x_a \in \mathbb{R}^{n_a}$, such that for any $R \in (0, R_0]$, there exist $\varepsilon, \delta, \rho, r, \mathsf{d}, \tau_0, C > 0$ such that

(1) *$\delta \leq \mathsf{d}/8$ and (3.13)–(3.15) hold,*
(2) *for any $\tau \geq \tau_0$, the Carleman estimate (5.5) holds for P, for all $u \in C^\infty_0(\mathbb{R}^n_+)$ with $\operatorname{supp} u \subset B(x^0, 4R)$.*

If moreover $\phi'_{x_n}(x^0) > 0$, then the Carleman estimate (5.6) holds for P for all $u \in C^\infty_0(\mathbb{R}^n_+)$ with $\operatorname{supp} u \subset B(x^0, 4R)$ and $u_{|x_n=0} = 0$.

The estimates can also be made uniform for $\tau > \tau_0 \max\{1, \|V\|^{2/3}_{L^\infty}, \|W\|^2_{L^\infty}\}$ if P is replaced by $P_{V,W}$, as in Corollary 5.3.

Proof. First, by noncharacteristicity, we have $\tilde{q}_{x_b}(\xi_b) \neq 0$ for $x_b = (x'_b, 0)$ and $\xi'_b = 0$, $\xi^n_b = 1$. We may thus reason in normal geodesic coordinates for $\tilde{q}_{x_b}$ in $\mathbb{R}^{n_b}$, in a sufficiently small neighborhood of $\{x_n = 0\}$. More precisely (see [Hör85, Appendix C.5]) there exists a local diffeomorphism Ψ_b from a neighborhood of x^0_b in $\mathbb{R}^{n_b}_+$ to a neighborhood of 0 in $\mathbb{R}^{n_b}_+$ such that, for $\Psi := \mathrm{Id}_{\mathbb{R}^{n_a}} \otimes \Psi_b$, the principal part of $P^\Psi := (\Psi^{-1})^* P \Psi^*$ takes the form $\pm(\xi^n_b)^2 + r(x_b, \xi_a, \xi'_b)$. From the function $\phi \circ \Psi^{-1}$ (still defining a strongly pseudoconvex surface for P^Ψ since this property is invariant), we can construct a quadratic polynomial $\tilde{\psi}$ exactly as in Lemma 3.4/Corollary 3.6 such that the Carleman estimates (5.5)–(5.6) hold for P^Ψ and $\tilde{\psi}$. Note also that the constructions imply that if $\phi'_{x_n}(x^0) > 0$, then the same property holds for $\phi \circ \Psi^{-1}$ and then $\tilde{\psi}$. We then use Corollary 5.4 and next Corollary 5.3 to allow, first, lower order terms analytic in x_a and next lower order terms independent of x_a with the right estimates (note that both properties are invariant by our change of coordinates in x_b). Applying then the diffeomorphism Ψ to come back to the original setting yields the sought estimate with

$\psi = \tilde{\psi} \circ \Psi$, which remains a quadratic polynomial with respect to the variable x_a (only) since $\Psi := \mathrm{Id}_{\mathbb{R}^{n_a}} \otimes \Psi_b$. This proves (2).

Finally, the geometric assertion of (1) comes from the application of Lemma 3.4 in geodesic coordinates. There, using the distance $N(x, y) = |\Psi^{-1}(x) - \Psi^{-1}(y)|$ allows us to obtain (3.13)–(3.15) with Euclidean balls as claimed in (1). ◻

The aim is now to prove the following two local results, namely local quantitative uniqueness up to and from the boundary.

Theorem 5.12 (Local quantitative uniqueness up to the boundary). *Let $x^0 \in \{x_n = 0\}$ and let P satisfy Assumption 5.1. Assume that $\{x_n = 0\}$ is noncharacteristic with respect to P. Assume that there is a function ϕ defined in a neighborhood of x^0 in $\mathbb{R}^n$ such that $\phi(x^0) = 0$, and $\{\phi = 0\}$ is a C^2 strongly pseudoconvex oriented surface at x^0 in the sense of Definition 1.7 and such that $\phi'_{x_n}(x^0) > 0$.*

Then there exists $R_0 > 0$ such that for any $R \in (0, R_0)$, there exist $r, \rho > 0$ such that for any $\vartheta \in C_0^\infty(\mathbb{R}^n)$ with $\vartheta(x) = 1$ on a neighborhood of $\{\phi \geq 2\rho\} \cap B(x^0, 3R)$, for all $c_1, \kappa > 0$ there exist $C, \kappa', \beta, \tilde{\tau}_0 > 0$ such that

$$\|M^{\beta\mu}_{c_1\mu}\sigma_{r,c_1\mu}u\|_{1,+} \leq Ce^{\kappa\mu}\big(\|M^{\mu}_{c_1\mu}\vartheta_{c_1\mu}u\|_{1,+} + \|Pu\|_{L^2(B(x^0,4R)\cap\mathbb{R}^n_+)}\big) + Ce^{-\kappa'\mu}\|u\|_{1,+}$$

for all $\mu \geq \tilde{\tau}_0$ and $u \in C_0^\infty(\mathbb{R}^n_+)$ such that $u_{|x_n=0} = 0$.

Moreover, under the same assumptions, there exist $C_0, \kappa', \beta, \tilde{\tau}_0 > 0$ such that for all $V \in L^\infty(\mathbb{R}^{n_b})$, $W \in L^\infty(\mathbb{R}^{n_b}; \mathbb{R}^n)$ the previous estimate is still true with P replaced by $P_{V,W} = P + W \cdot \nabla + V$ with C replaced by $C_0 \max\{1, \|W\|_{L^\infty}\}$, and uniformly for all $\mu \geq \tilde{\tau}_0 \max\{1, \|V\|^{2/3}_{L^\infty}, \|W\|^2_{L^\infty}\}$.

This theorem is proved similarly to the case without boundary. See the details in the proof of the related Theorem 5.13 below.

Theorem 5.13 (Local quantitative uniqueness from the boundary). *Let x^0 and P satisfy Assumption 5.1. Assume that $\{x_n = 0\}$ is noncharacteristic with respect to P, and the function $\phi(x) = -x_n$ satisfies the property of Definition 1.7 at x^0.*

Then there exists $R_0 > 0$ such that for any $R \in (0, R_0)$, there exists $r > 0$ such that for all $c_1, \kappa > 0$ there exist $C, \kappa', \beta, \tilde{\tau}_0 > 0$ such that

$$\begin{aligned}&\|M^{\beta\mu}_{c_1\mu}\sigma_{r,c_1\mu}u\|_{1,+}\\ &\quad\leq Ce^{\kappa\mu}\big(\|D_n u\|_{L^2(B(x^0,4R)\cap\{x_n=0\}} + \|Pu\|_{L^2(B(x^0,4R)\cap\mathbb{R}^n_+)}\big) + Ce^{-\kappa'\mu}\|u\|_{1,+}\end{aligned}$$

for all $\mu \geq \tilde{\tau}_0$ and $u \in C_0^\infty(\mathbb{R}^n_+)$ such that $u_{|x_n=0} = 0$.

The same dependence of the constants holds if P is replaced by $P_{V,W}$ as in Theorem 5.12.

Proof. The proof is very similar to the proof of Theorem 3.1 in Section 3, using the Carleman estimate (5.5) of Theorem 5.2 . We only sketch it and underline the differences from the boundaryless case. We moreover add the lower order terms V and $W \cdot \nabla$; we need to check that all estimates can be carried out uniformly with respect to these terms.

Step 1: The geometric setting. We start by choosing $\phi = -x_n$. The surface $\{\phi = 0\} = \{-x_n = 0\}$ is noncharacteristic by assumption, and according to Remark 1.10, is hence a strongly pseudoconvex oriented surface for P. Proposition 5.11 furnishes an appropriate convexified ψ, polynomial of degree two in the variable x_a, that satisfies the desired geometric conditions, together with the Carleman estimate (5.5). We now follow the proof of the boundaryless case.

Step 2: Using the Carleman estimate. The point is to use the Carleman estimate (5.5) with weight ψ, applied to the (compactly supported) function $w = \sigma_{2R}\sigma_{R,\lambda}\chi_{\delta,\lambda}(\psi)\widetilde{\chi}_\delta(\psi)u$.

Similarly, using the same support property $\operatorname{supp}\chi_\delta \subset (-8\delta, \delta)$, and Lemma 2.13, we write

$$\begin{aligned}\|Q^\psi_{\varepsilon,\tau} P_{V,W} w\|_{0,+} &\le \|Q^\psi_{\varepsilon,\tau}\sigma_{2R}\sigma_{R,\lambda}\chi_{\delta,\lambda}(\psi)\widetilde{\chi}_\delta(\psi) P_{V,W} u\|_{0,+}\\ &\quad + \|Q^\psi_{\varepsilon,\tau}[\sigma_{2R}\sigma_{R,\lambda}\chi_{\delta,\lambda}(\psi)\widetilde{\chi}_\delta(\psi), P_{V,W}]u\|_{0,+}\\ &\le e^{\tau^2/\lambda} e^{\delta\tau}\|P_{V,W}u\|_{L^2(B(x^0,4R)\cap\{x_n\ge 0\})}\\ &\quad + \|Q^\psi_{\varepsilon,\tau}[\sigma_{2R}\sigma_{R,\lambda}\chi_{\delta,\lambda}(\psi)\widetilde{\chi}_\delta(\psi), P_{V,W}]u\|_{0,+}.\end{aligned}$$

Next, Lemma 3.7 still holds in $\mathbb{R}^n_+$ since x_a is a tangential variable (see Remark 5.1). Hence, the commutator term is bounded by

$$\begin{aligned}&\|Q^\psi_{\varepsilon,\tau}[\sigma_{2R}\sigma_{R,\lambda}\chi_{\delta,\lambda}(\psi)\widetilde{\chi}_\delta(\psi), P]u\|_{0,+}\\ &\qquad \le Ce^{2\delta\tau}\|M^{2\mu}_\lambda \vartheta_\lambda u\|_{1,+} + C\lambda^{1/2}\tau^N\big(e^{-\frac{\varepsilon\mu^2}{4\tau}} + e^{-8\delta\tau} + e^{\delta\tau - c\mu}\big)e^{\tau^2/\lambda}e^{\delta\tau}\|u\|_{1,+},\end{aligned}$$

with some ϑ (equal to 1 in a neighborhood of $\{\phi \ge 2\rho\} \cap B(x^0, 3R)$) supported in $\{\phi > \rho\} = \{x_n < -\rho\}$.

Moreover, following Remark 3.8, we can get uniform estimates for the commutator of $P_{V,W}$ by replacing C by $C_0 \max\{1, \|W\|_{L^\infty(\mathbb{R}^{n_b})}\}$. We will not write it any more for clarity but it appears multiplicatively in all the estimates.

Since the operator $M^\mu_{c_1\mu}$ only applies in the tangential variable x_a, we have

$$\|M^\mu_{c_1\mu}\vartheta_{c_1\mu}u\|_{1,+} \le \|\vartheta_{c_1\mu}u\|_{1,+}.$$

Moreover, since ϑ is supported in $\{x_n < -\rho\}$ and $\vartheta_{c_1\mu} = e^{-\frac{|D_a|^2}{c_1\mu}}\vartheta$ is a regularization in the variable x_a, $\vartheta_{c_1\mu}$ is also supported in $\{x_n < -\rho\}$ and $\vartheta_{c_1\mu}(x) = 0$ if $x_n \ge 0$. In particular, $\|\vartheta_{c_1\mu}u\|_{1,+} = 0$. That is,

$$\begin{aligned}\|Q^\psi_{\varepsilon,\tau} P_{V,W} w\|_{0,+} &\le Ce^{\tau^2/\lambda}e^{\delta\tau}\|P_{V,W}u\|_{L^2(B(x^0,4R)\cap\mathbb{R}^n_+)}\\ &\quad + C\lambda^{1/2}\tau^N\big(e^{-\varepsilon\mu^2/4\tau} + e^{-8\delta\tau} + e^{\delta\tau - c\mu}\big)e^{\tau^2/\lambda}e^{\delta\tau}\|u\|_{1,+}.\end{aligned}$$

The other terms in the Carleman estimate that we have to check are

$$\tau|(D(Q^\psi_{\varepsilon,\tau}w))_{|x_n=0}|_0^2 + e^{-\mathsf{d}\tau}|e^{\tau\psi}Dw_{|x_n=0}|_0^2 \le C\tau|e^{\tau\psi}D_n w_{|x_n=0}|_0^2, \tag{5.36}$$

where we have used $u_{|x_n=0} = w_{|x_n=0} = 0$. This also implies

$$D_n w_{|x_n=0} = (\sigma_{2R}\sigma_{R,\lambda}\chi_{\delta,\lambda}(\psi)\widetilde{\chi}_\delta(\psi)D_n u)_{|x_n=0}.$$

Since $\|e^{\tau\psi}\chi_{\delta,\lambda}(\psi)\|_{L^\infty} \leq C\lambda^{1/2}e^{\delta\tau}e^{\tau^2/\lambda}$ thanks to Lemma 2.13, the left hand side of (5.36) is bounded by $C\lambda e^{2\delta\tau}e^{2\tau^2/\lambda}\tau|D_n u|^2_{L^2(B(x^0,4R)\cap\{x_n=0\})}$.

