Tunisian Journal of Mathematics

an international publication organized by the Tunisian Mathematical Society

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2023 vol. 5 no. 1

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TUNISIAN JOURNAL OF MATHEMATICS
Vol. 5 (2023), No. 1, pp. 125–170
DOI: 10.2140/tunis.2023.5.125

Uniform observation of semiclassical Schrödinger eigenfunctions on an interval

Camille Laurent and Matthieu Léautaud

We consider eigenfunctions of a semiclassical Schrödinger operator on an interval, with a single-well type potential and Dirichlet boundary conditions. We give upper and lower bounds on the L^2-density of the eigenfunctions that are uniform in both semiclassical and high energy limits. These bounds are optimal and are applied in an essential way in a companion paper to a controllability problem. The proofs rely on Agmon estimates and a Gronwall-type argument in the classically forbidden region, and on the description of semiclassical measures for boundary value problems in the classically allowed region. Limited regularity for the potential is assumed.

1. Introduction and main results

We investigate the localization of eigenfunctions of the semiclassical Schrödinger operator

$$P_\varepsilon := -\varepsilon^2 \partial_x^2 + V_\varepsilon(x), \tag{1-1}$$

MSC2020: 35B60, 35P20, 47F05, 93B07, 93C73.
Keywords: semiclassical Schrödinger operator, eigenfunctions, observability.

on the interval $[0, L]$, with Dirichlet boundary conditions, where $V_\varepsilon : [0, L] \to \mathbb{R}$ is a family of real-valued bounded potentials. In this setting, for any $\varepsilon > 0$, the operator P_ε endowed with domain $D(P_\varepsilon) = H^2([0, L]) \cap H_0^1([0, L])$ is a selfadjoint operator on $L^2(0, L)$, with compact resolvents. Its spectrum $\mathrm{Sp}(P_\varepsilon)$ thus consists only of countably many real eigenvalues with finite multiplicity (equal to 1 since this is a 1D problem). We are concerned with properties of eigenfunctions of P_ε, that is to say, solutions ψ to

$$P_\varepsilon \psi = E\psi, \quad \psi \in H^2([0, L]) \cap H_0^1([0, L]), \ \|\psi\|_{L^2([0,L])} = 1, \tag{1-2}$$

where, as already mentioned, E is necessarily a real number (depending on ε). We shall further assume that the potentials V_ε converge to a fixed potential V. The assumptions we make on V_ε and V are those of Assumption 1.1 or 1.2.

Assumption 1.1. *Assume*:

- $V \in C^0([0, L]; \mathbb{R})$, $V_\varepsilon \in L^\infty(0, L; \mathbb{R})$ *are real valued and* $\|V - V_\varepsilon\|_{L^\infty(0,L)} \to 0$.
- *There is* $x = \boldsymbol{x}_0 \in (0, L)$ *such that* V *is strictly decreasing on* $[0, \boldsymbol{x}_0]$ *and strictly increasing on* $[\boldsymbol{x}_0, L]$.

Assumption 1.2. *Assume*:

- $V_\varepsilon, V \in C^1([0, L]; \mathbb{R})$ *are real valued and* $\|V - V_\varepsilon\|_{C^1([0,L])} \to 0$.
- *The only* $x \in [0, L]$ *such that* $V'(x) = 0$ *is* $x = \boldsymbol{x}_0 \in (0, L)$ *and* $V(\boldsymbol{x}_0) = \min_{[0,L]} V$.

The typical shape of the potential V is illustrated on Figure 1.

Note that Assumption 1.2 implies Assumption 1.1. Alternatively, we shall also write $V_\varepsilon(x) = V(x) + q_\varepsilon(x)$ with $q_\varepsilon \to 0$ in L^∞ or C^1 topology as $\varepsilon \to 0$. That is to say, we consider the single well problem on the interval. We denote by E_0 the ground state energy, that is to say

$$E_0 = \min_{x \in [0,L]} (V(x)) = V(\boldsymbol{x}_0).$$

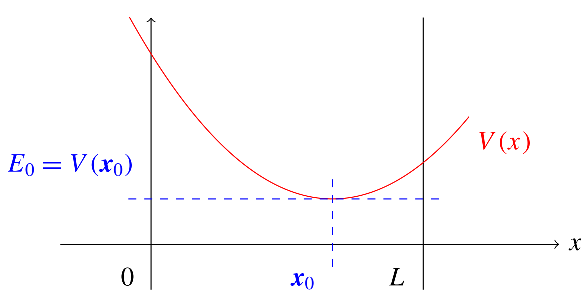

Figure 1. A typical potential V satisfying Assumption 1.2 (and thus Assumption 1.1).

The classically allowed region at energy E for the potential V is defined by

$$K_E = \{x \in [0, L],\, V(x) \le E\},$$

and the Agmon distance (see for example [Helffer 1988, Chapter 3]) to the set K_E at the energy level E by

$$\begin{aligned} d_{A,E}(x) &:= \inf_{y \in K_E} \left| \int_y^x \sqrt{(V(s)-E)_+}\, ds \right| \\ &= \left| \int_{y_E}^x \sqrt{(V(s)-E)_+}\, ds \right|, \qquad \text{if } E \ge E_0, \end{aligned} \tag{1-3}$$

where $(V(x)-E)_+ = \max(V(x)-E, 0)$ and where y_E is any point in K_E. Note in particular that $d_{A,E}$ vanishes identically on K_E (and only on this set). If $E < E_0$, we have $K_E = \varnothing$, so that the Agmon distance above is not well-defined; in that case, we shall use the convention that

$$d_{A,E}(x) = d_{A,E_0}(x), \qquad \text{if } E \le E_0.$$

This is the appropriate convention since, if ψ and $E \in \mathbb{R}$ satisfy (1-2), the L^2 inner product of (1-2) with ψ yields

$$E = \varepsilon^2 \|\psi'\|^2_{L^2([0,L])} + \int_{[0,L]} (V + q_\varepsilon)|\psi|^2, \tag{1-4}$$

and thus, under Assumption 1.1,

$$E \in \mathrm{Sp}(P_\varepsilon) \Longrightarrow E \ge E_0 - \|q_\varepsilon\|_\infty \underset{\varepsilon \to 0^+}{\longrightarrow} E_0. \tag{1-5}$$

Under Assumption 1.2, we prove upper and lower bounds that, roughly speaking, say that solutions of $P_\varepsilon \psi = E\psi$ behave, in the sense of L^2-density, like $|\psi(x)| \sim e^{-d_{A,E}(x)/\varepsilon}$ up to some loss $e^{\delta/\varepsilon}$. The upper bounds on the eigenfunctions of P_ε are expressed under the form of uniform Agmon estimates.

Theorem 1.3 (upper bounds on eigenfunctions: uniform Agmon-type estimates). *Let V, V_ε satisfy Assumption 1.1. Then, for all $\delta > 0$ there exist $\varepsilon_0 = \varepsilon_0(\delta) \in (0, 1]$ such that for all $E \in \mathbb{R}$ and solutions ψ to* (1-2), *we have for all $\varepsilon < \varepsilon_0$,*

$$\left\| e^{d_{A,E}/\varepsilon} \frac{\varepsilon}{\sqrt{|E|+1}} \psi' \right\|_{L^2} + \|e^{d_{A,E}/\varepsilon}\psi\|_{L^2} \le e^{\delta/\varepsilon}, \tag{1-6}$$

$$\frac{\varepsilon}{\sqrt{|E|+1}} |\psi'(0)| \le e^{-(d_{A,E}(0)-\delta)/\varepsilon}, \qquad \frac{\varepsilon}{\sqrt{|E|+1}} |\psi'(L)| \le e^{-(d_{A,E}(L)-\delta)/\varepsilon}. \tag{1-7}$$

The main result of this note is the following converse estimate:

Theorem 1.4 (lower bounds on eigenfunctions). *Let V, V_ε satisfy Assumption 1.2. Then, for any interval $U \subset [0, L]$ with nonempty interior and any $\delta > 0$, there is $\varepsilon_0 > 0$ such that for all $E \in \mathbb{R}$ and solutions ψ to* (1-2), *we have for all $\varepsilon < \varepsilon_0$,*

$$\|\psi\|_{L^2(U)} \geq e^{-(d_{A,E}(U)+\delta)/\varepsilon}, \quad d_{A,E}(U) = \inf_{x\in U} d_{A,E}(x), \tag{1-8}$$

$$\frac{\varepsilon}{\sqrt{|E|+1}}|\psi'(0)| \geq e^{-(d_{A,E}(0)+\delta)/\varepsilon}, \quad \frac{\varepsilon}{\sqrt{|E|+1}}|\psi'(L)| \geq e^{-(d_{A,E}(L)+\delta)/\varepsilon}. \tag{1-9}$$

Note that this lower bound is as precise as upper bound (1-6) (except for the δ loss) and thus, essentially optimal. Also, in these estimates, the loss $e^{-\delta/\varepsilon}$ can be removed/improved in several situations (see for instance Proposition 2.3 in the classically allowed region).

Note that Theorems 1.4 and 1.3 are counterparts to each other. They state essentially that, in this very particular one dimensional setting, an eigenfunction ψ associated to the energy E satisfies $|\psi(x)| \sim e^{-d_{A,E}(x)/\varepsilon}$ in the sense of L^2-density (and that this is uniform in E, x, ε).

Notice finally that, under Assumption 1.1, the set K_E is an interval given for $E \geq E_0$ by $K_E = [x_-(E), x_+(E)] \subset [0, L]$, where $x_\pm(E)$ are defined precisely below.

Definition 1.5. For $E \geq E_0$, let

- $x_-(E)$ be the solution to $V(x_-(E)) = E$ satisfying $x_-(E) \leq \boldsymbol{x}_0$ for $E \leq V(0)$ and $x_-(E) = 0$ for $E \geq V(0)$,
- $x_+(E)$ be the solution to $V(x_+(E)) = E$ satisfying $x_+(E) \geq \boldsymbol{x}_0$ for $E \leq V(L)$ and $x_+(E) = L$ for $E \geq V(L)$,

(with $\boldsymbol{x}_0 = x_-(E_0) = x_+(E_0)$ if $E = E_0$).

The proof of Theorem 1.4 relies on an explicit expression of semiclassical measures in the present context, which is of its own interest.

Theorem 1.6. *Assume that V_ε, V satisfy Assumption 1.2. Suppose that $\varepsilon_n \to 0$, $E_n \to E_* \in \mathbb{R} \cup \{+\infty\}$ as $n \to +\infty$, and ψ_n solves*

$$(P_{\varepsilon_n} - E_n)\psi_n = r_n, \quad \psi_n \in H^2([0, L]) \cap H_0^1([0, L]), \ \|\psi_n\|_{L^2([0,L])} = 1, \tag{1-10}$$

where $\|r_n\|_{L^2(0,L)} = \boldsymbol{o}(\varepsilon_n)$. Then, in the sense of weak-$*$ convergence of measures, we have $|\psi_n(x)|^2\, dx \rightharpoonup \mathfrak{m}_{E_*}$ for a nonnegative Radon measure $\mathfrak{m}_{E_*}$ on $[0, L]$ explicitly given by

$$\mathfrak{m}_{E_*} = \begin{cases} C_{E_*} \frac{\mathbb{1}_{(x_-(E),x_+(E))}(x)dx}{\sqrt{(E-V(x))_+}} & \text{if } E_0 < E_* < +\infty, \\ \delta_{\boldsymbol{x}_0} & \text{if } E_* = E_0, \\ dx/L & \text{if } E_* = +\infty, \end{cases}$$

where we have set $C_{E_*} = \left(\int_{x_-(E_*)}^{x_+(E_*)} \frac{dx}{\sqrt{E_*-V(x)}}\right)^{-1}$. *Moreover, in* $\mathbb{R}$ *we have*

$$\begin{aligned} |\varepsilon_n \psi_n'(0)|^2 &\to 2C_{E_*}\sqrt{E_* - V(0)}\mathbb{1}_{V(0)<E_*}, \\ |\varepsilon_n \psi_n'(L)|^2 &\to 2C_{E_*}\sqrt{E_* - V(L)}\mathbb{1}_{V(L)<E_*} \quad \text{if } E_* < +\infty, \\ E_n^{-1}|\varepsilon_n \psi_n'(0)|^2 \to \frac{2}{L}, &\quad E_n^{-1}|\varepsilon_n \psi_n'(L)|^2 \to \frac{2}{L} \quad \text{if } E_* = +\infty. \end{aligned}$$

Several remarks are in order. First, for a given E_*, the uniqueness of the limit measure implies that the whole sequence $|\psi_n(x)|^2\,dx$ converges. This is a very rare situation (probably linked to the simplicity of the spectrum and the regularity of the spectral gap in this 1D situation, but we do not use this information here).

Second, this theorem only describes the limit measures of $|\psi_n(x)|^2\,dx$. The latter are projections on the x-space of the semiclassical measure that live in the phase-space $(x,\xi) \in [0,L]\times\mathbb{R}$, and are as well described explicitly in the proof of Theorem 1.6. Their expression is slightly less readable, so we decided not to write them here.

Other possible approaches to this problem (which could in principle also lead to statements like those of Theorems 1.4 and 1.6) include WKB expansions (at least to leading order), see for instance [Grigis and Sjöstrand 1994, pp. 139–143] for the single well problem in $\mathbb{R}$ or [Duistermaat 1974] (in a much more general setting), or ODE methods, see for example [Olver 1974, Section 6 pp. 190–198; Berezin and Shubin 1991, Theorems 4.5 and 4.6; Fröman and Fröman 2002].