So, combining the Carleman estimate of Corollary 5.3 and the previous bounds, we have proved for all $\tau \geq \tau_0 \max\{1, \|V\|^{2/3}_{L^\infty}, \|W\|^2_{L^\infty}\}$, $\mu \geq 1$, $\frac{1}{C}\mu \leq \lambda \leq C\mu$,

$$
\begin{aligned}
\tau^{1/2}\|Q^{\psi}_{\varepsilon,\tau}\sigma_{2R}\sigma_{R,\lambda}\chi_{\delta,\lambda}(\psi)\widetilde{\chi}_\delta(\psi)u\|_{1,+,\tau}
&\leq Ce^{\tau^2/\lambda}e^{\delta\tau}\|P_{V,W}u\|_{L^2(B(x^0,4R)\cap\mathbb{R}^n_+)}\\
&\quad + C\lambda^{1/2}\tau^{1/2}e^{\delta\tau}e^{\tau^2/\lambda}|D_n u|_{L^2(B(x^0,4R)\cap\{x_n=0\})}\\
&\quad + C\lambda^{1/2}\tau^N\big(e^{-\frac{\varepsilon\mu^2}{4\tau}} + \tau e^{-8\delta\tau} + e^{\delta\tau-c\mu}\big)e^{\tau^2/\lambda}e^{\delta\tau}\|u\|_{1,+}.
\end{aligned}
$$

So, denoting $D = e^{\kappa\mu}(\|D_n u\|_{L^2(B(x^0,4R)\cap\{x_n=0\})} + \|Pu\|_{L^2(B(x^0,4R)\cap\mathbb{R}^n_+)})$, we can rewrite it as

$$
\begin{aligned}
\|Q^{\psi}_{\varepsilon,\tau}\sigma_{2R}\sigma_{R,\lambda}\chi_{\delta,\lambda}(\psi)\widetilde{\chi}_\delta(\psi)u\|_{1,+,\tau} &\leq C\mu^{1/2}e^{\delta\tau}e^{C\tau^2/\mu}e^{-\kappa\mu}D\\
&\quad + C\mu^{1/2}\tau^N\big(e^{-\frac{\varepsilon\mu^2}{4\tau}} + \tau e^{-8\delta\tau} + e^{\delta\tau-c\mu}\big)e^{C\tau^2/\lambda}e^{\delta\tau}\|u\|_{1,+}.
\end{aligned}
$$

Step 3: A complex analysis argument. We now proceed exactly as in the boundaryless case. For any test function $f \in C^\infty_0(\mathbb{R}^n_+)$, we define the distribution h_f (with $\beta > 0$ to be chosen later) by

$$
\langle h_f, w\rangle_{\mathcal{E}'(\mathbb{R}),C^\infty(\mathbb{R})} := \langle \sigma_{2R}\sigma_{R,\lambda}\chi_{\delta,\lambda}(\psi)\widetilde{\chi}_\delta(\psi)w(\psi)u, (M^{\beta\mu}f)\rangle_{H^1_0(\mathbb{R}^n_+),H^{-1}(\mathbb{R}^n_+)}.
$$

We proceed similarly, noticing at the end that $C^\infty_0(\mathbb{R}^n_+)$ is dense in the dual space $H^{-1}(\mathbb{R}^n_+)$ and that all operations are tangential. The analogue of Lemma 3.10 is proved with the same complex analysis argument (which does not involve the x-space, but only complexifies the Carleman large parameter τ), using Lemma 3.11. This yields the analogous result for $\mu \geq C\tau_0 \max\{1, \|V\|^{2/3}_{L^\infty}, \|W\|^2_{L^\infty}\}$.

Finally, it remains to transfer the estimate of $\|Q^{\psi}_{\varepsilon,\tau}\sigma_{2R}\sigma_{R,\lambda}\chi_{\delta,\lambda}(\psi)\widetilde{\chi}_\delta(\psi)u\|_{1,+,\tau}$ to an estimate of $\|M^{\beta\mu}_{c_1\mu}\sigma_{r,c_1\mu}u\|_{1,+}$. The computations of the end of Section 3.3 remain valid in the present context for the following two reasons: (a) the operators $M^{\beta\mu}_{c_1\mu}$ are tangential and the associated estimates of Section 2.4.1 still hold; (b) these computations only rely on the geometric fact that $\sigma_R = \chi_\delta(\psi) = \widetilde{\chi}_\delta(\psi) = \eta_\delta(\psi) = 1$ on a neighborhood of $\operatorname{supp}\sigma_r$, which now follows from Proposition 5.11. ◻

5.4. The semiglobal estimate with boundary

In this section, we prove a version of Theorem 1.11/4.7 adapted to the boundary value problem. More precisely, the following result considers, under the assumptions of the above uniqueness results, the Dirichlet boundary condition at the bottom and the top of the graph, with an observation at the bottom.

Recall that in the present context, the analytic variable is supposed to be tangential to the boundary. In the following results (as opposed to the boundaryless case), this translates into the fact that we assume that, in the splittings $x = (x', x_n) \in \mathbb{R}^{n-1} \times [0, \ell_0]$ and $x = (x_a, x_b) \in \mathbb{R}^{n_a} \times \mathbb{R}^{n_b}$, the variable $x_n = x_b^n$ always belongs to the x_b variables.

In Theorem 5.14 below, we state the semiglobal estimate with an observation from the boundary (i.e. the first hypersurface S_0 is a Dirichlet boundary) and when the last hypersurface S_1 touches a (Dirichlet) boundary. This is the most intricate situation. The proof is the same in the cases where the last hypersurface does not touch the boundary, or if we have an internal observation around the first surface. We do not state these cases for the sake of concision.

Theorem 5.14. *Let D be a bounded open subset of $\mathbb{R}^{n-1}$ with smooth boundary. Let $G = G(x', \varepsilon)$ be a C^2 function defined in a neighborhood of $\overline{D} \times [0, 1]$ such that*

- *for all $\varepsilon \in (0, 1]$, we have $\{x' \in \mathbb{R}^{n-1} : G(x', \varepsilon) \geq 0\} = \overline{D}$,*
- *for all $x' \in D$, the function $\varepsilon \mapsto G(x', \varepsilon)$ is strictly increasing,*
- *for all $\varepsilon \in (0, 1]$, we have $\{x' \in \mathbb{R}^{n-1} : G(x', \varepsilon) = 0\} = \partial D$.*

Set

$$\ell_0 = \max_{x' \in \overline{D}} G(x', 1), \qquad G(x', 0) = 0, \qquad S_0 = \overline{D} \times \{x_n = 0\},$$

and, for $\varepsilon \in (0, 1]$,

$$\begin{aligned} S_\varepsilon &= \{(x', x_n) \in \mathbb{R}^n : x_n \geq 0 \text{ and } G(x', \varepsilon) = x_n\} \\ &= (\overline{D} \times \mathbb{R}) \cap \{(x', x_n) \in \mathbb{R}^n : G(x', \varepsilon) = x_n\}, \\ K &= \{x \in \mathbb{R}^n : 0 \leq x_n \leq G(x', 1)\}. \end{aligned}$$

Let Ω be a neighborhood of K in $\mathbb{R}^{n-1} \times [0, \ell_0]$ and $\widetilde{D}$ a neighborhood of $\overline{D}$ in $\mathbb{R}^{n-1}$. Let P satisfy Assumption 5.1. Assume that $\{x_n = 0\}$ and $\{x_n = \ell_0\}$ are noncharacteristic with respect to P. Assume also that for any $\varepsilon \in [0, 1]$, the function

$$\phi_\varepsilon(x', x_n) := G(x', \varepsilon) - x_n$$

is strongly pseudoconvex in $\{\xi_a = 0\}$ with respect to P on the whole S_ε.

Then there exist a neighborhood U of K and constants $\kappa, C, \mu_0 > 0$ such that for all $u \in C_0^\infty(\mathbb{R}^{n-1} \times [0, \ell_0])$ satisfying

$$u_{|x_n=0} = u_{|x_n=\ell_0} = 0 \quad \text{on } \widetilde{D},$$

we have

$$\|u\|_{L^2(U)} \leq Ce^{\kappa\mu}(\|D_n u_{|x_n=0}\|_{L^2(\widetilde{D})} + \|Pu\|_{L^2(\Omega)}) + \frac{C}{\mu}\|u\|_{H^1(\mathbb{R}^{n-1}\times[0,\ell_0])}$$

for all $\mu \geq \mu_0$.

Moreover, under the same assumptions, there exist $C_0, \kappa', \beta, \tilde{\tau}_0 > 0$ such that for all $V \in L^\infty(\mathbb{R}^{n_b})$, $W \in L^\infty(\mathbb{R}^{n_b}; \mathbb{R}^n)$ the previous estimate is still true with P replaced by $P_{V,W} = P + W \cdot \nabla + V$, with C replaced by $C_0 \max\{1, \|W\|_{L^\infty}\}$, and uniformly for all $\mu \geq \tilde{\tau}_0 \max\{1, \|V\|_{L^\infty}^{2/3}, \|W\|_{L^\infty}^2\}$.

Proof. For simplicity, we first argue for $V = 0$, $W = 0$, and we will check the dependence on V, W at the end.

We will use the same scheme of proof as for Theorem 4.7. We first note that the notion of $\lhd$ can be extended to the case when there is a boundary and the variables ξ_a are tangential to the boundary. Then, the local uniqueness results of Corollary 4.6, and Theorem 5.12, can be written as

$$B(x^0, r) \lhd \left[\{\phi > \rho\} \cap B(x^0, 4R)\right] \tag{5.37}$$

as long as $B(x^0, 4R) \cap \{x_n = 0\} = \emptyset$. Indeed, in (5.37), the case where $B(x^0, 4R) \cap \{x_n = \ell_0\} = \emptyset$ follows from the internal quantitative uniqueness result (e.g. Corollary 4.6), whereas the case "up to the boundary" $B(x^0, 4R) \cap \{x_n = \ell_0\} \neq \emptyset$ follows from Theorem 5.12. To apply this theorem in this context, one needs to make the change of variables $x_n \mapsto \ell_0 - x_n$, which transforms $\{x_n \leq \ell_0\}$ into $\mathbb{R}^n_+$ and $\phi_\varepsilon = G(x', \varepsilon) - x_n$ to $\tilde{\phi}_\varepsilon := G(x', \varepsilon) - (\ell_0 - x_n)$. The condition $\partial_{x_n}\tilde{\phi}_\varepsilon = -\partial_{x_n}\phi_\varepsilon = 1 > 0$ is satisfied, the surface $\{x_n = 0\}$ (new coordinates) remains noncharacteristic; the pseudoconvexity assumption is invariant as well.

Claim. *For any open neighborhood $\widetilde{\omega}$ of $S_0 = \overline{D} \times \{x_n = 0\}$, there exists an open neighborhood U of K (for the topology of $\mathbb{R}^{n-1} \times [0, \ell_0]$) such that*

$$U \lhd \widetilde{\omega}.$$

The claim can be proved with almost the same proof as that of Theorem 4.7, but using in addition Theorem 5.12 instead of only Theorem 3.1. So, we have to ensure that in the proof, we only apply Theorem 5.12 for some points $x_i^{\varepsilon_j}$ with $B(x_i^{\varepsilon_j}, 4R_i^{\varepsilon_j}) \cap \{x_n = 0\} = \emptyset$. This is the point of Remark 4.10, which then allows us to prove the Claim as in Theorem 4.7.

Now, let $x^0 \in \overline{D} \times \{x_n = 0\}$. We apply Theorem 5.13 with R_x small enough that $\mathbb{R}^{n-1} \times \{x_n = 0\} \cap B(x, R_x) \subset \{x_n = 0\} \times \widetilde{D}$ and $B(x, R_x) \subset \Omega$. This gives r_x such that for some $\beta, \kappa, C, \kappa', \mu_0 > 0$,

$$\|M^{\beta\mu}_{c_1\mu}\sigma^{x^0}_{r,c_1\mu}u\|_{1,+} \leq Ce^{\kappa\mu}\left(\|D_n u_{|x_n=0}\|_{L^2(\widetilde{D})} + \|Pu\|_{L^2(\Omega)}\right) + Ce^{-\kappa'\mu}\|u\|_{1,+}.$$

where $\sigma^{x^0}_r$ is centered at x^0. By compactness of $\overline{D}$, we can cover it by a finite number of such balls $(B(x^i, r^i))_{i\in I}$. Pick $\vartheta \in C^\infty_0(\mathbb{R}^{n-1} \times [0, \ell_0])$ with $\operatorname{supp}\vartheta \subset \bigcup_{i\in I} B(x^i, r^i)$ so that $\vartheta = 1$ in a neighborhood $\widetilde{\omega}$ of S_0. Lemma 2.12 gives, for functions $\sigma^{x^i}_{r^i}$ equal to 1 on $B(x^i, r^i)$, the estimate

$$\|M^{2\beta\mu}_{\mu}\vartheta_\mu u\|_{m-1} \leq \sum_{i\in I}\|M^{\beta\mu}_{c_1\mu}\sigma^{x^i}_{r^i,c_1\mu}u\|_{1,+} + Ce^{-c\mu}\|u\|_{1,+}.$$

Now, apply the Claim with the selected $\widetilde{\omega}$ and for some $\widetilde{\vartheta} \in C^\infty_0(U \cap \mathbb{R}^{n-1} \times [0, \ell_0])$ equal to 1 in a neighborhood of K. For some $\kappa_1 < \min(c/2, \kappa')$, there exist $C_1, \kappa_1' > 0$ such that

$$\|M^{\alpha\mu}_{\mu}\widetilde{\vartheta}_\mu u\|_{1,+} \leq Ce^{\kappa_1\mu}\left(\|M^{2\beta\mu}_{\mu}\vartheta_\mu u\|_{m-1} + \|Pu\|_{L^2(\Omega)}\right) + Ce^{-\kappa_1'\mu}\|u\|_{1,+}.$$

This implies, for some $\kappa_2, \kappa_2', C > 0$,

$$\|M_\mu^{\alpha\mu}\widetilde{\vartheta}_\mu u\|_{1,+} \leq Ce^{\kappa_2\mu}\big(\|D_n u_{|x_n=0}\|_{L^2(\widetilde{D})} + \|Pu\|_{L^2(\Omega)}\big) + Ce^{-\kappa_2'\mu}\|u\|_{1,+}.$$

We finish the proof as in Theorem 1.11 once Theorem 4.7 is proved, taking into account Remark 5.1.

Now, if P is replaced by $P_{V,W}$, we want to obtain uniformity with respect to the size of V and W. It is clear that the proof of the theorem involves a finite number of applications of Theorems 5.12 and 5.13. Indeed, the scheme of proof of Theorem 4.7 only involves a finite number of applications of the geometric propagation of the property $\lhd$. They can be divided into two categories: the general ones described in Proposition 4.5 that are completely independent of the operator P (so the constants will be independent of V and W), and those using Theorems 5.12 and 5.13 where the dependence of the constants μ_0 and C is explicitly described. Note also that all properties (propagation, transitivity, simplification...) that we prove about the relations $\lhd$ and $\unlhd$ in Propositions 4.3 and 4.5 satisfy the following: once κ is fixed, the associated μ_0 provided by $\lhd$ and $\unlhd$ is always transformed into some linear combination (with universal constants) of the μ_0 corresponding to the previous ones. The same holds for the constants C involved in $\lhd$ and $\unlhd$. Finally, a finite number of applications of these rules will always conclude with the restriction of the form $\mu \geq \tilde{\tau}_0 \max\{1, \|V\|_{L^\infty}^{2/3}, \|W\|_{L^\infty}^2\}$ and C of the form $C_0 \max\{1, \|W\|_{L^\infty}\}$, once κ is fixed. ◻

6. Applications

We now give applications of the above main results, namely Theorem 1.11 and, in the case with boundary, Theorem 5.14, to the wave and Schrödinger operators. In these applications, we study an evolution equation in the analytic variable. We thus have $n_a = 1$, $n_b = n - 1 = \dim(\mathcal{M})$ and we denote accordingly by $t = x_a$ the time variable and by $x = x_b$ the space variable. In this section, we prove general versions of Theorems 1.1 and 1.5: we add (complex valued) lower order terms that are analytic in time. We also provide uniform estimates with respect to these lower order terms if they are time-independent. The proof consists each time in

- first applying the quantitative estimates of Theorem 5.14;
- then using energy estimates to relate time-space H^1 norms of the solution to the energy of the initial data and the norm of the source term.