The study of eigenvalues and eigenfunctions for 1D Schrödinger operators in the semiclassical limit is a classical topic; we refer, e.g., to the seminal papers [Simon 1983] and [Helffer and Sjöstrand 1984] for the bottom energy and [Helffer and Robert 1984] for higher energies, as well as the books [Helffer 1988; Dimassi and Sjöstrand 1999]. In particular, the proof of Theorem 1.3 consists of a rather classical Agmon estimate [Helffer and Sjöstrand 1984; Helffer 1988; Dimassi and Sjöstrand 1999], and we essentially need to check here the limited regularity of the potential and the uniform dependence on the energy levels E. This uniformity is necessary for the proof of Theorem 1.6 in [Laurent and Léautaud 2023].

The literature on lower bounds (such as those given in Theorem 1.4) and semiclassical measures (such as those given in Theorem 1.6) for a boundary value problem is slightly poorer. We mention the article [Allibert 1998], where an analogue of Theorem 1.4 is stated in which the lower bounds in the right hand-sides of (1-8) and (1-9) are given in terms of the Agmon distance to the ground energy d_{A,E_0}. Similar (but less precise) estimates have also been used by the authors in [Laurent and Léautaud 2021a; 2021b] for applications to eigenfunctions on surfaces of revolution.

The exponential bounds obtained in both Theorem 1.3 and 1.4 could certainly be refined under additional assumptions (analyticity of $V_\varepsilon = V$, non degeneracy of V at $\boldsymbol{x}_0, \dots$), especially for the bottom energy E_0, using for instance some of the techniques developed in [Helffer and Sjöstrand 1984; 1986; Helffer 1988; Dimassi and Sjöstrand 1999; Helffer and Nier 2006].

Note finally that there are very few situations in which semiclassical measures of eigenfunctions/quasimodes can be described explicitly, see for example [Jakobson 1997] for the torus or [Anantharaman et al. 2016] for the disk. It is therefore satisfactory to be able to express all semiclassical measures in this very simple geometric situation. We refer to [Helffer et al. 1987, Section 4] (relying on [Duistermaat 1974]) for a related statement in a boundaryless setting with $V_\varepsilon = V$ smooth, linked to quantum ergodicity. Note, by the way, that the proof of Theorem 1.6 below implies in particular that the operator in (1-1) is quantum uniquely ergodic at all energy levels under Assumption 1.2.

The plan of the article is thus as follows: Section 2 is devoted to the proofs of the above results. The proof of Theorem 1.3, a consequence of Agmon estimates, is first given in Section 2A below as a warmup. Then, we focus on the proof of Theorem 1.4, which relies on three key lemmata:

- A geometric control estimate in the classically allowed region, proved in Section 2B. The latter essentially reduces to the description of semiclassical measures as stated in Theorem 1.6, and Section 2B is thus dedicated to the proof of Theorem 1.6.
- A tunneling estimate into the classically forbidden region (inspired by [Allibert 1998]), with sharp tunneling rate, proved in Section 2C.
- A rough Gronwall estimate used to patch the previous two estimates in the transition between the classically allowed and forbidden regions (that is, near the two turning points), also proved in Section 2C.

The last two points use arguments inspired by Section 3.2 pp. 1541–1546 of [Allibert 1998]. There are three main differences with that reference. First, we have $d_{A,E}$ in the exponent of Theorem 1.4, where Allibert only had d_{A,E_0}. Second, our estimate is uniform with respect to the energy level E. Third, the potential has limited regularity and can be perturbed by lower order terms (denoted q_ε here). This uniformity is actually a source of some complications in the proofs. Yet, it is necessary for the proof of the cost of controllability in Theorem 1.6 in [Laurent and Léautaud 2023]. We finally prove Theorem 1.4 from the three key lemmata in Section 2D.

Section 3 is devoted to the proof of several technical properties of semiclassical measures for boundary-value problems (and in dimension one only), that are prerequisites to the proof of Theorem 1.6. The results are summarized in Proposition 2.4.

The plan of Section 3 is as follows: We start by proving an a priori estimate and the so-called hidden regularity of traces in Section 3A. This allows us to define semiclassical measures associated to the eigenfunctions $\psi_n(x)$ (as well as limits of the Neumann traces) that are lifts to the phase space $(x, \xi) \in [0, L] \times \mathbb{R}$ of the measures $\mathfrak{m}_{E_*}$ appearing in Theorem 1.6. We then prove that these semiclassical measures are supported on the energy layer $\{\xi^2 + V(x) = E_*\}$ in Section 3B. Next, we prove in Section 3C that the measure satisfies an appropriate transport equation (charged at the boundary). Invariance properties near the boundary are finally deduced in Section 3D.

Most arguments in Section 3 are essentially inspired from the seminal paper of Gérard and Leichtnam [1993], where eigenfunctions of the Laplace operator are considered in any dimension, in domains with boundary having limited smoothness. We believe it is useful to provide here a detailed argument in our context for two reasons: First, the results of [Gérard and Leichtnam 1993] do not apply here since they only deal with the flat Laplacian without potential. Second, the proofs of [Gérard and Leichtnam 1993] (as well as other references on boundary propagation for semiclassical measures, e.g., [Lebeau 1996; Burq 1997a; 1997b; Robbiano and Zuily 2009) are highly technical because of the geometry and the weak regularity of the boundary. Many arguments simplify considerably in our 1D context. We thus take this as an opportunity to write a proof as detailed and pedagogical as possible, which we hope can be read as an elementary introduction to boundary propagation.

Note that although the problem is one dimensional, the fact that we consider a semiclassical Schrödinger operator makes it a very good toy model that encompasses part of the richness of propagation theory for boundary value problems [Melrose and Sjöstrand 1978]. Indeed, we shall see that elliptic, hyperbolic and glancing points all arise on the energy layer $\xi^2 + V(x) = E_*$ for certain values of the energy E_* (see Section 3D).

Note finally that all proofs of the present article are completely self-contained except for the standard semiclassical calculus in $\mathbb{R}$.

2. Proofs

Before turning to the proofs, we start with two simple observations that will be used in the proofs. The first aims at reducing the proofs to the energies E that are greater than or equal to E_0 and concerns the a priori regularity of the functions $x_\pm$ of Definition 1.5 and $d_{A,E}$ defined in (1-3).

Lemma 2.1. *Under Assumption 1.1, the functions $x_\pm : [E_0, \infty) \to [0, L]$ are uniformly continuous function. The function $\mathbb{R} \times [0, L] \to \mathbb{R}$ defined by $(E, x) \mapsto d_{A,E}(x)$ is uniformly continuous and $x \mapsto d_{A,E}(x)$ is C-Lipschitz with C independent of E.*

Proof. The first statement comes from continuity of V^{-1} on the compact interval $[\boldsymbol{x}_0, L]$ (and similarly on $[0, \boldsymbol{x}_0]$). The second statement follows from the explicit expression

$$\begin{aligned}
&d_{A,E}(x) = \int_{x_+(E)}^{x} \sqrt{V(s)-E}\,ds, && \text{if } E \geq E_0,\ x \geq x_+(E),\\
&d_{A,E}(x) = 0, && \text{if } E \geq E_0,\ x \in [x_-(E), x_+(E)],\\
&d_{A,E}(x) = \int_{x}^{x_-(E)} \sqrt{V(s)-E}\,ds, && \text{if } E \geq E_0,\ x \leq x_-(E),\\
&d_{A,E}(x) = d_{A,E_0}(x) = \Bigl|\int_{\boldsymbol{x}_0}^{x} \sqrt{V(s)-E}\,ds\Bigr|, && \text{if } E \leq E_0,\ x \in [0, L],
\end{aligned}$$

and in particular, $d_{A,E}(x) = 0$ for $E \geq \max V$ and $d_{A,E}(x) = d_{A,E_0}(x)$ for $E \leq E_0$. Moreover, we see that $d_{A,E}$ is C-Lipschitz with

$$C = \max\{\sqrt{V(x)-E} : E \in [E_0, \max V],\ x \in [0, L]\}. \qquad \square$$

The second observation concerns the reduction of the statements for all energy levels $E \in \mathbb{R}$ to only $E \geq E_0$.