Note that the first step, the quantitative unique continuation itself, does not involve the lower order terms. For instance, Theorem 6.7 below is equally valid for the Schrödinger operator $i\partial_t + \Delta_g$, the heat operator $\partial_t - \Delta_g$, Ginzburg–Landau operators $e^{i\theta}\partial_t + \Delta_g$, etc. The second step however uses the well-posedness properties of the evolution problem (conservation of energies...), and is not so well-adapted to dissipative equations.

6.1. The wave equation

Our result for the wave equation can be formulated as follows. We recall that the geometric constant $\mathcal{L}(\mathcal{M},\omega)$ is introduced in (1.3).

Theorem 6.1. *Let $\mathcal{M}$ be a compact Riemannian manifold with (or without) boundary, Δ_g the Laplace–Beltrami operator on $\mathcal{M}$, and*

$$P = \partial_t^2 - \Delta_g + W_0\partial_t + W_1\cdot\nabla + V$$

with $V, W_0, W_1, \operatorname{div}(W_1)$ bounded and depending analytically on the variable $t\in(-T,T)$ (see Remark 6.4).

For any nonempty open subset ω of $\mathcal{M}$ and any $T > \mathcal{L}(\mathcal{M},\omega)$, there exist $C,\kappa,\mu_0 > 0$ such that for any $(u_0,u_1)\in H_0^1(\mathcal{M})\times L^2(\mathcal{M})$, $f\in L^2((-T,T)\times\mathcal{M})$ and associated solution u of

$$\begin{cases} Pu = f & \text{in } (-T,T)\times \operatorname{Int}(\mathcal{M}),\\ u=0 & \text{in } (-T,T)\times\partial\mathcal{M},\\ (u,\partial_t u)_{|t=0} = (u_0,u_1) & \text{in } \operatorname{Int}(\mathcal{M}),\end{cases} \tag{6.1}$$

we have, for any $\mu\geq\mu_0$,

$$\begin{aligned}&\|(u_0,u_1)\|_{L^2\times H^{-1}}\\ &\quad\leq Ce^{\kappa\mu}\big(\|u\|_{L^2((-T,T);H^1(\omega))} + \|f\|_{L^2((-T,T)\times\mathcal{M})}\big) + \frac{C}{\mu}\|(u_0,u_1)\|_{H^1\times L^2}.\end{aligned} \tag{6.2}$$

If moreover $\partial\mathcal{M}=\emptyset$ and all coefficients of P are analytic in both t and x (i.e. the manifold $\mathcal{M}$, the metric g and the lower order terms W_0, W_1, V are analytic), then there exists $\tilde{\varphi}\in C_0^\infty((-T,T)\times\omega)$ such that for any $s\in\mathbb{R}$, we have

$$\begin{aligned}&\|(u_0,u_1)\|_{L^2\times H^{-1}}\\ &\quad\leq Ce^{\kappa\mu}\big(\|\tilde{\varphi}u\|_{H^{-s}((-T,T)\times\mathcal{M})} + \|f\|_{L^2((-T,T)\times\mathcal{M})}\big) + \frac{C}{\mu}\|(u_0,u_1)\|_{H^1\times L^2}.\end{aligned}$$

If $\partial\mathcal{M}\neq\emptyset$ and Γ is a nonempty open subset of $\partial\mathcal{M}$, for any $T>\mathcal{L}(\mathcal{M},\Gamma)$ there exist $C,\kappa,\mu_0>0$ such that for any $(u_0,u_1)\in H_0^1(\mathcal{M})\times L^2(\mathcal{M})$, $f\in L^2((-T,T)\times\mathcal{M})$ and associated solution u of (6.1), we have

$$\begin{aligned}&\|(u_0,u_1)\|_{L^2\times H^{-1}}\\ &\quad\leq Ce^{\kappa\mu}\big(\|\partial_\nu u\|_{L^2((-T,T)\times\Gamma)} + \|f\|_{L^2((-T,T)\times\mathcal{M})}\big) + \frac{C}{\mu}\|(u_0,u_1)\|_{H^1\times L^2}.\end{aligned} \tag{6.3}$$

Finally, if V, W_0 and W_1 are time-independent then we have the following stronger result. There exist $C_0,\kappa,\mu_0>0$ such that for any $V, W_0, W_1, \operatorname{div}(W_1)$ bounded (all independent of t), for any $(u_0,u_1)\in H_0^1(\mathcal{M})\times L^2(\mathcal{M})$, $f\in L^2((-T,T)\times\mathcal{M})$ and u the solution of (6.1), estimates (6.2) and (6.3) hold uniformly for all $\mu\geq \mu_0\max\{1,\|V\|_{L^\infty}^{2/3},\|W_0\|_{L^\infty}^2,\|W_1\|_{L^\infty}^2\}$ with constant

$$C = C_0\exp\big(C_0\max\{\|V\|_{L^\infty(\mathcal{M})},\|W_0\|_{L^\infty(\mathcal{M})},\|W_1\|_{L^\infty(\mathcal{M})},\|\operatorname{div}(W_1)\|_{L^\infty(\mathcal{M})}\}\big). \tag{6.4}$$

Remark 6.2. Using Lemma A.3 and the admissibility condition $\|\partial_\nu u\|_{L^2((-T,T)\times\Gamma)} \leq C\|(u_0,u_1)\|_{H^1\times L^2}$, we can write the previous estimates as in Corollary 1.2 with some constants depending explicitly on the norms of the lower order terms.

Note that refinements of the rough energy estimates made in the proof of Theorem 6.1 lead to improved dependences of the constant in (6.4) (see e.g. [LL16, Section 3]).

Theorem 6.1 above is a consequence of the following result, together with basic energy estimates for solutions to the wave equation.

Theorem 6.3. *Let $\mathcal{M}$ be a compact Riemannian manifold with (or without) boundary, Δ_g the Laplace–Beltrami operator on $\mathcal{M}$, and $P = \partial_t^2 - \Delta_g + R$ with $R = R(t,x,\partial_t,\partial_x)$ a differential operator of order 1 on $(-T,T)\times\mathcal{M}$, with coefficients bounded and depending analytically on the variable $t \in (-T,T)$ (see Remark 6.4 below).*

For any nonempty open subset ω of $\mathcal{M}$ and any $T > \mathcal{L}(\mathcal{M},\omega)$, there exist $\varepsilon, C, \kappa, \mu_0 > 0$ such that for any $u \in H^1((-T,T)\times\mathcal{M})$ and $f \in L^2((-T,T)\times\mathcal{M})$ solving

$$\begin{cases} Pu = f & \text{in } (-T,T)\times \mathrm{Int}(\mathcal{M}), \\ u = 0 & \text{in } (-T,T)\times\partial\mathcal{M}, \end{cases} \tag{6.5}$$

we have, for any $\mu \geq \mu_0$,

$$\|u\|_{L^2((-\varepsilon,\varepsilon)\times\mathcal{M})} \leq Ce^{\kappa\mu}\big(\|u\|_{L^2((-T,T);H^1(\omega))} + \|f\|_{L^2((-T,T)\times\mathcal{M})}\big) + \frac{C}{\mu}\|u\|_{H^1((-T,T)\times\mathcal{M})}.$$

If moreover $\mathcal{M}$, the metric g and the lower order terms R are analytic, and $\partial\mathcal{M} = \emptyset$, then there exists $\tilde{\varphi} \in C_0^\infty((-T,T)\times\omega)$ such that for any $s \in \mathbb{R}$, we have

$$\|u\|_{L^2((-\varepsilon,\varepsilon)\times\mathcal{M})} \leq Ce^{\kappa\mu}\big(\|\tilde{\varphi}u\|_{H^{-s}((-T,T)\times\mathcal{M})} + \|f\|_{L^2((-T,T)\times\mathcal{M})}\big) + \frac{C}{\mu}\|u\|_{H^1((-T,T)\times\mathcal{M})}.$$

If $\partial\mathcal{M} \neq \emptyset$ and Γ is a nonempty open subset of $\partial\mathcal{M}$, then for any $T > \mathcal{L}(\mathcal{M},\Gamma)$ there exist $\varepsilon, C, \kappa, \mu_0 > 0$ such that for any $u \in H^1((-T,T)\times\mathcal{M})$ and $f \in L^2((-T,T)\times\mathcal{M})$ solving (6.5), we have

$$\|u\|_{L^2((-\varepsilon,\varepsilon)\times\mathcal{M})} \leq Ce^{\kappa\mu}\big(\|\partial_\nu u\|_{L^2((-T,T)\times\Gamma)} + \|f\|_{L^2((-T,T)\times\mathcal{M})}\big) + \frac{C}{\mu}\|u\|_{H^1((-T,T)\times\mathcal{M})}.$$

Finally, if all lower order terms are time-independent, that is, $R = W_0\partial_t + W_1\cdot\nabla + V$ does not depend on t, then we have the following stronger result. There exist $\varepsilon, C_0, \kappa, \mu_0 > 0$ that such for any $V, W_0 \in L^\infty(\mathcal{M})$ and W_1 an L^∞ vector field on $\mathcal{M}$, for any $u \in H^1((-T,T)\times\mathcal{M})$ and $f \in L^2((-T,T)\times\mathcal{M})$ solving (6.5) all the above estimates hold uniformly for all $\mu \geq \mu_0 \max\{1, \|V\|_{L^\infty}^{2/3}, \|W_0\|_{L^\infty}^2, \|W_1\|_{L^\infty}^2\}$ and C replaced by $C_0\max\{1, \|W_0\|_{L^\infty}, \|W_1\|_{L^\infty}\}$.

Remark 6.4. In the above theorems, a function is said to be "bounded and depending analytically on the variable $t \in (-T, T)$" if it is bounded in $N \times \mathcal{M}$ where N is a complex neighborhood of $(-T, T)$, and depending analytically on the variable $t \in N$ for almost every $x \in \mathcal{M}$.

We first prove Theorem 6.3 and then conclude with the proof of Theorem 6.1.

Proof of Theorem 6.3. We only prove here the more complicated case of the boundary observation. The internal observation case is simpler and follows the same proof. To transport information from one point x^0 to another point x^1, the idea is to build nice coordinates in a neighborhood of a path between x^0 and x^1. In these coordinates, we construct an appropriate foliation in order to apply our semiglobal estimate. To construct these coordinates, we follow the presentation of Lebeau [Leb92, pp. 21–22].

We fix a point $x^1 \in \mathcal{M}$. We can find $x^0 \in \Gamma$ and a smooth path $\gamma : [0, 1] \to \mathcal{M}$ of length ℓ_0 with $\mathcal{L}(\mathcal{M}, \Gamma) < \ell_0 < T$ (see the definition of $\mathcal{L}(\mathcal{M}, \Gamma)$ in (1.3)) so that $\gamma(0) = x^0$ and $\gamma(1) = x^1$. Moreover, we can require that

$$\begin{cases} \gamma \text{ does not have self-intersections,} \\ \gamma(s) \in \operatorname{Int}(\mathcal{M}) \text{ for } s \in (0, 1), \\ \dot\gamma(0) \text{ is orthogonal to } \partial\mathcal{M}, \\ \dot\gamma(1) \text{ is orthogonal to } \partial\mathcal{M} \text{ in case } \gamma(1) = x^1 \in \partial\mathcal{M}. \end{cases}$$

According to Lemma 6.5 below, we can find local coordinates (w, x_n) near γ in which $\mathcal{M}$ is defined by $0 \le x_n \le \ell_0$, the path γ by $\gamma(s) = (0, s\ell_0)$ and the (co)metric is given by the matrix $m(w, x_n) \in M_n(\mathbb{R})$ with

$$m(w, x_n) = \begin{pmatrix} m'(x_n) & 0 \\ 0 & 1 \end{pmatrix} + O_{M_n(\mathbb{R})}(|w|) \quad \text{for } w \in B_{\mathbb{R}^{n-1}}(0, \delta),\ \delta > 0, \tag{6.6}$$

with $m'(x_n) \in M_{n-1}(\mathbb{R})$ symmetric and (uniformly) positive definite. With these coordinates in the space variable, and still using the straight time variable, the symbol of the wave operator is given by

$$p(t, w, x_n, \tau, \xi_w, \xi_n) = p(w, x_n, \tau, \xi_w, \xi_n) = -\tau^2 + \langle m(w, x_n)\xi, \xi\rangle, \qquad \xi = (\xi_w, \xi_n), \tag{6.7}$$

where we have used τ for the dual of the time variable and ξ_w, ξ_n for the duals to $w \in B_{\mathbb{R}^{n-1}}(0, \delta)$ and $x_n \in [0, \ell_0]$.

We now aim to apply Theorem 5.14. Pick again t_0 with $\ell_0 < t_0 < T$. For $b < \delta$ small, to be fixed later, we define

$$x_n = l, \qquad x' = (t, w), \qquad D = \left\{(t, w) : \left(\frac{w}{b}\right)^2 + \left(\frac{t}{t_0}\right)^2 < 1\right\},$$

$$G(t, w, \varepsilon) = \varepsilon\ell_0\theta\left(\sqrt{\left(\frac{w}{b}\right)^2 + \left(\frac{t}{t_0}\right)^2}\right), \quad \phi_\varepsilon(t, w, x_n) := G(t, w, \varepsilon) - x_n, \quad \varepsilon \in [0, 1],$$

where θ is such that, for some $\eta_0, \eta_1 > 0$,

$$\theta : [-1-\eta_0, 1+\eta_0] \to [-\eta_1, 1], \quad \text{smooth and even}, \quad \theta(\pm 1) = 0, \quad \theta(0) = 1,$$
$$\theta(s) \geq 0 \text{ if and only if } s \in [-1, 1], \quad |\theta'| \leq \alpha \text{ on } [-1-\eta_0, 1+\eta_0],$$

with $1 < \alpha < t_0/\ell_0$. This is possible since $t_0/\ell_0 > 1$.

Note also that the point $(t = 0, w = 0, x_n = \ell_0)$ corresponding in the local coordinates to x^1 belongs to $\{\phi_1 = 0\}$. We have

$$\begin{aligned} &d\phi_\varepsilon(t, w, x_n) \\ &\quad = \varepsilon \ell_0 \left(\left(\frac{w}{b}\right)^2 + \left(\frac{t}{t_0}\right)^2 \right)^{-1/2} \theta'\left(\sqrt{\left(\frac{w}{b}\right)^2 + \left(\frac{t}{t_0}\right)^2} \right) \left(\frac{t\,dt}{t_0^2} + \frac{w\,dw}{b^2} \right) - dx_n. \end{aligned}$$

Given the form of the principal symbol of the wave operator in these coordinates (see (6.6)–(6.7)), we obtain

$$\begin{aligned} p(w, x_n, d\phi_\varepsilon(t, w, x_n)) = &-\varepsilon^2 \ell_0^2 \frac{t^2}{t_0^4} \left(\left(\frac{w}{b}\right)^2 + \left(\frac{t}{t_0}\right)^2 \right)^{-1} |\theta'|^2 \\ &+ \ell_0^2 \frac{\varepsilon^2}{b^4} \langle m'(x_n) w, w \rangle \left(\left(\frac{w}{b}\right)^2 + \left(\frac{t}{t_0}\right)^2 \right)^{-1} |\theta'|^2 + 1 \\ &+ O(|w|^2) \left(1 + \frac{\varepsilon^2 \ell_0^2}{b^4} |w|^2 \left(\left(\frac{w}{b}\right)^2 + \left(\frac{t}{t_0}\right)^2 \right)^{-1} |\theta'|^2 \right), \end{aligned}$$

where $|\theta'|^2$ is taken at the point $\sqrt{(w/b)^2 + (t/t_0)^2}$. Now, since $\alpha < t_0/\ell_0$ and $m'(x_n)$ is uniformly (for $x_n \in [0, \ell_0]$) positive definite, there is $\eta > 0$ such that for $|w| \leq b$ small enough, we have

$$1 + O(|w|^2) \geq \alpha^2 \frac{\ell_0^2}{t_0^2} + \eta,$$
$$\langle m'(x_n) w, w \rangle + O(|w|^2)|w|^2 \geq \tfrac{1}{2} \langle m'(x_n) w, w \rangle \geq 0.$$

Hence, there is a sufficiently small neighborhood (taking again b small enough) of the path (i.e. of $w = 0$), in which we have, for any $\varepsilon \in [0, 1]$ and any $(t, w, x_n) \in \overline{D} \times [0, \ell_0]$,

$$\begin{aligned} p(w, x_n, d\phi_\varepsilon(t, w, x_n)) &\geq -\frac{\varepsilon^2}{t_0^2} \ell_0^2 \left(\frac{t}{t_0}\right)^2 \left(\left(\frac{w}{b}\right)^2 + \left(\frac{t}{t_0}\right)^2 \right)^{-1} |\theta'|^2 + \alpha^2 \frac{\ell_0^2}{t_0^2} + \eta \\ &\geq -\frac{\ell_0^2}{t_0^2} |\theta'|^2 + \alpha^2 \frac{\ell_0^2}{t_0^2} + \eta \geq \eta. \end{aligned}$$

So, the surface $\{\phi_\varepsilon = 0\}$ is noncharacteristic for any $\varepsilon \in [0, 1]$, and therefore strongly pseudoconvex with respect to the wave operator (see Remark 1.10).

Moreover, since b can be chosen arbitrarily small and Γ open with $x^0 \in \Gamma$, we can select b small enough so that in the chosen coordinates, we have $D \subset [-t_0, t_0] \times \Gamma$.

Therefore, applying Theorem 5.14 (in case $x^1 \in \partial\mathcal{M}$; see the remark preceding Theorem 5.14 in case $x^1 \notin \partial\mathcal{M}$) in the chosen coordinates and writing (with a slight abuse of notation) the final result in an invariant way, we get

$$\|u\|_{L^2(U)} \leq Ce^{\kappa\mu}\big(\|\partial_\nu u\|_{L^2((-T,T)\times\Gamma)} + \|Pu\|_{L^2((-T,T)\times\mathcal{M})}\big) + \frac{C}{\mu}\|u\|_{H^1((-T,T)\times\mathcal{M})}, \tag{6.8}$$

where U is a neighborhood (in the local coordinates) of $\{\phi_1 = 0\}$ and in particular a neighborhood of x^1 (in the global coordinates). Note that we actually apply the theorem to χu with $\chi \in C^\infty((-T,T)\times\mathcal{M})$ so that in the coordinate charts, $\chi u \in C_0^\infty([0,\ell_0]\times\mathbb{R}^{n-1})$ and $\chi = 1$ on a neighborhood of the Ω defined in Theorem 5.14. We have therefore $\|P\chi u\|_{L^2(\Omega)} = \|Pu\|_{L^2(\Omega)} \leq C\|Pu\|_{L^2((-T,T)\times\mathcal{M})}$ and $\|\chi u\|_{H^1([0,\ell_0]\times\mathbb{R}^{n-1})} \leq \|u\|_{H^1((-T,T)\times\mathcal{M})}$ (where we have switched from some coordinate set to another with a slight abuse of notation).

Since the previous property is true for any $x^1 \in \mathcal{M}$, we deduce by compactness (taking the worst of all the constants κ, C, μ_0), using this estimate only a finite number of times, that there exists $\varepsilon > 0$ such that

$$\begin{aligned}&\|u\|_{L^2((-\varepsilon,\varepsilon)\times\mathcal{M})}\\ &\quad\leq Ce^{\kappa\mu}\big(\|\partial_\nu u\|_{L^2((-T,T)\times\Gamma)} + \|Pu\|_{L^2((-T,T)\times\mathcal{M})}\big) + \frac{C}{\mu}\|u\|_{H^1((-T,T)\times\mathcal{M})}.\end{aligned}$$

This concludes the proof of the theorem in the general (boundary) case.

For the last analytic case, we apply the same reasoning as before using the case $n_a = n$ of Theorem 1.11 and taking care about having some analytic change of coordinates. For instance, we need to have an analytic path. So, this leads to an observation term $\|\varphi u\|_{H^{-s}}$ where $\varphi = 1$ on all the cutoff functions obtained by the theorem.

The lower order terms depending analytically on time are treated using Corollary 5.4 and Remark 3.8.

The uniform dependence with respect to time-independent lower order terms follows from the fact that we only use Theorem 5.14 a finite number of times. □

With Theorem 6.3, we now conclude the proof of Theorem 6.1, using energy estimates to relate $\|(u_0,u_1)\|_{H_0^1\times L^2(\mathcal{M})}$ to $\|u\|_{H^1((-T,T)\times\mathcal{M})}$, and $\|(u_0,u_1)\|_{L^2\times H^{-1}(\mathcal{M})}$ to $\|u\|_{L^2((-T,T)\times\mathcal{M})}$. These estimates are very classical in the selfadjoint case (which we omit here) and need a little care in the general case. They can be refined in various ways (see e.g. [LL16, Section 3]).

Proof of Theorem 6.1. We consider a perturbation of order 1, $R(t,x,\partial_t,\partial_x)u = V(t,x)u + W_0(t,x)\partial_t u + W_1(t,x)\cdot\nabla u$, and perform the energy estimates. For $s \in [-T,T]$, we have the pointwise-in-time estimate

$$\|R(s)u(s)\|_{L^2} \leq C_R\big(\|u(s)\|_{H^1(\mathcal{M})} + \|\partial_t u(s)\|_{L^2(\mathcal{M})}\big)$$

with

$$C_R = \|V\|_{L^\infty([-T,T]\times\mathcal{M})} + \|W_0\|_{L^\infty([-T,T]\times\mathcal{M})} + \|W_1\|_{L^\infty([-T,T]\times\mathcal{M})}.$$

Using the Duhamel formula and the Gronwall lemma gives

$$\|(u,\partial_t u)(t)\|_{H^1\times L^2(\mathcal{M})} \leq Ce^{CC_R}\big(\|(u_0,u_1)\|_{H^1\times L^2(\mathcal{M})} + \|f\|_{L^1([-T,T];L^2(\mathcal{M}))}\big),$$

and in particular, after integrating in time,

$$\|u\|_{H^1((-T,T)\times\mathcal{M})} \leq Ce^{CC_R}\big(\|(u_0,u_1)\|_{H^1\times L^2(\mathcal{M})} + \|f\|_{L^1([-T,T];L^2(\mathcal{M}))}\big). \tag{6.9}$$

Let $R^*(t,x,\partial_t,D_x)u = V(t,x)u - \partial_t(W_0(t,x)u) - \operatorname{div}(W_1(t,x)u)$ be the formal (space-time) adjoint of R (we take the real duality for simplicity).

If $(v_0,v_1)\in H^1\times L^2$, let v be the associated solution of $\Box v + R^*v = 0$. We have

$$\|R^*(s)v(s)\|_{L^2} \leq C_{R^*}\big(\|v(s)\|_{H^1(\mathcal{M})} + \|\partial_t v(s)\|_{L^2(\mathcal{M})}\big)$$

for $s\in[0,\varepsilon]$, with

$$\begin{aligned} C_{R^*} &= \|V\|_{L^\infty([0,\varepsilon]\times\mathcal{M})} + \|W_0\|_{W^{1,\infty}([0,\varepsilon];L^\infty(\mathcal{M}))} + \|W_1\|_{L^\infty([0,\varepsilon]\times\mathcal{M})} \\ &\quad + \|\operatorname{div}(W_1)\|_{L^\infty([0,\varepsilon]\times\mathcal{M})}. \end{aligned}$$

Similar energy estimates applied to v give

$$\|v\|_{H^1((0,\varepsilon)\times\mathcal{M})} \leq Ce^{C\varepsilon C_{R^*}}\|(v_0,v_1)\|_{H^1\times L^2(\mathcal{M})}. \tag{6.10}$$

We now choose $\chi\in C^\infty([0,\varepsilon])$ such that $\chi(0)=1$, $\dot\chi(0)=0$, $\chi(\varepsilon)=0$, and $\dot\chi(\varepsilon)=0$. Then $w=\chi(t)v$ is the solution of

$$\begin{cases} \Box w + R^*w = 2\dot\chi(t)\partial_t v + \dot\chi(t)W_0 v + \ddot\chi(t)v =: g,\\ w_{|\partial\mathcal{M}} = 0,\\ (w,\partial_t w)_{|t=0} = (v_0,v_1). \end{cases}$$

Thus, g is a (trivial) control that drives (v_0,v_1) to zero, i.e. $(w,\partial_t w)_{|t=\varepsilon}=(0,0)$, with, according to (6.10), $\|g\|_{L^2((0,\varepsilon)\times\mathcal{M})} \leq Ce^{CC_{R^*}}\|(v_0,v_1)\|_{H^1\times L^2}$. So, the usual computation yields, after integrating by parts,

$$\begin{aligned} \int_{(0,\varepsilon)\times\mathcal{M}} ug &= \int_{(0,\varepsilon)\times\mathcal{M}} u(\Box + R^*)w \\ &= \int_{\mathcal{M}} u_1v_0 - \int_{\mathcal{M}} u_0v_1 - \int_{\mathcal{M}} W_0(0,x)u_0v_0 + \int_{(0,\varepsilon)\times\mathcal{M}} fw, \end{aligned}$$

and in particular

$$\begin{aligned} &\langle (u_0,u_1),(-v_1,v_0)\rangle \\ &\quad \leq C\|u\|_{L^2((0,\varepsilon)\times\mathcal{M})}\|g\|_{L^2((0,\varepsilon)\times\mathcal{M})} + C\|f\|_{L^2((0,\varepsilon)\times\mathcal{M})}\|w\|_{L^2((0,\varepsilon)\times\mathcal{M})} \\ &\quad \leq Ce^{CR^*}\|(v_0,v_1)\|_{H^1\times L^2}\big(\|u\|_{L^2((0,\varepsilon)\times\mathcal{M})} + \|f\|_{L^2((0,\varepsilon)\times\mathcal{M})}\big), \end{aligned}$$

where $\langle\cdot,\cdot\rangle$ is the twisted duality $\langle (u_0,u_1),(v_1,v_0)\rangle_{L^2\times H^{-1},L^2\times H^1} = \int_{\mathcal{M}} u_1v_0 - \int_{\mathcal{M}} u_0v_1 - \int_{\mathcal{M}} W_0(0,x)u_0v_0$.

By specifying to $v_0 = 0$ and $\|v_1\|_{L^2} = 1$, this gives first by duality

$$\|u_0\|_{L^2} = \sup_{\|v_1\|_{L^2}=1} \int_{\mathcal{M}} u_0 v_1 \le C e^{C_R^*}\big(\|u\|_{L^2((0,\varepsilon)\times\mathcal{M})} + \|f\|_{L^2((0,\varepsilon)\times\mathcal{M})}\big).$$

Then, with $v_1 = 0$ and $\|v_0\|_{H^1} = 1$, we obtain

$$\begin{aligned}
\|u_1\|_{H^{-1}} &= \sup_{\|v_0\|_{H^1}=1} \int_{\mathcal{M}} u_1 v_0 \\
&\le \sup_{\|v_0\|_{H^1}=1} \int_{\mathcal{M}} \Big(u_1 v_0 - \int_{\mathcal{M}} W_0(0,x) u_0 v_0 + \int_{\mathcal{M}} W_0(0,x) u_0 v_0 \Big) \\
&\le \sup_{\|v_0\|_{H^1}=1} \langle (u_0,u_1),(0,v_0)\rangle_{L^2\times H^{-1},L^2\times H^1} + \sup_{\|v_0\|_{H^1}=1} \int_{\mathcal{M}} W_0(0,x) u_0 v_0 \\
&\le C e^{C_R^*}\big(\|u\|_{L^2((0,\varepsilon)\times\mathcal{M})} + \|f\|_{L^2((0,\varepsilon)\times\mathcal{M})}\big) + C\|W_0\|_{L^\infty}\|u_0\|_{L^2}.
\end{aligned}$$

So, finally, we have

$$\|(u_0,u_1)\|_{L^2\times H^{-1}} \le C e^{C_R^*}\big(\|u\|_{L^2((0,\varepsilon)\times\mathcal{M})} + \|f\|_{L^2((0,\varepsilon)\times\mathcal{M})}\big). \tag{6.11}$$

In the particular case where the perturbation is independent of time, we have

$$C_R + C_{R^*} \le C \max\{\|V\|_{L^\infty(\mathcal{M})}, \|W_0\|_{L^\infty(\mathcal{M})}, \|W_1\|_{L^\infty(\mathcal{M})}, \|\mathrm{div}(W_1)\|_{L^\infty(\mathcal{M})}\}.$$

The combination of Theorem 6.3 with (6.9) and (6.11) gives the sought result. □

We now give a brief proof of Theorem 1.4 (so-called "penetration into shadow for waves"), which is very close to that of Theorem 6.3.

Proof of Theorem 1.4. Following exactly the same proof as for Theorem 6.3 but stopping at estimate (6.8) and using the internal observation instead, we find that for any $x_1 \in \omega_1$, there exist $\varepsilon > 0$ and C, κ, μ_0 such that

$$\|u\|_{L^2((-\varepsilon,\varepsilon)\times B(x_1,\varepsilon))} \le C e^{\kappa\mu}\|u\|_{L^2((-T,T);H^1(\omega_0))} + \frac{C}{\mu}\|u\|_{H^1((-T,T)\times\mathcal{M})}, \tag{6.12}$$

uniformly for $\mu \ge \mu_0$. Since ω_1 is compact, we can cover it by a finite number of such balls, $\omega_1 \Subset \bigcup_{i=1}^N B(x_i,\varepsilon_i)$. In particular, we can find $\widetilde{\varepsilon} > 0$ small such that $\widetilde{\varepsilon} < \varepsilon_i$ for any $1 \le i \le N$ and $\mathrm{Nhd}(\omega_1,\widetilde{\varepsilon}) \subset \bigcup_{i=1}^N B(x_i,\varepsilon_i)$. This gives $\|u\|_{L^2((-\widetilde{\varepsilon},\widetilde{\varepsilon})\times \mathrm{Nhd}(\omega_1,\widetilde{\varepsilon}))} \le C\sum_{i=1}^N \|u\|_{L^2((-\varepsilon_i,\varepsilon_i)\times B(x_i,\varepsilon_i))}$.