Remark 2.2. We notice that it suffices to prove the statements of Theorems 1.3–1.4 for $E \geq E_0$ (and not for all $E \in \mathbb{R}$).

Indeed, if $P_\varepsilon \psi = E\psi$, and if we set $E_\varepsilon = E + \|q_\varepsilon\|_\infty$, we then have $E_\varepsilon \geq E_0$ from (1-5). Moreover, with $\widetilde{P}_\varepsilon = P_\varepsilon + \|q_\varepsilon\|_\infty$ (which is equal to P_ε with q_ε replaced by $\tilde{q}_\varepsilon = q_\varepsilon + \|q_\varepsilon\|_\infty \geq 0$, which is such that $\|\tilde{q}_\varepsilon\|_{L^\infty} \to 0$ under Assumption 1.1 or $\|\tilde{q}_\varepsilon\|_{C^1} \to 0$ under Assumption 1.2) we have $\widetilde{P}_\varepsilon \psi = E_\varepsilon \psi$.

The results of Theorems 1.3–1.4 apply to $\widetilde{P}_\varepsilon$ and $E_\varepsilon \geq E_0$ with $d_{A,E}$ replaced by d_{A,E_ε}. The conclusion for all $E \in \mathbb{R}$ follows from Lemma 2.1 above: for any $\delta > 0$ there is $\varepsilon_0 > 0$ such that $d_{A,E} - \delta \leq d_{A,E_\varepsilon} \leq d_{A,E} + \delta$ uniformly on $x \in [0, L]$ and $\varepsilon < \varepsilon_0$.

2A. ***Uniform Agmon estimates: proof of Theorem 1.3.*** We follow, e.g., [Helffer 1988] for the proof of Theorem 1.3.

Proof of Theorem 1.3. Notice first that according to Remark 2.2, it suffices to consider $E \geq E_0$. Next, consider the range $E \geq \max_{[0,L]} V$. In that case, $(V-E)_+ = 0$ and the Agmon distance $d_{A,E}$ vanishes identically on $[0, L]$. Hence, statement (1-6) can be written as

$$\frac{\varepsilon}{\sqrt{|E|+1}} \|\psi'\|_{L^2} + \|\psi\|_{L^2} \leq e^{\delta/\varepsilon},$$

which, for ε sufficiently small, is a consequence of $\|\psi\|_{L^2([0,L])} = 1$ together with

$$\varepsilon^2 \|\psi'\|^2_{L^2([0,L])} \leq (|E| + \|V\|_\infty + 1),$$

which follows from (1-4). Next, estimate (1-7) holds uniformly on compact sets of energies E as a consequence of the hidden regularity estimate (3-2) in Lemma 3.1

below, with $h = \varepsilon$ and $\mathcal{V} = \mathcal{V}_1 = V_\varepsilon - E$. For $E \geq 1$, we use estimate (3-2) with $h = \varepsilon/\sqrt{E}$, $\mathcal{V}_1 = V_\varepsilon/E = (h^2/\varepsilon^2)V_\varepsilon$ and $\mathcal{V}_2 = -1$. This implies that

$$(\varepsilon/\sqrt{E})|\psi'(0)| = h|\psi'(0)| \leq Ch^{-1}\|\mathcal{V}_1\|_{L^\infty} + C \leq (h/\varepsilon^2)\|V_\varepsilon\|_{L^\infty} + C \leq \varepsilon^{-2}C_{V,q_\varepsilon}$$

uniformly in E, ε, and in particular (1-7) holds in this range of energies.

We finally consider the most substantial case, $E \in [\min_{[0,L]} V - 1, \max_{[0,L]} V]$, and proceed with the proof of the Agmon estimates. We start with the following integration by parts formula: for all $\phi \in W^{1,\infty}(0, L)$ and $u \in H^2 \cap H_0^1(0, L)$ we have

$$\int_0^L (\varepsilon^2|\partial_x(e^{\phi/\varepsilon}u)|^2 - |\partial_x\phi|^2 e^{2\phi/\varepsilon}|u|^2) = \operatorname{Re} \int_0^L e^{2\phi/\varepsilon}(-\varepsilon^2\partial_x^2 u)\bar{u}.$$

We use this identity with $u = \psi$ a solution to $-\varepsilon^2\psi'' + V\psi + q_\varepsilon\psi = P_\varepsilon\psi = E\psi$. This yields

$$\int_0^L \varepsilon^2|\partial_x(e^{\phi/\varepsilon}\psi)|^2 + \int_0^L (V - E - |\partial_x\phi|^2 + q_\varepsilon)e^{2\phi/\varepsilon}|\psi|^2 = 0.$$

We now write $(0, L) = \Omega_\alpha^+ \sqcup \Omega_\alpha^-$ with $\Omega_\alpha^+ = \{V - E \geq \alpha^2\}$ and $\Omega_\alpha^- = \{V - E < \alpha^2\}$ for some $0 < \alpha \leq 1$ to be chosen later. We obtain

$$\int_0^L \varepsilon^2|\partial_x(e^{\phi/\varepsilon}\psi)|^2 + \int_{\Omega_\alpha^+} (V - E - |\partial_x\phi|^2 + q_\varepsilon)e^{2\phi/\varepsilon}|\psi|^2 \\ \leq \sup_{\Omega_\alpha^-} |V - E - |\partial_x\phi|^2 + q_\varepsilon| \int_{\Omega_\alpha^-} e^{2\phi/\varepsilon}|\psi|^2. \quad \text{(2-1)}$$

We now choose the weight $\phi = (1 - \delta)\, d_{A,E}$ for $\delta \in (0, 1)$ (where $d_{A,E}$ is defined in (1-3) and is Lipschitz continuous according to Lemma 2.1).