Note then that the wave equation with Dirichlet boundary conditions is well-posed under the assumptions of Theorem 1.4 (that the operator $(-\Delta_g$, Dirichlet$)$ is essentially selfadjoint on $L^2(\mathcal{M})$ follows e.g. from an adaptation of [Str83]). This allows one to perform energy estimates as in the compact case. Hence, since (u_0,u_1) are supported in ω_1, the finite speed of propagation implies that $u(t)$ is supported in $\mathrm{Nhd}(\omega_1,\widetilde{\varepsilon})$ for $|t| < \widetilde{\varepsilon}$ (where we use the distance coming from the Riemannian metric to define balls).

That is, $\|u\|_{L^2((-\widetilde{\varepsilon},\widetilde{\varepsilon})\times \mathrm{Nhd}(\omega_1,\widetilde{\varepsilon}))} = \|u\|_{L^2((-\widetilde{\varepsilon},\widetilde{\varepsilon})\times \mathcal{M})}$. Now, we conclude as before using the inequalities $\|u\|_{L^2((-\widetilde{\varepsilon},\widetilde{\varepsilon})\times \mathcal{M})} \geq C\|(u_0,u_1)\|_{L^2\times H^{-1}}$ and $\|u\|_{H^1((-T,T)\times\mathcal{M})} \leq C\|(u_0,u_1)\|_{H^1\times L^2}$ which only rely on energy estimates and duality. □

The following lemma is contained in [Leb92, p. 22] (see also [ABB12, Lemma 11.38, p. 221]. We give the proof for completeness.

Lemma 6.5. *Let $\gamma : [0,1] \to \mathcal{M}$ be a smooth path without self-intersections, of length ℓ_0, such that*

$$\begin{cases} \gamma(s) \in \mathrm{Int}(\mathcal{M}) \text{ for } s \in (0,1), \\ \gamma(0) \text{ and } \gamma(1) \text{ belong to } \partial\mathcal{M}, \\ \dot\gamma(0) \text{ and } \dot\gamma(1) \text{ are orthogonal to } \partial\mathcal{M}. \end{cases}$$

Then there are some coordinates $(w,l) \in B_{\mathbb{R}^{n-1}}(0,\varepsilon) \times [0,\ell_0]$ in an open neighborhood U near $\gamma([0,1])$ such that

- $\gamma([0,1]) = \{w=0\} \times [0,\ell_0]$,
- *the metric g is of the form* $m(l,w) = \begin{pmatrix} 1 & 0 \\ 0 & m'(l)\end{pmatrix} + O_{M_n(\mathbb{R})}(|w|)$,
- *in coordinates, we have $\mathcal{M} \cap U = B_{\mathbb{R}^{n-1}}(0,\varepsilon) \times [0,\ell_0]$ for some $\varepsilon > 0$.*

Proof. The path γ is of length ℓ_0, so we can reparametrize it by $\gamma : [0,\ell_0] \to \mathcal{M}$ such that γ is unitary (that is, $g_{\gamma(s)}(\dot\gamma(s),\dot\gamma(s)) = 1$ for all $s \in [0,\ell_0]$). Moreover, since γ does not have self-intersections, there exist a neighborhood U (in the topology of $\mathcal{M}$) of γ and a diffeomorphism Ψ (in the structure of $\mathcal{M}$) such that

- $\Psi(U) \subset \{(x,y) \in \mathbb{R}^n : x \in [-\varepsilon, \ell_0+\varepsilon],\ |y| \leq \varepsilon\}$,
- $\Psi(\gamma(s)) = (s,0)$,
- $\Psi(U) = \{(x,y) \in \mathbb{R}^n : f_1(y) \leq x \leq f_2(y),\ x \in [-\varepsilon,\ell_0+\varepsilon], |y| \leq \varepsilon\}$ for some smooth functions f_i locally defined.

Up to making the change of variable $(x,y) \mapsto (x - f_1(y), y)$, we can moreover require $f_1 = 0$ and change f_2 to $f_2 - f_1$.

Then, we make some change of variable to diagonalize the metric on γ. By unitarity of the coordinates, the metric on γ has the form

$$m(x,0) = \begin{pmatrix} 1 & l(x) \\ {}^t l(x) & G(x) \end{pmatrix},$$

where l is a row vector and G is a positive definite matrix. We perform the change of variable $\Phi : (x,y) \mapsto (\widetilde{x},\widetilde{y}) = (x - a(x)\cdot y, y)$. In $y=0$, we have $D\Phi(x,0) = \begin{pmatrix} 1 & -a(x) \\ 0 & \mathrm{Id}\end{pmatrix}$ with ${}^t D\Phi(x,0) = \begin{pmatrix} 1 & 0 \\ -{}^t a(x) & \mathrm{Id}\end{pmatrix}$ (in particular, the change of variable is valid for small y) and $D\Phi(x,0)^{-1} = \begin{pmatrix} 1 & a(x) \\ 0 & \mathrm{Id}\end{pmatrix}$ with ${}^t D\Phi(x,0)^{-1} = \begin{pmatrix} 1 & 0 \\ {}^t a(x) & \mathrm{Id}\end{pmatrix}$. Moreover, in the new coordinates, the set is $\{\widetilde{y} = 0\}$ and the metric there is given by

$${}^t D\Phi(x,0)^{-1} m(x,0) D\Phi(x,0)^{-1} = \begin{pmatrix} 1 & l(x) + a(x) \\ {}^t l(x) + {}^t a(x) & * \end{pmatrix}.$$

So, we choose $a(x) = -l(x)$ so that in these new coordinates

$$m(x, 0) = \begin{pmatrix} 1 & 0 \\ 0 & * \end{pmatrix}. \tag{6.13}$$

We notice that since $\dot\gamma(0)$ is orthogonal to ∂M which is defined locally by $\{x = 0\}$, we have $l(0) = 0$ (since $\dot\gamma(0) = (1, 0)$, this implies ${}^t(0, y)m(0, 0)\dot\gamma(0) = {}^t l(0)y$ for all y). In particular, Φ restricted to $\{x = 0\}$ is the identity.

This implies that in these new coordinates, $\mathcal{M}$ is still defined near γ by $0 \le x \le f_2(y)$ (now, we still write (x, y) for $(\tilde x, \tilde y)$). We still have $f_2(0) = \ell_0$. Morever, since $\dot\gamma(\ell_0) = (1, 0)$ is orthogonal to $\partial\mathcal{M}$ which is defined locally by $\{x = f_2(y)\}$, and using the fact that $m(x, 0)$ is of the form (6.13), we get $df_2(0) = 0$.

Finally, making the change of variable $(x, y) \mapsto (\ell_0/f_2(y)x, y)$, which is the identity on γ, we see that $\mathcal{M}$ is given by $0 \le x \le \ell_0$. Moreover, since $df_2(0) = 0$, the metric is unchanged on γ.

The expected property of m is then obtained by the mean value theorem using the diagonal form (6.13) on γ. □

6.2. The Schrödinger equation

Now, we turn to the Schrödinger equation. The results are quite similar to those for the wave equation except for two facts.

The first one is that there is no minimal time. This is quite natural with the infinite speed of propagation. In the proof, this appears in the fact that the principal symbol of the Schrödinger operator $i\partial_t + \Delta_g$ is $|\xi|_g^2$. Therefore, the hypersurface $\{\varphi(t, x) = 0\}$ is noncharacteristic if $\nabla_x\varphi \neq 0$, without any assumption on the time derivative.

The second difference is that the remainder term involving the $H^1((-T, T) \times \mathcal{M})$ norm involves some derivatives in time and space which do not have the same weight. Hence, since $\partial_t u = i\Delta_g u$, this term will actually count for two derivatives in space.

Theorem 6.6. *Let $\mathcal{M}$ be a compact Riemannian manifold with (or without) boundary, Δ_g the Laplace–Beltrami operator on $\mathcal{M}$, and*

$$P = i\partial_t + \Delta_g + V,$$

with V bounded and depending analytically on the variable $t \in (-T, T)$ (see Remark 6.4). Assume moreover that $V \in L^\infty((-T, T); W^{2,\infty}(\mathcal{M}))$.

For any nonempty open subset ω of $\mathcal{M}$ and any $T > 0$, there exist $C, \kappa, \mu_0 > 0$ such that for any $u_0 \in H^2 \cap H_0^1(\mathcal{M})$, $f \in L^2((-T, T); H^2(\mathcal{M}))$ and u the solution of

$$\begin{cases} i\partial_t u + \Delta_g u + Vu = f & \text{in } (-T, T) \times \operatorname{Int}(\mathcal{M}), \\ u = 0 & \text{in } (T, T) \times \partial\mathcal{M}, \\ u(0) = u_0 & \text{in } \operatorname{Int}(\mathcal{M}), \end{cases} \tag{6.14}$$

we have, for any $\mu \ge \mu_0$,

$$\|u_0\|_{L^2} \le Ce^{\kappa\mu}\big(\|u\|_{L^2((-T,T);H^1(\omega))} + \|f\|_{L^2((-T,T);H^2(\mathcal{M}))}\big) + \frac{C}{\mu}\|u_0\|_{H^2}. \tag{6.15}$$

If moreover $\partial\mathcal{M} = \emptyset$ and all coefficients of P are analytic in both t and x (i.e. the manifold $\mathcal{M}$, the metric g and the lower order terms W_0, W_1, V are analytic), then there exists $\tilde{\varphi} \in C_0^\infty((-T, T) \times \omega)$ such that for any $s \in \mathbb{R}$, we have

$$\|u_0\|_{L^2} \leq Ce^{\kappa\mu}\big(\|\varphi u\|_{H^{-s}((-T,T)\times\mathcal{M})} + \|f\|_{L^2((-T,T);H^2(\mathcal{M}))}\big) + \frac{C}{\mu}\|u_0\|_{H^2}. \quad (6.16)$$

If $\partial\mathcal{M} \neq \emptyset$ and Γ is a nonempty open subset of $\partial\mathcal{M}$, then for any $T > 0$, there exist $C, \kappa, \mu_0 > 0$ such that for any $u_0 \in H^2 \cap H_0^1(\mathcal{M})$, and u the solution of (6.14), we have, for any $\mu \geq \mu_0$,

$$\|u_0\|_{L^2} \leq Ce^{\kappa\mu}\big(\|\partial_\nu u\|_{L^2((-T,T)\times\Gamma)} + \|f\|_{L^2((-T,T);H^2(\mathcal{M}))}\big) + \frac{C}{\mu}\|u_0\|_{H^2}. \quad (6.17)$$

Finally, if V is time-independent then we have the following stronger result. There exist $C_0, \kappa, \mu_0 > 0$ such that for any V bounded, for any $u_0 \in H^2 \cap H_0^1(\mathcal{M})$, $f \in L^2((-T, T) \times \mathcal{M})$ and u the solution of (6.14), estimates (6.15) and (6.17) hold uniformly for all $\mu \geq \mu_0 \max\{1, \|V\|_{L^\infty}^{2/3}\}$ with constant

$$C = C_0 \exp(C_0\|V\|_{W^{2,\infty}(\mathcal{M})}).$$

As in the case of the wave equation, the above theorem is a combination of the theorem below and energy estimates for the Schrödinger equation.

Theorem 6.7. *Let $\mathcal{M}$ be a compact Riemannian manifold with (or without) boundary, Δ_g the Laplace–Beltrami operator on $\mathcal{M}$, and $P = \Delta_g + R$ with $R = R(t, x, \partial_t, \partial_x)$ is a differential operator of order 1 on $(-T, T) \times \mathcal{M}$, with coefficients bounded and depending analytically on the variable $t \in (-T, T)$ (see Remark 6.4).*

For any nonempty open subset ω of $\mathcal{M}$ and any $T > 0$, there exist $\varepsilon, C, \kappa, \mu_0 > 0$ such that for any $u \in H^1((-T, T) \times \mathcal{M})$ and $f \in L^2((-T, T) \times \mathcal{M})$ solving

$$\begin{cases} Pu = f & \text{in } (-T, T) \times \mathrm{Int}(\mathcal{M}), \\ u = 0 & \text{in } (-T, T) \times \partial\mathcal{M}, \end{cases} \quad (6.18)$$

the same three estimates as in Theorem 6.3 hold.

In the case that $R = W_0\partial_t + W_1 \cdot \nabla + V$ does not depend on t, the dependence on the size of the coefficients of R remains the same as in Theorem 6.3.

Proof. The proof is quite similar to the one for the wave equation, so we only sketch the main steps. The main difference will be that T can be chosen arbitrary. Pick t_0 arbitrary with $t_0 < T$, this time without any relation to ℓ_0.

We use the same coordinate charts as defined in the proof of Theorem 6.1 for the wave equation. Then the principal symbol of the Schrödinger operator will be

$$p(w, x_n, \tau, \xi_w, \xi_n) = -\langle m(w, x_n)\xi, \xi\rangle, \qquad \xi = (\xi_w, \xi_n).$$

Therefore, p is a quadratic form with real coefficients that is definite on the set $\{\tau = 0\}$. Remark 1.10 implies that any noncharacteristic hypersurface is strongly pseudoconvex. So, with the same definition of ϕ_ε, we obtain

$$\begin{aligned} p(w, x_n, d\phi_\varepsilon(t, w, x_n)) = -\ell_0^2 \frac{\varepsilon^2}{b^4} \langle m'(x_n) w, w\rangle \left(\left(\frac{w}{b}\right)^2 + \left(\frac{t}{t_0}\right)^2 \right)^{-1} |\theta'|^2 - 1 \\ + O(|w|^2)\left(1 + \frac{\varepsilon^2 \ell_0^2}{b^4} |w|^2 \left(\left(\frac{w}{b}\right)^2 + \left(\frac{t}{t_0}\right)^2 \right)^{-1} |\theta'|^2 \right). \end{aligned}$$

But, for w small enough, we still have

$$-1 + O(|w|^2) \le -1/2, \qquad -\langle m'(x_n) w, w\rangle + O(|w|^2)|w|^2 \le 0.$$

In particular, with the same notations as for the wave equation, there exists b small enough such that for any $\varepsilon \in [0, 1]$, and any $(t, w, x_n) \in \overline{D} \times [0, \ell_0]$, we have

$$p(w, x_n, d\phi_\varepsilon(t, w, x_n)) \le -1/2.$$

So, applying the same reasoning as for the wave equation, we obtain the existence of some $\kappa, C, \mu_0, \eta > 0$ such that

$$\|u\|_{L^2((-\eta,\eta)\times\mathcal{M})} \le C e^{\kappa\mu} \|\partial_\nu u\|_{L^2((-T,T)\times\Gamma)} + \frac{C}{\mu} \|u\|_{H^1((-T,T)\times\mathcal{M})}$$

for any $\mu \ge \mu_0$.