On Ω_α^+, noticing that $|d'_{A,E}|^2 = (V - E)_+ = V - E$, we have

$$V - E - |\partial_x\phi|^2 + q_\varepsilon = (V - E)(1 - (1 - \delta)^2) + q_\varepsilon \geq \alpha^2\delta(2 - \delta) - \|q_\varepsilon\|_\infty,$$

hence providing a lower bound for the left-hand side of (2-1). Concerning the right-hand side of (2-1), we write for $E \in [\min_{[0,L]} V, \max_{[0,L]} V]$,

$$\sup_{\Omega_\alpha^-} |V - E - |\partial_x\phi|^2 + q_\varepsilon| \leq 4(\|V\|_\infty + 1) + 1 =: C_V.$$

We fix $\varepsilon_0 = \varepsilon_0(\delta, \alpha)$ such that $\|q_\varepsilon\|_\infty \leq \frac{1}{2}\alpha^2\delta$ for all $\varepsilon \in (0, \varepsilon_0)$. Coming back to (2-1), we have obtained for $\delta \in (0, 1)$ and $\varepsilon \leq \varepsilon_0$,

$$\int_0^L \varepsilon^2|\partial_x(e^{\phi/\varepsilon}\psi)|^2 + \tfrac{1}{2}\alpha^2\delta \int_{\Omega_\alpha^+} e^{2\phi/\varepsilon}|\psi|^2 \leq C_V \int_{\Omega_\alpha^-} e^{2\phi/\varepsilon}|\psi|^2.$$

This implies

$$\int_0^L \varepsilon^2 |\partial_x (e^{\phi/\varepsilon}\psi)|^2 + \tfrac{1}{2}\alpha^2\delta \int_0^L e^{2\phi/\varepsilon}|\psi|^2 \leq (C_V+1)\int_{\Omega_\alpha^-} e^{2\phi/\varepsilon}|\psi|^2. \tag{2-2}$$

To conclude the proof, we now estimate the right-hand side of (2-2). We write $\Omega_\alpha^- = (\Omega_\alpha^- \cap [0, \boldsymbol{x}_0)) \sqcup (\Omega_\alpha^- \cap [\boldsymbol{x}_0, L])$ and split the integral accordingly, using that V is injective on each part. We now only consider the second term, the first one being treated similarly. Uniform continuity of V^{-1} on the compact interval $[\boldsymbol{x}_0, L]$ implies the existence of $\alpha = \alpha(\delta) \in (0, 1]$ such that

$$(E \in \mathbb{R},\ x, y \in \{z;\ E \leq V(z) \leq E + \alpha^2\} \cap [\boldsymbol{x}_0, L]) \Longrightarrow |x-y| \leq \delta. \tag{2-3}$$

As a consequence, we have for $x \in \Omega_\alpha^- \cap [\boldsymbol{x}_0, L]$ (and $V(x) \geq E$, otherwise $\phi(x) = 0$ and the same estimate is true),

$$\begin{aligned}\phi(x) = (1-\delta)d_{A,E}(x) &= (1-\delta)\int_{x_+(E)}^x \sqrt{(V(s)-E)_+}\,ds \\ &\leq (1-\delta)(x - x_+(E))\alpha \leq (1-\delta)\delta\alpha,\end{aligned}$$

using (2-3) (where $x_+(E) \in K_E$ is the solution in $[\boldsymbol{x}_0, L]$ of $V(x_+(E)) = E$). Coming back to (2-2), we now have

$$\begin{aligned}\int_0^L \varepsilon^2 |\partial_x (e^{\phi/\varepsilon}\psi)|^2 + \tfrac{1}{2}\alpha^2\delta \int_0^L e^{2\phi/\varepsilon}|\psi|^2 &\leq (C_V+1)e^{2\delta\alpha/\varepsilon}\int_{\Omega_\alpha^-} |\psi|^2 \\ &\leq (C_V+1)e^{2\delta\alpha/\varepsilon}.\end{aligned}$$

We now want to replace ϕ by $d_{A,E}$. Recall that $\phi = (1-\delta)d_{A,E}$, and that $0 \leq d_{A,E}(x) \leq L D_V$ for another constant $D_V := \sqrt{\max_{[0,L]} V - \min_{[0,L]} V + 1}$ uniformly in x, E, so that we may write

$$\begin{aligned}&\int_0^L |\partial_x (e^{d_{A,E}/\varepsilon}\psi)|^2 + \int_0^L e^{2d_{A,E}/\varepsilon}|\psi|^2 \\ &\quad= \int_0^L |\partial_x (e^{\delta d_{A,E}/\varepsilon} e^{\phi/\varepsilon}\psi)|^2 + \int_0^L |e^{\delta d_{A,E}/\varepsilon} e^{\phi/\varepsilon}\psi|^2 \\ &\quad\leq \left(1 + \frac{\delta D_V}{\varepsilon} + \frac{2}{\alpha^2\delta}\right) e^{\delta L D_V/\varepsilon}\left[\int_0^L |\partial_x (e^{\phi/\varepsilon}\psi)|^2 + \tfrac{1}{2}\alpha^2\delta\int_0^L e^{2\phi/\varepsilon}|\psi|^2\right].\end{aligned}$$

Combining the above two estimates implies

$$\int_0^L \varepsilon^2 |\partial_x (e^{d_{A,E}/\varepsilon}\psi)|^2 + \int_0^L e^{2d_{A,E}/\varepsilon}|\psi|^2 \leq (C_V+1)\left(1 + \frac{\delta D_V}{\varepsilon} + \frac{2}{\alpha^2\delta}\right) e^{(\delta/\varepsilon)(2+LD_V)},$$

which proves (1-6), up to changing $\delta(2 + LD_V)$ into δ.

To obtain the bound on the normal trace, we need an H^2 bound on $e^{d_{A,E}/\varepsilon}\psi$. To this aim, we follow, e.g., [Helffer 1988, Remark 3.3] and first regularize $d_{A,E}$. We consider $\rho_\delta = (1/\delta)\rho((\cdot/\delta)) \in C_c^\infty(-\delta,\delta)$, a nonnegative smooth approximation of the identity, and define $d_{A,E}^\delta = \rho_\delta * d_{A,E}$ for δ small enough, where V, q_ε (and $d_{A,E}$ accordingly) have been extended in a fixed neighborhood of $[0,L]$. We have $0 \le d_{A,E}^\delta \le \sup_{x\in[-\delta,L+\delta]} d_{A,E}(x) \le 2LD_V$, and, uniformly for $x \in [0,L]$,

$$\begin{aligned}|d_{A,E}^\delta(x) - d_{A,E}(x)| &\le \int |d_{A,E}(x-y) - d_{A,E}(x)|\rho_\delta(y)\,dy \le D_V \int |y|\rho_\delta(y)\,dy \\ &\le \delta D_V \int |y|\rho(y)\,dy, \end{aligned} \tag{2-4}$$

where we used that $|d_{A,E}'| = \sqrt{(V-E)_+} \le D_V$. As a consequence, from (1-6), we now obtain for a constant D_V depending only on V, for $\varepsilon < \varepsilon_0 = \varepsilon_0(\delta)$, setting $\|f\|_{H_\varepsilon^1} = \varepsilon\|f'\|_{L^2} + \|f\|_{L^2}$ and $\Psi_\varepsilon = e^{d_{A,E}^\delta/\varepsilon}\psi$,

$$\|\Psi_\varepsilon\|_{H_\varepsilon^1} \le 2\|e^{(1/\varepsilon)(d_{A,E}^\delta - d_{A,E})}\|_{W^{1,\infty}}\|e^{d_{A,E}/\varepsilon}\psi\|_{H_\varepsilon^1} \le C_\delta \varepsilon^{-1} e^{D_V(\delta/\varepsilon)} e^{\delta/\varepsilon}. \tag{2-5}$$

The function Ψ_ε is a solution of

$$(P_\varepsilon - E)\Psi_\varepsilon = -2\varepsilon(e^{d_{A,E}^\delta/\varepsilon})'\varepsilon\psi' - \varepsilon^2(e^{d_{A,E}^\delta/\varepsilon})''\psi, \quad \Psi_\varepsilon(0) = \Psi_\varepsilon(L) = 0.$$

According to inequality (2-5) above and the bounds on $d_{A,E}^\delta$, we obtain $\|\Psi_\varepsilon''\|_{L^2} \le C_\delta e^{(D_V+2)(\delta/\varepsilon)}$ uniformly for $E \in [\min_{[0,L]} V - 1, \max_{[0,L]} V + 1]$ and $\varepsilon \le \varepsilon_0(\delta)$. This together with (2-5) directly implies $|e^{d_{A,E}^\delta(0)/\varepsilon}\psi'(0)| = |\Psi_\varepsilon'(0)| \le C_\delta e^{D_V(\delta/\varepsilon)}$ (and similarly at L). Using (2-4) again, $e^{d_{A,E}^\delta(0)/\varepsilon}$ is finally replaced by $e^{d_{A,E}(0)/\varepsilon}$ in this estimate with an additional $e^{CD_V(\delta/\varepsilon)}$ loss, thus implying (1-7) (after having changed $CD_V\delta$ into δ). This concludes the proof of the theorem. □

2B. ***Lower estimates in the classically allowed region.*** In this section, we first deduce the following "geometric control estimate" from the description of semiclassical measures in Theorem 1.6. We then give a proof of Theorem 1.6, relying on technical statements for semiclassical measures for one-dimensional boundary-value problems, proved in Section 3 below.

Proposition 2.3 (geometric control in the classically allowed region). *Let $V \in C^1([0,L])$ satisfy Assumption 1.2 and $q_\varepsilon \to 0$ in $C^1([0,L])$. Then for any family $(\lambda_\varepsilon)_{\varepsilon\in(0,1)}$, $\lambda_\varepsilon \in \mathbb{R}$, converging to zero as $\varepsilon \to 0^+$, for any $\nu > 0$, there are constants $C, \varepsilon_0 > 0$ such that for all $y \in [0,L]$, all $\varepsilon \in (0,\varepsilon_0]$, all $E \in \mathbb{R}$ and ψ*

satisfying (1-2), *we have*

$$\|\psi\|_{L^2(U)} \geq C, \quad \textit{with } U = (y-\nu, y+\nu) \cap [0, L], \ \textit{if } E \geq V(y) - \lambda_\varepsilon, \tag{2-6}$$

$$\frac{\varepsilon}{\sqrt{|E|+1}} |\psi'(0)| \geq C, \quad \textit{if } E \geq V(0) + \nu, \tag{2-7}$$

$$\frac{\varepsilon}{\sqrt{|E|+1}} |\psi'(L)| \geq C, \quad \textit{if } E \geq V(L) + \nu. \tag{2-8}$$

Some remarks are in order:

- Note that the lemma states "observability inequalities" for eigenfunctions (1-2) from a neighborhood of a point y, assuming a "geometric control condition", which is here formulated as $E \geq V(y)$ (internal case) or $E \geq V(0) + \nu$ (observation from the boundary 0) or $E \geq V(L) + \nu$ (observation from the boundary L). The latter condition ensures that all classical trajectories with energy E intersect the region $(y-\nu, y+\nu)$ (internal case) or 0 (observation from the boundary 0) or L (observation from the boundary L).

- Note that the proof below proceeds by contradiction and uses semiclassical measures, following the general strategy introduced by Lebeau [1996].

- Note that the explicit expression of the measures in Theorem 1.6 can be used to describe for instance the asymptotic values of the constants C in (2-6), (2-7), (2-8). Note also that the eigenfunction in (1-2) can be "relaxed" to a quasimode equation as in the statement of Theorem 1.6.

Proof of Proposition 2.3 from Theorem 1.6. We proceed to the proof by contradiction, following the strategy introduced by Lebeau [1996]. Given $\lambda_\varepsilon \to 0$ and $\nu > 0$, if the statement of the lemma is not satisfied, the following holds: for all $n \in \mathbb{N}$, there exist $y_n \in [0, L]$, $\varepsilon_n \in (0, 1/n]$, $E_n \in \mathbb{R}$, ψ_n satisfying (1-10) with $r_n = 0$, together with

$$\|\psi_n\|_{L^2(y_n-\nu, y_n+\nu)} < \frac{1}{n}, \quad \text{in case } E_n \geq V(y_n) - \lambda_{\varepsilon_n}, \tag{2-9}$$

$$\frac{\varepsilon_n}{\sqrt{|E_n|+1}} |\psi_n'(0)| < \frac{1}{n}, \quad \text{in case } E_n \geq V(0) + \nu, \tag{2-10}$$

$$\frac{\varepsilon_n}{\sqrt{|E_n|+1}} |\psi_n'(L)| < \frac{1}{n}, \quad \text{in case } E_n \geq V(L) + \nu.$$

We may now extract from the sequence $(y_n, \varepsilon_n, E_n, \psi_n)_{n\in\mathbb{N}}$ a subsequence (which we do not relabel, with a slight abuse of notation) such that

$$\varepsilon_n \to 0, \quad y_n \to y_* \in [0, L], \quad E_n \to E_* \in [V(y_*), +\infty], \quad |\psi_n(x)|^2 \, dx \rightharpoonup \mathfrak{m}_{E_*},$$

where the last convergence holds in the sense of weak-$*$ convergence of measures. The measure $\mathfrak{m}_{E_*}$ is described explicitly in Theorem 1.6. Note that the assumptions yield $E_* \geq V(y_*) \geq V(\boldsymbol{x}_0) = \min V$. This implies $y_* \in [x_-(E_*), x_+(E_*)]$ and in particular, $\mathfrak{m}_{E_*}\big((y_* - \frac{1}{2}\nu, y_* + \frac{1}{2}\nu)\big) > 0$ in all three cases of the definitions of $\mathfrak{m}_{E_*}$ in Theorem 1.6.