The dependence on the lower order term R follows as for the wave equation. □

Proof of Theorem 6.6. Since multiplication by V acts on H_0^1 and H^2 if $V \in W^{2,\infty}(\mathcal{M})$, using the Duhamel formula and a Gronwall argument yields, for $s \in [-T, T]$,

$$\begin{aligned} \|u_0\|_{L^2(\mathcal{M})} &\le C e^{C\|V\|_{L^\infty(\mathcal{M})}} \big(\|u(s)\|_{L^2(\mathcal{M})} + \|f\|_{L^2((-T,T)\times\mathcal{M})} \big), \\ \|u(s)\|_{H^2(\mathcal{M})} &\le C e^{C\|V\|_{W^{2,\infty}(\mathcal{M})}} \big(\|u_0\|_{H^2} + \|f\|_{L^2((-T,T);H^2(\mathcal{M}))} \big). \end{aligned}$$

Integrating in time gives

$$\begin{aligned} \|u_0\|_{L^2(\mathcal{M})} &\le C e^{C\|V\|_{L^\infty(\mathcal{M})}} \big(\|u\|_{L^2((-\varepsilon,\varepsilon)\times\mathcal{M})} + \|f\|_{L^2((-T,T)\times\mathcal{M})} \big) \\ \|u\|_{L^2((-T,T);H^2(\mathcal{M}))} &\le C e^{C\|V\|_{W^{2,\infty}}} \big(\|u_0\|_{H^2} + \|f\|_{L^2((-T,T);H^2(\mathcal{M}))} \big). \end{aligned}$$

To estimate $\partial_t u$, we notice that $\partial_t u = i(\Delta + V)u - if$. Therefore, we only need to estimate $\|\Delta_g u\|_{L^2}$. We have

$$\begin{aligned} \|\partial_t u\|_{L^2((-T,T)\times\mathcal{M})} &\le C\|u\|_{L^2((-T,T);H^2)} + C\|V\|_{L^\infty(\mathcal{M})} \|u\|_{L^2((-T,T)\times\mathcal{M})} \\ &\quad + \|f\|_{L^2((-T,T)\times\mathcal{M})} \\ &\le C e^{C\|V\|_{W^{2,\infty}(\mathcal{M})}} \big(\|u_0\|_{H^2} + \|f\|_{L^2((-T,T);H^2(\mathcal{M}))} \big). \end{aligned}$$

Hence,

$$\|u\|_{H^1((-T,T)\times\mathcal{M})} \le C e^{C\|V\|_{W^{2,\infty}(\mathcal{M})}} \big(\|u_0\|_{H^2} + \|f\|_{L^2((-T,T);H^2(\mathcal{M}))} \big).$$

When combined with Theorem 6.7, this gives the estimates of the theorem. □

Appendix A. Two elementary technical lemmata

In the above proof, we used the following elementary lemma (see e.g. [LRL12]).

Lemma A.1. *Let K be a compact set and f, g, h three continuous real valued functions on K. Assume that $f \geq 0$ on K, and $g > 0$ on $\{f = 0\}$. Then there exist $A_0, C > 0$ such that for all $A \geq A_0$, we have $g + Af - \frac{1}{A}h \geq C$ on K.*

Lemma A.1 is a consequence of the following variant.

Lemma A.2. *Let K be a compact set and f a continuous real valued function on K. Let g and h be two bounded functions defined on K. Assume that $f \geq 0$ on K, and there exists an open neighborhood V of $\{f = 0\}$ in K such that $g > c$ on V for some constant $c > 0$. Then there exist $A_0, C > 0$ such that for all $A \geq A_0$, we have $g + Af - \frac{1}{A}h \geq C$ on K.*

We also used the following classical result.

Lemma A.3. *Consider the following three assertions, for $C_1, C_2, \alpha, D_1, D_2 > 0$ and $a, b, c > 0$:*

$$b \leq C_2 c, \quad a \leq c, \quad \text{and} \quad a \leq e^{C_1\mu} b + c/\mu^\alpha \quad \text{for all } \mu \geq \mu_0, \tag{A.1}$$

$$a \leq \frac{D_1}{\log(c/b + 1)^\alpha} c, \tag{A.2}$$

$$c \leq e^{D_2 (c/a)^{1/\alpha}} b. \tag{A.3}$$

Then:

- *for any $C_1, C_2, \alpha > 0$, there exists $K \geq 1$ such that for all $\mu_0 > 0$, (A.1) implies (A.2) with $D_1 = (2C_1)^\alpha \max\{K, \mu_0^\alpha\}$;*
- *(A.2) implies (A.3) with $D_2 = D_1^{1/\alpha}$;*
- *(A.3) together with $a \leq c$ and $b \leq C_2 c$ implies (A.1) with $\mu_0 = 0$ (and all $\mu > 0$) and $C_1 = D_2$.*

Note in particular that (A.1) for some large μ_0 implies (A.1) for $\mu_0 = 0$, but with a loss in the exponent (namely C_1 replaced by $2C_1 \max\{K^{1/\alpha}, \mu_0\}$).

Proof of Lemma A.2. Let us prove the first two statements, namely (A.1)⇒(A.2)⇒(A.3) with appropriate constants. Dividing all inequalities by c, and setting $y = a/c > 0$ and $x = b/c > 0$, it suffices to prove

$$\left[x \leq C_2,\ y \leq 1,\ y \leq e^{C_1\mu} x + \mu^{-\alpha} \text{ for all } \mu \geq \mu_0\right] \Rightarrow\ y \leq \frac{D_1}{\log(1/x + 1)^\alpha} \ \Rightarrow\ \frac{1}{x} \leq e^{(D_1/y)^{1/\alpha}}.$$

Note that the second implication is straightforward since the second assertion is equivalent to $1/x \leq e^{(D_1/y)^{1/\alpha}} - 1$. To prove the first implication, we set

$$\mu(x) := \frac{1}{2C_1} \log\left(\frac{1}{x} + 1\right),$$

so that $e^{C_1\mu(x)}x = (1/x+1)^{1/2}x = (1+x)^{1/2}x^{1/2}$. Denoting now $C_3 = C_3(C_1, C_2, \alpha) = \sup_{x\le C_2}(1+x)^{1/2}x^{1/2}\mu(x)^\alpha < +\infty$, we have $e^{C_1\mu(x)}x \le C_3/\mu(x)^\alpha$. As a consequence, if $\mu(x) \ge \mu_0$, then $y \le (C_3+1)/\mu(x)^\alpha$, which is the sought estimate.

If now $\mu(x) \le \mu_0$, that is, $\frac{1}{2C_1}\log(1/x+1) \le \mu_0$, we have $1 \le \big(\frac{2C_1\mu_0}{\log(1/x+1)}\big)^\alpha$. Then the assumption $y \le 1$ directly implies $y \le \big(\frac{2C_1\mu_0}{\log(1/x+1)}\big)^\alpha$. This concludes the proof of the first two statements of the lemma for $D_1 = (2C_1)^\alpha \max\{C_3+1, \mu_0^\alpha\}$.

To prove the last statement, fix $\mu > 0$.Then either $c/a \le \mu^\alpha$, in which case, according to (A.3), $a \le c \le e^{D_2\mu}b$, or $c/a \ge \mu^\alpha$, in which case $a \le c/\mu^\alpha$. In any case, $a \le e^{D_2\mu}b + c/\mu^\alpha$, which proves (A.1). □

Appendix B. Elementary complex analysis

We recall that we identify $\mathbb{C}$ and $\mathbb{R}^2$ with $z = x + iy = (x, y)$ and denote

$$Q_1 = \{z \in \mathbb{C} : \operatorname{Re} z > 0,\ \operatorname{Im} z > 0\} = \mathbb{R}_+^* + i\mathbb{R}_+^*.$$

Lemma B.1. *Let $f_0, f_1 \in W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}_+)$ be such that $|f_0'(x)|, |f_1'(x)| \le C$ for some $C > 0$ and almost all $x \in \mathbb{R}_+$. Then the function defined for $(x, y) \in Q_1$ by*

$$\begin{aligned} f(x,y) &= \frac{4xy}{\pi}\int_0^\infty \frac{\xi f_0(\xi)}{((x-\xi)^2+y^2)((x+\xi)^2+y^2)}\,d\xi \\ &\quad + \frac{4xy}{\pi}\int_0^\infty \frac{\eta f_1(\eta)}{(x^2+(y+\eta)^2)(x^2+(y-\eta)^2)}\,d\eta \end{aligned} \tag{B.1}$$

satisfies $|f(z)| \le 2C(1+|z|)$ in $\overline{Q}_1$ together with

$$\Delta f = 0 \quad \text{in } Q_1, \qquad f(x,0) = f_0(x), \qquad f(0,y) = f_1(y), \qquad x, y \in \mathbb{R}_+^*.$$

If moreover $f_0(0) = f_1(0)$, then f is continuous on $\overline{Q}_1$.

Note that this theorem provides an existence result for the Poisson problem on Q_1 associated to Lipschitz boundary conditions. The Phragmén–Lindelöf theorem B.4 below provides an associated uniqueness result in the class of functions having a subquadratic growth at infinity.

The next lemma is a key point in the proof of the local estimate (see Section 3.3).

Lemma B.2. *Let $R, \delta, \kappa, \varepsilon, c_1 > 0$. Then there exists $d_0 = d_0(\delta, \kappa, R, \varepsilon, c_1)$ such that for any $d < d_0$, there exists $\beta_0(\delta, \kappa, R, \varepsilon, c_1, d)$ such that for any $0 < \beta < \beta_0$, the following two assertions hold:*

- *the function*

$$f_1(y) = Ry\mathbb{1}_{[0,\gamma)}(y) + \mathbb{1}_{[\gamma,+\infty)}(y)\min\{Ry, \max(-\kappa, -9\delta y, -\varepsilon/y) + c_1y^2 + \beta^2/y\}$$

  is continuous for all $\gamma \le \beta/(R+9\delta)^{1/2}$ (in the application $\gamma = \tau_0/\mu$),
- *the function f given by Lemma B.1 associated to f_1 and $f_0 = 0$ satisfies*

$$f(x,y) \le -8\delta y \quad \text{for } d/4 \le |(x,y)| \le 2d.$$

Proof of Lemma B.1. Let us first justify the form (B.1) of the solution. From the Green function $G_{\mathbb{C}}(z, z') = (2\pi)^{-1} \log |z' - z|$ in $\mathbb{C}$, we first construct a Green function in Q_1 using the so-called "image points" $\bar{z}$, $-z$ and $-\bar{z}$. This yields

$$G_{Q_1}(z, z') := \frac{1}{2\pi} \log |z' - z| - \frac{1}{2\pi} \log |z' - \bar{z}| - \frac{1}{2\pi} \log |z' + \bar{z}| + \frac{1}{2\pi} \log |z' + z|,$$

that is, with $z = (x, y)$ and $z' = (\xi, \eta)$,

$$\begin{aligned} G_{Q_1}((x, y), (\xi, \eta)) := &\frac{1}{4\pi} \log((\xi - x)^2 + (\eta - y)^2) - \frac{1}{4\pi} \log((\xi - x)^2 + (\eta + y)^2) \\ &- \frac{1}{4\pi} \log((\xi + x)^2 + (\eta - y)^2) + \frac{1}{4\pi} \log((\xi + x)^2 + (\eta + y)^2). \end{aligned}$$

For fixed $z \in Q_1$, the last three terms are smooth in $z' \in Q_1$ so that $-\Delta_{z'} G_{Q_1}(z, z') = \delta_{z'=z}$. Moreover, for $z' = (\xi, \eta) \in \partial Q_1$, either $\xi = 0$ or $\eta = 0$, so that $G_{Q_1} = 0$ for $z' \in \partial Q_1$.

Now we compute

$$\begin{aligned} \left.\frac{\partial G_{Q_1}}{\partial \xi}\right|_{\xi=0} &= -\frac{4xy}{\pi} \frac{\eta}{(x^2 + (y + \eta)^2)(x^2 + (y - \eta)^2)}, \\ \left.\frac{\partial G_{Q_1}}{\partial \eta}\right|_{\eta=0} &= -\frac{4xy}{\pi} \frac{\xi}{((x - \xi)^2 + y^2)((x + \xi)^2 + y^2)}. \end{aligned}$$

The representation formula for solutions of $\Delta f = 0$ in Q_1 and $f|_{\partial Q_1} = \tilde{f}$ reads

$$f(z) = \int_{\partial Q_1} \left.\frac{\partial G_{Q_1}}{\partial \nu_{\partial Q_1}}(z, z')\right|_{z' \in \partial Q_1} \tilde{f}(z')\, dz',$$

which justifies (B.1).

Let us now estimate for $(x, y) \in Q_1$ the term

$$\begin{aligned} &\left| \frac{4xy}{\pi} \int_0^\infty \frac{\eta f_1(\eta)}{(x^2 + (y + \eta)^2)(x^2 + (y - \eta)^2)}\, d\eta \right| \\ &\qquad \leq \frac{4xy}{\pi} \int_0^\infty \frac{\eta C(1 + \eta)}{(x^2 + (y + \eta)^2)(x^2 + (y - \eta)^2)}\, d\eta \\ &\qquad \leq C((2/\pi) \arctan(y/x) + y) \\ &\qquad \leq C(1 + y), \end{aligned}$$

where we have used Lemma B.3 in the second inequality. The other term containing f_0 can be estimated as well in Q_1 by $C(1 + x)$, so that

$$|f(z)| \leq C(2 + x + y) \leq 2C(1 + |z|), \qquad z = (x, y) \in Q_1.$$

That $\Delta f = 0$ follows from the definition of G_{Q_1} as a Green function, and it only remains to check the boundary values of f. For this, by symmetry, it suffices to prove that for all $x_0, y_0 > 0$, we have

$$\lim_{(x,y)\to(x_0,0)} (T f_1)(x, y) = 0, \qquad \lim_{(x,y)\to(0,y_0)} (T f_1)(x, y) = f_1(y_0), \tag{B.2}$$

with

$$(Tf_1)(x,y) := \frac{4xy}{\pi}\int_0^\infty \frac{\eta f_1(\eta)}{(x^2+(y+\eta)^2)(x^2+(y-\eta)^2)}\,d\eta.$$

Since $f_1' \in L^\infty(\mathbb{R}_+)$, we have

$$|f_1(\eta)| \le |f_1(0)| + \eta\|f_1'\|_{L^\infty}.$$

Hence, according to the definition of T, we obtain

$$|Tf_1| \le |f_1(0)|T(1) + \|f_1'\|_{L^\infty} T(\eta). \tag{B.3}$$

In view of Lemma B.3, this implies

$$|(Tf_1)(x,y)| \le |f_1(0)|(2/\pi)\arctan(y/x) + \|f_1'\|_{L^\infty} y,$$

and thus $(Tf_1)(x,y) \to 0$ as $(x,y) \to (x_0,0)$, which yields the first part of (B.2).