Note also that dominated convergence in (2-9) implies that

$$\|\psi_n\|_{L^2(y_*-2\nu/3, y_*+2\nu/3)} \to 0, \qquad \text{as } n \to +\infty. \tag{2-11}$$

We obtain a contradiction with $\mathfrak{m}_{E_*}\big((y_* - \frac{1}{2}\nu, y_* + \frac{1}{2}\nu)\big) > 0$ by taking a bump function $\varphi \in C_c^0\big((y_* - \frac{2}{3}\nu, y_* + \frac{2}{3}\nu), [0, 1]\big)$ equal to one on $\big(y_* - \frac{1}{2}\nu, y_* + \frac{1}{2}\nu\big)$, which yields

$$\begin{aligned}
&\|\psi_n\|^2_{L^2(y_*-2\nu/3, y_*+2\nu/3)} \\
&\quad \geq \|\varphi\psi_n\|^2_{L^2(0,L)} \xrightarrow[n\to+\infty]{} \int_{[0,L]} \varphi(x)^2 \, d\mathfrak{m}_{E_*}(x) \geq \mathfrak{m}_{E_*}\big((y_* - \tfrac{1}{2}\nu, y_* + \tfrac{1}{2}\nu)\big) > 0,
\end{aligned}$$

and contradicts (2-11). This proves the internal observability estimate (2-6) and we are now left to prove the boundary observability. We only treat the case at the left boundary $x = 0$, that is to prove that (2-10) gives a contradiction.

To this aim, we now consider the cases $E_* = +\infty$ and $E_* < +\infty$ separately. If $E_* < +\infty$, then Theorem 1.6 gives $|\varepsilon_n\psi_n'(0)|^2 \to 2C_{E_*}\sqrt{E_* - V(0)}\mathbb{1}_{V(0)<E_*}$. Moreover, taking the limit in the second part of (2-10) gives $E_* \geq V(0) + \nu$. This implies $2C_{E_*}\sqrt{E_* - V(0)}\mathbb{1}_{V(0)<E_*} > 0$ and therefore $\lim_{n\to+\infty} |\varepsilon_n\psi_n'(0)|^2 > 0$, which is a contradiction to (2-10).

If now $E_* = +\infty$, Theorem 1.6 gives $E_n^{-1}|\varepsilon_n\psi_n'(0)|^2 \to 2/L$. Yet, since $E_n^{-1} \leq 2/(|E_n| + 1)$ for n large, (2-10) gives $E_n^{-1}|\varepsilon_n\psi_n'(0)|^2 \to 0$, which is a contradiction and ends the proof of the lemma. □

We are now left to prove Theorem 1.6. It relies on Proposition 2.4 in which we describe fine localization properties and transport equations satisfied by semiclassical measures for solutions to 1D boundary value problems. The proof of Proposition 2.4 is given in Section 3 below. In the statement of Proposition 2.4, we change slightly the current notation: we focus on the energy level $E = 0$ for a potential $\mathcal{V}_n \to \mathcal{V}$, and consider the semiclassical parameter $h_n \to 0$. When deducing a proof of Theorem 1.6, we will use Proposition 2.4 both with

- $h_n = \varepsilon_n$ and $\mathcal{V}_n = V_n - E_n$ which converges to $\mathcal{V} = V - E_*$ (in case E_n has a finite limit E_*),
- $h_n = \varepsilon_n/\sqrt{E_n}$ and $\mathcal{V}_n = -1 + V/E_n + q_{\varepsilon_n}/E_n$ which converges to $\mathcal{V} = -1$ (in case $E_n \to +\infty$),

and in both cases, we describe the energy level $\mathcal{V} = 0$.

Proposition 2.4. *Let $\mathcal{V}_n, \mathcal{V} \in C^1([0, L])$ be real valued so that $\mathcal{V}_n \to \mathcal{V}$ in $C^1([0, L])$. Let $h_n \to 0$ and ψ_n be such that*

$$\psi_n \in H^2(0, L) \subset C^1([0, L]), \quad \psi_n(0) = \psi_n(L) = 0, \quad \|\psi_n\|_{L^2(0,L)} = 1,$$
$$-h_n^2 \psi_n'' + \mathcal{V}_n(x)\psi_n = r_n, \quad \text{in } \mathcal{D}'((0, L)), \tag{2-12}$$

and, given a function u defined on $[0, L]$, denote by $\underline{u}$ the function satisfying $\underline{u} = u$ on $[0, L]$ and $\underline{u} = 0$ on $[0, L]^c$. Assume that $r_n = \mathcal{O}_{L^2(0,L)}(h_n)$, then there exist

- *a subsequence of indices (still denoted by n),*
- *a probability measure μ on $T^*\mathbb{R} = \mathbb{R}_x \times \mathbb{R}_\xi$, supported in $[0, L] \times \mathbb{R}_\xi$, such that*

$$(\mathrm{Op}_{h_n}(a)\underline{\psi}_n, \underline{\psi}_n)_{L^2} \to \langle \mu, a \rangle, \quad \text{for all } a \in C_c^\infty(T^*\mathbb{R}), \tag{2-13}$$

- *two nonnegative numbers ℓ_0 and ℓ_L so that*

$$|h_n \psi_n'(0^+)|^2 \to \ell_0, \quad |h_n \psi_n'(L^-)|^2 \to \ell_L, \tag{2-14}$$

- *a probability measure $\mathfrak{m}$ on $\mathbb{R}$ such that $|\underline{\psi}_n(x)|^2\, dx \rightharpoonup \mathfrak{m}$, in the sense of weak-$*$ convergence of measures on $\mathbb{R}$.*

Moreover, writing $p(x, \xi) := \xi^2 + \mathcal{V}(x)$, the following statements hold:

(1) *We have* $\mathrm{supp}(\mu) \subset \{p(x, \xi) = 0\} \cap [0, L] \times \mathbb{R}_\xi$.

(2) *If $r_n = \boldsymbol{o}(h_n)_{L^2(0,L)}$, then μ satisfies $H_p\mu = 0$ in $\mathcal{D}'((0, L) \times \mathbb{R}_\xi)$.*

(3) *If $r_n = \boldsymbol{o}(h_n)_{L^2(0,L)}$, then depending on the value $\mathcal{V}(0)$, we have:*

- *Elliptic case: if $\mathcal{V}(0) > 0$, then $\ell_0 = 0$ and there is $\delta > 0$ such that $\mu = 0$ in $(-\delta, \delta) \times \mathbb{R}$.*
- *Glancing case: if $\mathcal{V}(0) = 0$, then $H_p\mu = -\ell_0 \delta_{x=0} \otimes \delta'_{\xi=0}$ for x close to 0. If moreover $\mathcal{V}'(0) \le 0$, then $\ell_0 = 0$ and $H_p\mu = 0$ in $\mathcal{D}'((-\infty, L) \times \mathbb{R})$.*
- *Hyperbolic case: if $\mathcal{V}(0) < 0$, then*

$$H_p\mu = \frac{\ell_0}{2\sqrt{-\mathcal{V}(0)}} \delta_{x=0} \otimes (\delta_{\xi=\sqrt{-\mathcal{V}(0)}} - \delta_{\xi=-\sqrt{-\mathcal{V}(0)}}) \quad \text{in } \mathcal{D}'((-\infty, L) \times \mathbb{R}).$$

(Also, symmetric relations are true close to L).