To prove the second part of (B.2), we write

$$|f_1(\eta) - f_1(y_0)| \le |\eta - y_0|\,\|f_1'\|_{L^\infty}.$$

This implies

$$\begin{aligned} |Tf_1(x,y) - (2/\pi)\arctan(y/x) f_1(y_0)| &= |Tf_1 - T(f_1(y_0))|(x,y) \\ &\le \|f_1'\|_{L^\infty} T(|\eta - y_0|)(x,y). \end{aligned} \tag{B.4}$$

We now study the term

$$\begin{aligned} T(|\eta-y_0|)(x,y) &= \frac{4xy}{\pi}\int_0^\infty \frac{\eta|\eta-y_0|}{(x^2+(y+\eta)^2)(x^2+(y-\eta)^2)}\,d\eta \\ &= \frac{4xy}{\pi}\int_0^{y_0} \frac{\eta(y_0-\eta)}{(x^2+(y+\eta)^2)(x^2+(y-\eta)^2)}\,d\eta \\ &\quad + \frac{4xy}{\pi}\int_{y_0}^\infty \frac{\eta(\eta-y_0)}{(x^2+(y+\eta)^2)(x^2+(y-\eta)^2)}\,d\eta \\ &= 2\frac{4xy}{\pi}\int_0^{y_0} \frac{\eta(y_0-\eta)}{(x^2+(y+\eta)^2)(x^2+(y-\eta)^2)}\,d\eta \\ &\quad + \frac{4xy}{\pi}\int_0^\infty \frac{\eta(\eta-y_0)}{(x^2+(y+\eta)^2)(x^2+(y-\eta)^2)}\,d\eta \\ &= 2\frac{4xy}{\pi}\int_0^{y_0} \frac{\eta(y_0-\eta)}{(x^2+(y+\eta)^2)(x^2+(y-\eta)^2)}\,d\eta + T(\eta-y_0)(x,y). \end{aligned}$$

With Lemma B.3, we have $T(\eta-y_0)(x,y) = y - (2/\pi)\arctan(y/x)y_0 \to 0$ as $(x,y) \to (0,y_0)$. Moreover, we have

$$\begin{aligned} &\frac{4xy}{\pi}\int_0^{y_0} \frac{\eta(y_0-\eta)}{(x^2+(y+\eta)^2)(x^2+(y-\eta)^2)}\,d\eta \\ &\qquad = \frac{1}{\pi}\int_0^{y_0}\left(-\frac{x(y_0-\eta)}{x^2+(y+\eta)^2} + \frac{x(y_0-\eta)}{x^2+(y-\eta)^2}\right)d\eta \end{aligned}$$

(see the proof of Lemma B.3). The term $\int_0^{y_0} \frac{x(y_0-\eta)}{x^2+(y+\eta)^2}\, d\eta$ vanishes when $(x, y) \to (0, y_0)$. Concerning the second term, we have

$$\begin{aligned}\frac{1}{\pi}\int_0^{y_0} \frac{x(y_0-\eta)}{x^2+(y-\eta)^2}\, d\eta &= \frac{1}{\pi}\int_{-y/x}^{(y_0-y)/x} (y_0-y-xs)\frac{ds}{1+s^2}\\ &= \frac{y_0-y}{\pi}\left(\arctan\left(\frac{y_0-y}{x}\right)+\arctan\left(\frac{y}{x}\right)\right) - \frac{x}{2\pi}\log\left(\frac{x^2+(y_0-y)^2}{x^2+y^2}\right),\end{aligned}$$

which vanishes when $(x, y) \to (0, y_0)$. The last three estimates prove $T(|\eta - y_0|)(x, y) \to 0$ as $(x, y) \to (0, y_0)$. In view of (B.4), this implies

$$\lim_{(x,y)\to(0,y_0)} |Tf_1(x, y) - (2/\pi)\arctan(y/x) f_1(y_0)| = 0,$$

which is the second part of (B.2).

For the continuity, by symmetry and translation by a constant, it is sufficient to prove that if $f_1(0) = 0$, then $Tf_1(x, y)$ converges to zero as (x, y) converges to zero. This is implied by (B.3). This concludes the proof of the lemma. □

Proof of Lemma B.2. Let us define

$$I_\beta := \left[\beta\sqrt{\frac{4}{\delta}}, \min\left(\frac{\delta}{4c_1}, \frac{\kappa}{9\delta}, \frac{\sqrt{\varepsilon}}{3\sqrt{\delta}}\right)\right],$$

and notice that $I_\beta \neq \emptyset$ for $\beta \leq \beta_0$ with $\beta_0 = \beta_0(\delta, \kappa, c_1, \varepsilon)$ sufficiently small. We first prove that for all $\gamma \leq \beta\sqrt{4/\delta}$, we have

$$f_1(y) = -9\delta y + c_1 y^2 + \beta^2/y \quad \text{on } I_\beta, \tag{B.5}$$

and

$$I_\beta \subset \{f_1 \leq -8.5\delta y\}, \tag{B.6}$$

and a fortiori for $\gamma \leq \beta/(R + 9\delta)^{1/2} \leq \beta\sqrt{4/\delta}$.

For this, notice that $y \in I_\beta$ implies $y \leq \delta/(4c_1)$ and $y \geq \beta\sqrt{4/\delta}$, which yields

$$-\delta y^2/2 + c_1 y^3 \leq -\delta y^2/4 \leq -\beta^2.$$

As a consequence, for $y \in I_\beta$, we have

$$-\delta y/2 + c_1 y^2 + \beta^2/y \leq 0, \quad \text{and so} \quad -9\delta y + c_1 y^2 + \beta^2/y \leq -8.5\delta y \leq 0 \leq Ry. \tag{B.7}$$

In particular, (B.5) implies (B.6). Moreover, for $y \in I_\beta$, we have $-\kappa \leq -9\delta y$ together with $-\varepsilon/y \leq -9\delta y$, so that $\max(-\kappa, -9\delta y, -\varepsilon/y) = -9\delta y$. This proves (B.5) with the help of (B.7).

Let us now check the continuity of f_1. First we remark that both

$$y \mapsto Ry \quad \text{and} \quad y \mapsto \min\{Ry, \max(-\kappa, -9\delta y, -\varepsilon/(4y)) + c_1 y^2 + \beta^2/y\}$$

are continuous. Second, we prove that both functions coincide for $y \le \gamma$, which provides the continuity of f_1. For $0 \le y \le \gamma \le \beta/((R+9\delta)^{1/2})$, we have $(9\delta + R - c_1 y)y^2 \le \beta^2$ and we obtain $Ry \le -9\delta y + c_1 y^2 + \beta^2/y$. For $\beta \le \beta_0$ we have $I_\beta \ne \emptyset$ so that $y \le \beta\sqrt{4/\delta} \le \min\big(\frac{\kappa}{9\delta}, \frac{\sqrt{\varepsilon}}{3\sqrt{\delta}}\big)$, and $\max(-\kappa, -9\delta y, -\varepsilon/y) = -9\delta y$ for $y \le \gamma$. As a consequence, we have

$$Ry = \min\big\{Ry, \max(-\kappa, -9\delta y, -\varepsilon/y) + c_1 y^2 + \beta^2/y\big\} \quad \text{for } 0 \le y \le \gamma,$$

and f_1 is continuous for all $\beta \le \beta_0$ and $\gamma \le \beta/(R+9\delta)^{1/2}$.

Since f_1 is continuous, piecewise smooth, and linear at infinity, it is globally Lipschitz. Hence, it satisfies all assumptions of Lemma B.1 (and $f_0 = 0$), so that we can define f by

$$f(x,y) = \frac{4xy}{\pi}\int_0^\infty \frac{\eta f_1(\eta)}{(x^2+(y+\eta)^2)(x^2+(y-\eta)^2)}\,d\eta.$$

Setting $\widetilde{f} = f + 8.5\delta y$, we now prove an upper bound for $\widetilde{f}$. Using the second formula of Lemma B.3, we have

$$\begin{aligned}\widetilde{f}(x,y) &= \frac{4xy}{\pi}\int_0^\infty \frac{\eta(f_1(\eta)+8.5\delta\eta)}{(x^2+(y+\eta)^2)(x^2+(y-\eta)^2)}\,d\eta \\ &= \frac{4xy}{\pi}\int_{\mathbb{R}_+\setminus I_\beta} \cdots\, d\eta + \frac{4xy}{\pi}\int_{I_\beta} \cdots\, d\eta.\end{aligned}$$

According to (B.6), we have

$$\frac{4xy}{\pi}\int_{I_\beta} \frac{\eta(f_1(\eta)+8.5\delta\eta)}{(x^2+(y+\eta)^2)(x^2+(y-\eta)^2)}\,d\eta \le 0. \tag{B.8}$$

Next, for small β, we have $\mathbb{R}_+ \setminus I_\beta = [0, D_\beta] \cup [D, +\infty]$, with $D_\beta := \beta\sqrt{4/\delta} < D := \min\big(\frac{\delta}{4c_1}, \frac{\kappa}{9\delta}, \frac{\sqrt{\varepsilon}}{3\sqrt{\delta}}\big)$. Since $f_1(y) \le Ry$, we have

$$\frac{4xy}{\pi}\int_D^\infty \cdots \le \frac{4xy}{\pi}\int_D^\infty \frac{(R+8.5\delta)\eta^2}{(x^2+(y+\eta)^2)(x^2+(y-\eta)^2)}\,d\eta.$$

If $0 \le y \le D/2$ and $\eta \ge D$, we have $(y-\eta)^2 \ge (\eta - D/2)^2$ and $(y+\eta)^2 \ge \eta^2$, so

$$\frac{4xy}{\pi}\int_D^\infty \cdots \le \frac{16xy}{\pi}\int_D^\infty \frac{(R+8.5\delta)\eta^2}{\eta^2(\eta-D/2)^2}\,d\eta = C(\delta,\kappa,R,\varepsilon,c_1)xy.$$

Hence, if $x \le \nu D$ and $y \le D/2$, this implies

$$\frac{4xy}{\pi}\int_D^\infty \cdots \le \nu C(\delta,\kappa,R,\varepsilon,c_1)D(\delta,\kappa,\varepsilon,c_1)y \le \delta y/4 \tag{B.9}$$

as soon as $\nu \le \delta/(4CD)$. Now we fix $2d_0 := 2d_0(\delta,\kappa,R,\varepsilon,c_1) = \min\{\nu D, D/2\}$. For any $d \le d_0$, we have (B.9) for all (x,y) such that $|(x,y)| \le 2d$.

Finally, we study the term $(4xy/\pi)\int_0^{D_\beta}\cdots\,d\eta$. For β sufficiently small (namely $\beta \leq d\sqrt{\delta}/16$), we have $d/4 - D_\beta \geq d/8$ (recall $D_\beta = \beta\sqrt{4/\delta}$). As a consequence, for (x, y) such that $d/4 \leq |(x, y)| \leq 2d$, and for all $\eta \in [0, D_\beta]$, the triangle inequality yields

$$(x^2 + (y+\eta)^2) \geq (d/4 - D_\beta)^2 \geq d^2/8^2, \qquad (x^2 + (y-\eta)^2) \geq (d/4 - D_\beta)^2 \geq d^2/8^2.$$

Still using $f_1(y) \leq Ry$, we have

$$\begin{aligned}\frac{4xy}{\pi}\int_0^{D_\beta}\cdots &\leq \frac{4xy}{\pi}\int_0^{D_\beta}\frac{(R+8.5\delta)\eta^2}{(x^2+(y+\eta)^2)(x^2+(y-\eta)^2)}\,d\eta\\ &\leq \frac{4xy}{\pi}\left(\frac{8}{d}\right)^4(R+8.5\delta)\int_0^{D_\beta}\eta^2\,d\eta\\ &\leq \frac{4xy}{\pi}\left(\frac{8}{d}\right)^4(R+8.5\delta)\frac{D_\beta^3}{3} \leq C'(R,\delta,d)\beta^3 y.\end{aligned}$$

Now, for all $\beta \leq \left(\frac{\delta}{4C'(R,\delta,d)}\right)^{1/3}$ this is less than $\delta y/4$.

This together with (B.8) and (B.9) implies that $\tilde{f}(x, y) \leq \delta y/2$ for (x, y) such that $d/4 \leq |(x, y)| \leq 2d$, that is,

$$f(x, y) \leq -8\delta y \qquad \text{for } d/4 \leq |(x, y)| \leq 2d.$$

This concludes the proof of the lemma. □

Lemma B.3. *For all $x, y > 0$, we have*

$$\begin{aligned}\frac{4xy}{\pi}\int_0^\infty \frac{\eta}{(x^2+(y+\eta)^2)(x^2+(y-\eta)^2)}\,d\eta &= (2/\pi)\arctan(y/x),\\ \frac{4xy}{\pi}\int_0^\infty \frac{\eta^2}{(x^2+(y+\eta)^2)(x^2+(y-\eta)^2)}\,d\eta &= y.\end{aligned}$$

Proof. First notice that

$$\frac{4xy\eta}{(x^2+(y+\eta)^2)(x^2+(y-\eta)^2)} = -\frac{x}{x^2+(y+\eta)^2} + \frac{x}{x^2+(y-\eta)^2}.$$

Hence, we obtain

$$\begin{aligned}&4xy\int_0^N \frac{\eta}{(x^2+(y+\eta)^2)(x^2+(y-\eta)^2)}\,d\eta\\ &\quad = \int_0^N\left(-\frac{x}{x^2+(y+\eta)^2} + \frac{x}{x^2+(y-\eta)^2}\right)d\eta\\ &\quad = -\int_{y/x}^{(N+y)/x}\frac{1}{1+s^2}\,ds + \int_{(y-N)/x}^{y/x}\frac{1}{1+s^2}\,ds\\ &\quad = -\arctan((N+y)/x)) + \arctan(y/x) + \arctan(y/x) - \arctan((y-N)/x))\\ &\quad \to 2\arctan(y/x) \qquad \text{as } N \to \infty,\end{aligned}$$

since $x, y > 0$.

Concerning the second equation, we have

$$\begin{aligned}\int_0^N \frac{4xy\eta^2}{(x^2+(y+\eta)^2)(x^2+(y-\eta)^2)}\,d\eta &= \int_0^N \left(-\frac{x\eta}{x^2+(y+\eta)^2}+\frac{x\eta}{x^2+(y-\eta)^2}\right)d\eta\\ &= -\int_{-N}^{N}\frac{x\eta}{x^2+(y+\eta)^2}\,d\eta = -\int_{-N+y}^{N+y}\frac{x(s-y)}{x^2+s^2}\,ds\\ &= -\int_{-N+y}^{N+y}\frac{xs}{x^2+s^2}\,ds+\int_{-N+y}^{N+y}\frac{xy}{x^2+s^2}\,ds.\end{aligned}$$

The integrand of the first term is an odd function, so that

$$\int_{-N+y}^{N+y}\frac{xs}{x^2+s^2}\,ds = -\int_{-N-y}^{-N+y}\frac{xs}{x^2+s^2}\,ds,$$

which converges to zero as $N\to\infty$. Moreover, the second term satisfies

$$\int_{-N+y}^{N+y}\frac{xy}{x^2+s^2}\,ds = y\int_{(-N+y)/x}^{(N+y)/x}\frac{1}{1+s^2}\,ds\to\pi y\quad\text{ as } N\to\infty,$$

which concludes the proof of the lemma. □

The following is a version of the Phragmén–Lindelöf principle for subharmonic functions in a sector of the complex plane. We prove it as a consequence of the maximum principle for subharmonic functions in bounded domains. Note that the usual Phragmén–Lindelöf theorem (see [PL08] or [SS03, Theorem 3.4]) can be deduced from this one.

Lemma B.4. *Let ϕ be a subharmonic function in Q_1, continuous in $\overline{Q}_1$. Assume that there exist $\varepsilon, C>0$ such that*

$$\begin{aligned}&\phi(z)\le C(1+|z|^{2-\varepsilon}),\qquad z\in Q_1,\\ &\phi(z)\le 0,\qquad z\in\partial Q_1=\mathbb{R}_+\cup i\mathbb{R}_+.\end{aligned}$$

Then $\phi(z)\le 0$ for all $z\in Q_1$.

Note that the power $2-\varepsilon$ with $\varepsilon>0$ is sharp: the result is false for $\varepsilon=0$, as showed by the harmonic function $(x,y)\mapsto xy$.

Proof of Lemma B.4. First note that the sector Q_1 can be rotated, say to the quadrant

$$Q=\{z\in\mathbb{C}:\arg z\in[-\pi/4,\pi/4]\}.$$

We set $v:=\operatorname{Re} z^{2-\varepsilon/2}$ (with the principal determination of the logarithm) which is harmonic in Q. We have $v(r,\theta)=r^{2-\varepsilon/2}\cos((2-\varepsilon/2)\theta)\ge r^{2-\varepsilon/2}\cos((2-\varepsilon/2)\pi/4)$ with $\cos((2-\varepsilon/2)\pi/4)>0$. Let

$$u_\delta(z)=\phi(z)-\delta v(z),$$

which is also subharmonic in Q. We have $\limsup_{z \in Q,\, |z| \to \infty} u(z) = -\infty$. As a consequence, there exists $R > 0$ such that $u_\delta(z) < 0$ on $\{|z| \geq R\} \cap Q$. Now, on the bounded set $Q^R = Q \cap \{|z| \leq R\}$, we apply the maximum principle to the function u_δ, satisfying $u_\delta \leq 0$ on ∂Q^R. This yields $u_\delta \leq 0$ on Q^R and hence $u_\delta \leq 0$ on Q. Finally, letting δ tend to zero, we obtain the sought result. □

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References

[ABB12] Agrachev, A., Barilari, D., Boscain, U.: Introduction to Riemannian and sub-Riemannian geometry. Lecture notes, unpublished: http://webusers.imj-prg.fr/~davide.barilari/Notes.php, 2012.

[Ali83] Alinhac, S.: Non-unicité du problème de Cauchy. Ann. of Math. (2) **117**, 77–108 (1983) Zbl 0516.35018 MR 0683803

[AB79] Alinhac, S., Baouendi, M. S.: Construction de solutions nulles et singulières pour des opérateurs de type principal. In: Séminaire Goulaouic-Schwartz (1978/1979), exp. 22, École Polytech., Palaiseau, 6 pp. (1979) Zbl 0427.35082 MR 0557533

[AB95] Alinhac, S., Baouendi, M. S.: A nonuniqueness result for operators of principal type. Math. Z. **220**, 561–568 (1995) Zbl 0851.35003 MR 1363855

[Bah87] Bahouri, H.: Dépendance non linéaire des données de Cauchy pour les solutions des équations aux dérivées partielles. J. Math. Pures Appl. (9) **66**, 127–138 (1987) Zbl 0565.35009 MR 0896184

[BLR92] Bardos, C., Lebeau, G., Rauch, J.: Sharp sufficient conditions for the observation, control, and stabilization of waves from the boundary. SIAM J. Control Optim. **30**, 1024–1065 (1992) Zbl 0786.93009 MR 1178650

[Bel87] Belishev, M.: An approach to multidimensional inverse problems for the wave equation. Dokl. Akad. Nauk SSSR **297**, 524–527 (1987) (in Russian) Zbl 0661.35084 MR 0924687

[Bel07] Belishev, M.: Recent progress in the boundary control method. Inverse Problems **23**, R1–R67 (2007) Zbl 1126.35089 MR 2353313

[BLR15] Bellassoued, M., Le Rousseau, J.: Carleman estimates for elliptic operators with complex coefficients. Part I: Boundary value problems. J. Math. Pures Appl. (9) **104**, 657–728 (2015) Zbl 1329.35079 MR 3394613

[Bur98] Burq, N.: Décroissance de l'énergie locale de l'équation des ondes pour le problème extérieur et absence de résonance au voisinage du réel. Acta Math. **180**, 1–29 (1998) Zbl 0918.35081 MR 1618254

[Cal58] Calderón, A. P.: Uniqueness in the Cauchy problem for partial differential equations. Amer. J. Math. **80**, 16–36 (1958) Zbl 0080.30302 MR 0104925

[Car39] Carleman, T.: Sur un problème d'unicité pour les systèmes d'équations aux dérivées partielles à deux variables indépendantes. Ark. Mat. Astr. Fys. **26**, no. 17, 9 pp. (1939) JFM65.0394.03 MR 0000334

[Hol01] Holmgren, E.: Über Systeme von linearen partiellen Differentialgleichungen. Öfversigt af Kongl. Vetenskaps-Acad. Förh. **58**, 91–103 (1901) JFM 32.0357.02

[Hör63] Hörmander, L.: Linear Partial Differential Operators. Springer, Berlin (1963) Zbl 0108.09301 MR 0161012

[Hör85] Hörmander, L.: The Analysis of Linear Partial Differential Operators, Vol. III. Springer, Berlin (1985) Zbl 0601.35001 MR 1313500

[Hör90] Hörmander, L.: The Analysis of Linear Partial Differential Operators, Vol. I. 2nd ed., Springer, Berlin (1990) Zbl 1028.35001 MR 1065136

[Hör92] Hörmander, L.: A uniqueness theorem for second order hyperbolic differential equations. Comm. Partial Differential Equations **17**, 699–714 (1992) Zbl 0815.35063 MR 1177289

[Hör94] Hörmander, L.: The Analysis of Linear Partial Differential Operators, Vol. IV. Grundlehren Math. Wiss. 275, Springer, Berlin (1994) Zbl 0612.35001 MR 1481433

[Hör97] Hörmander, L.: On the uniqueness of the Cauchy problem under partial analyticity assumptions. In: Geometrical Optics and Related Topics (Cortona, 1996), Progr. Nonlinear Differential Equations Appl. 32, Birkhäuser Boston, Boston, MA, 179–219 (1997) Zbl 0907.35002 MR 2033496

[Joh49] John, F.: On linear partial differential equations with analytic coefficients. Unique continuation of data. Comm. Pure Appl. Math. **2**, 209–253 (1949) Zbl 0035.34601 MR 0036930

[Joh60] John, F.: Continuous dependence on data for solutions of partial differential equations with a presribed bound. Comm. Pure Appl. Math. **13**, 551–585 (1960) Zbl 0097.08101 MR 0130456

[KKL01] Katchalov, A., Kurylev, Y., Lassas, M.: Inverse Boundary Spectral Problems. Chapman & Hall/CRC Monogr. Surveys Pure Appl. Math. 123, Chapman & Hall/CRC, Boca Raton, FL (2001) Zbl 1037.35098 MR 1889089

[LL16] Laurent, C., Léautaud, M.: Uniform observability estimates for linear waves. ESAIM Control Optim. Calc. Var. **22**, 1097–1136 (2016) Zbl 1368.35163 MR 3570496

[LL17] Laurent, C., Léautaud, M.: Tunneling estimates and approximate controllability for hypoelliptic equations. arXiv:1703.10797 (2017)

[Leb92] Lebeau, G.: Contrôle analytique. I. Estimations a priori. Duke Math. J. **68**, 1–30 (1992) Zbl 0780.93053 MR 1185816

[Leb96] Lebeau, G.: Équation des ondes amorties. In: Algebraic and Geometric Methods in Mathematical Physics (Kaciveli, 1993), Math. Phys. Stud. 19, Kluwer, Dordrecht, 73–109 (1996) Zbl 0863.58068 MR 1385677

[LR95] Lebeau, G., Robbiano, L.: Contrôle exact de l'équation de la chaleur. Comm. Partial Differential Equations **20**, 335–356 (1995) Zbl 0819.35071 MR 1312710

[LR97] Lebeau, G., Robbiano, L.: Stabilisation de l'équation des ondes par le bord. Duke Math. J. **86**, 465–491 (1997) Zbl 0884.58093 MR 1432305

[Le72] Lebedev, N. N.: Special Functions and their Applications. Dover Publ., New York (1972) Zbl 0271.33001 MR 0350075

[Ler88] Lerner, N.: Uniqueness for an ill-posed problem. J. Differential Equations **71**, 255–260 (1988) Zbl 0699.35162 MR 0927001

[Ler10] Lerner, N.: Metrics on the Phase Space and Non-Selfadjoint Pseudo-Differential Operators. Birkhäuser, Basel (2010) Zbl 1186.47001 MR 2599384

[LRL12] Le Rousseau, J., Lebeau, G.: On Carleman estimates for elliptic and parabolic operators. Applications to unique continuation and control of parabolic equations. ESAIM Control Optim. Calc. Var. **18**, 712–747 (2012) Zbl 1262.35206 MR 3041662

[Lio88a] Lions, J.-L.: Contrôlabilité exacte, perturbations et stabilisation de systèmes distribués. Tome 1. Recherches Math. Appl. 8, Masson, Paris (1988) Zbl 0653.93002 MR 0953547

[Lio88b] Lions, J.-L.: Contrôlabilité exacte, perturbations et stabilisation de systèmes distribués. Tome 2. Recherches Math. Appl. 9, Masson, Paris (1988) Zbl 0653.93003 MR 0963060

[Mar02] Martinez, A.: An Introduction to Semiclassical and Microlocal Analysis. Universitext, Springer, New York (2002) Zbl 0994.35003 MR 1872698

[PL08] Phragmén, E., Lindelöf, E.: Sur une extension d'un principe classique de l'analyse et sur quelques propriétés des fonctions monogènes dans le voisinage d'un point singulier. Acta Math. **31**, 381–406 (1908) JFM 39.0465.01 MR 1555044

[Phu01] Phung, K. D.: Observability and control of Schrödinger equations. SIAM J. Control Optim. **40**, 211–230 (2001) Zbl 0995.93037 MR 1855313

[Phu10] Phung, K. D.: Waves, damped wave and observation. In: Some Problems on Nonlinear Hyperbolic Equations and Applications, Ser. Contemp. Appl. Math. 15, Higher Ed. Press, 386–412 (2010) Zbl 1223.35215 MR 2816460

[RT73] Rauch, J., Taylor, M.: Penetrations into shadow regions and unique continuation properties in hyperbolic mixed problems. Indiana Univ. Math. J. **22**, 277–285 (1972/73) Zbl 0227.35064 MR 0303098

[RT74] Rauch, J., Taylor, M.: Exponential decay of solutions to hyperbolic equations in bounded domains. Indiana Univ. Math. J. **24**, 79–86 (1974) Zbl 0281.35012 MR 0361461

[Rob91] Robbiano, L.: Théorème d'unicité adapté au contrôle des solutions des problèmes hyperboliques. Comm. Partial Differential Equations **16**, 789–800 (1991) Zbl 0735.35086 MR 1113107

[Rob95] Robbiano, L.: Fonction de coût et contrôle des solutions des équations hyperboliques. Asymptotic Anal. **10**, 95–115 (1995) Zbl 0882.35015 MR 1324385

[RZ98] Robbiano, L., Zuily, C.: Uniqueness in the Cauchy problem for operators with partially holomorphic coefficients. Invent. Math. **131**, 493–539 (1998) Zbl 0909.35004 MR 1614547

[Rus71a] Russell, D. L.: Boundary value control of the higher-dimensional wave equation. SIAM J. Control **9**, 29–42 (1971) Zbl 0216.55501 MR 0274917

[Rus71b] Russell, D. L.: Boundary value control theory of the higher-dimensional wave equation. II. SIAM J. Control **9**, 401–419 (1971) Zbl 0204.46201 MR 0294896

[Sjö82] Sjöstrand, J.: Singularités analytiques microlocales. Astérisque **95**, 1–166 (1982) MR 0699623

[SS03] Stein, E. M., Shakarchi, R.: Complex Analysis. Princeton Lectures in Analysis 2, Princeton Univ. Press, Princeton, NJ (2003) Zbl 1020.30001 MR 1976398

[Str83] Strichartz, R. S.: Analysis of the Laplacian on the complete Riemannian manifold. J. Funct. Anal. **52**, 48–79 (1983) Zbl 0515.58037 MR 0705991

[Sym83] Symes, W. W.: A trace theorem for solutions of the wave equation, and the remote determination of acoustic sources. Math. Methods Appl. Sci. **5**, 131–152 (1983) Zbl 0528.35085 MR 0703950

[Tat95] Tataru, D.: Unique continuation for solutions to PDE's; between Hörmander's theorem and Holmgren's theorem. Comm. Partial Differential Equations **20**, 855–884 (1995) Zbl 0846.35021 MR 1326909

[Tat96] Tataru, D.: Carleman estimates and unique continuation for solutions to boundary value problems. J. Math. Pures Appl. (9) **75**, 367–408 (1996) Zbl 0896.35023 MR 1411157

[Tat99a] Tataru, D.: Carleman estimates, unique continuation and applications. Lecture notes, unpublished; https://math.berkeley.edu/~tataru/papers/ucpnotes.ps (1999)

[Tat99b] Tataru, D.: Unique continuation for operators with partially analytic coefficients. J. Math. Pures Appl. (9) **78**, 505–521 (1999) Zbl 0936.35038

[Zui83] Zuily, C.: Uniqueness and Nonuniqueness in the Cauchy Problem. Progr. Math. 33, Birkhäuser Boston, Boston, MA (1983) Zbl 0521.35003 MR 0701544

[Zwo12] Zworski, M.: Semiclassical Analysis. Grad. Stud. Math. 138, Amer. Math. Soc., Providence, RI (2012) Zbl 1252.58001 MR 2952218