(4) *The measures $\mathfrak{m}$ and μ are linked by $\mathfrak{m} = \pi^*\mu$, where $\pi : \mathbb{R}_x \times \mathbb{R}_\xi \to \mathbb{R}_x$ is the canonical projection, that is to say $\int_{\mathbb{R}} \varphi(x)\, d\mathfrak{m} = \int_{\mathbb{R}^2} \varphi \circ \pi\, d\mu$ for all $\varphi \in C_c^0(\mathbb{R}^2)$.*

Note that since H_p is only assumed to be a C^0 vector field, $H_p\mu$ is defined by duality, which makes sense since μ is a measure (and not only a distribution), see Lemma 3.5 below.

Let us now prove Theorem 1.6 from Proposition 2.4. Note that the regularity assumption on $V \in C^1([0, L])$ requires some care in the propagation estimates for semiclassical measures (in the proof of Theorem 1.6 as well as in the proof of Proposition 2.4). One reason for this is that the Cauchy–Lipschitz theorem does not apply to the continuous Hamiltonian vector field $2\xi\partial_x - V'(x)\partial_\xi$.

Proof of Theorem 1.6. We consider the cases $E_* = +\infty$ and $E_* < +\infty$ separately. In each case, we will compute a semiclassical measure, but with respect to a different small parameter, namely $h_n = \varepsilon_n/\sqrt{E_n}$ or $h_n = \varepsilon_n$, respectively. In the present proof, we shall describe the full semiclassical measure μ_{E_*} in phase space, associated to the sequence of eigenfunctions ψ_n (extended by zero outside $[0, L]$) and the scale h_n. Then, the measure $\mathfrak{m}_{E_*}$ will be the (restriction to $[0, L]$ of the) projection in x of the semiclassical measure $\pi^*\mu_{E_*} = \mathbb{1}_{[0,L]}\mathfrak{m}_{E_*}$, where $\pi : \mathbb{R}_x \times \mathbb{R}_\xi \to \mathbb{R}_x$ is the canonical projection (see the last item of Proposition 2.4). The limits of the respective boundary terms will result from the computation of ℓ_0 and ℓ_L in the same Proposition 2.4.

Case 1: $E_* = +\infty$. We rewrite the first equation in (1-10) as

$$(-h_n^2\partial_x^2 + \mathcal{V}_n)\psi_n = r_n E_n^{-1},$$

where we have set $h_n = \varepsilon_n/\sqrt{E_n} \to 0^+$ and $\mathcal{V}_n = -1 + V/E_n + q_{\varepsilon_n}/E_n$. Extending ψ_n by 0 outside of $[0, L]$ (without changing notation), Proposition 2.4 can be applied with $\mathcal{V}_n = -1 + V/E_n + q_{\varepsilon_n}/E_n$ and $\mathcal{V} = -1$ with $\mathcal{V}_n \to \mathcal{V}$ in $C^1([0, L])$ and $r_n/E_n = \boldsymbol{o}(h_n)$ since $r_n = \boldsymbol{o}(\varepsilon_n)$. It provides a semiclassical measure μ such that, up to a subsequence,

$$(\mathrm{Op}_{h_n}(a)\psi_n, \psi_n)_{L^2} \to \langle \mu, a\rangle, \qquad \text{for all } a \in C_c^\infty(\mathbb{R}\times\mathbb{R}).$$

Moreover, according to Proposition 2.4, the measure μ is supported by $[0, L]_x \times \{\pm 1\}_\xi$ and locally invariant by the flow of the vector field $2\xi\partial_x$ in $(0, L)_x \times \{\pm 1\}_\xi$; we necessarily have

$$\begin{aligned}\mu = \theta_1 \frac{\mathbb{1}_{[0,L]}dx}{L} \otimes \delta_{\xi=1} + \theta_2 \frac{\mathbb{1}_{[0,L]}dx}{L} \otimes \delta_{\xi=-1} \\ + \theta_3\delta_{(0,1)} + \theta_4\delta_{(0,-1)} + \theta_5\delta_{(L,1)} + \theta_6\delta_{(L,-1)}, \qquad \text{with } \theta_j \in [0, 1],\ \sum_j \theta_j = 1.\end{aligned}$$

(see below for a justification of this decomposition in a slightly more intricate setting). Also, the second part of Proposition 2.4 gives

$$2\xi\partial_x\mu = \left(\tfrac{1}{2}\ell_0\delta_{x=0} - \tfrac{1}{2}\ell_L\delta_{x=L}\right) \otimes (\delta_{\xi=1} - \delta_{\xi=-1}) \quad \text{on } \mathbb{R}_x \times \mathbb{R}_\xi, \tag{2-15}$$

where ℓ_0 and ℓ_L are the limits of the normal traces $h_n^2|\psi_n'(0)|^2$ and $h_n^2|\psi_n'(L)|^2$, respectively.

In particular, the derivative of μ is a measure. This implies that $\mu(\{(0, 1)\}) = \mu(\{(0, -1)\}) = \mu(\{(L, 1)\}) = \mu(\{(L, -1)\}) = 0$, and thus the θ_j above vanish for all $j \geq 3$. Therefore, there is $\theta \in [0, 1]$ such that

$$\mu = \theta \frac{\mathbb{1}_{[0,L]} dx}{L} \otimes \delta_{\xi=1} + (1-\theta) \frac{\mathbb{1}_{[0,L]} dx}{L} \otimes \delta_{\xi=-1} \quad \text{on } \mathbb{R} \times \mathbb{R}.$$

Now, we compute the derivative of this measure, namely

$$2\xi \partial_x \mu = \frac{2\theta}{L} (\delta_{x=0} - \delta_{x=L}) \otimes \delta_{\xi=1} - \frac{2(1-\theta)}{L} (\delta_{x=0} - \delta_{x=L}) \otimes \delta_{\xi=-1}.$$

Identifying this with (2-15) yields

$$\theta = 1/2 \quad \text{and} \quad \ell_0 = \ell_L = 2/L. \tag{2-16}$$

We can now finally compute $\pi^* \mu = \mathbb{1}_{[0,L]} \, dx/L$ which gives $\mathfrak{m}_{E_*} = dx/L$ after restriction to $[0, L]$. Since the limit is the same for any subsequence, we deduce that the convergence holds for the full sequence. Recalling that $h_n = \varepsilon_n / \sqrt{E_n}$, the values of ℓ_0 and ℓ_L in (2-16) and the convergence result of (2-14) gives the expected limit for the boundary terms.

Case 2: $V(\boldsymbol{x}_0) \leq E_* < +\infty$.

This time, we consider semiclassical operators scaled with the small parameter $h_n = \varepsilon_n \to 0^+$, namely for $a \in C_c^\infty(\mathbb{R}_x \times \mathbb{R}_\xi)$, $\mathrm{Op}_{\varepsilon_n}(a) = a(x, \varepsilon_n D_x)$.

Proposition 2.4 applied with $h_n = \varepsilon_n$, $\mathcal{V}_n = V - E_n$ and $\mathcal{V} = V - E_*$ gives again a subsequence of indices (still denoted by n) and a nonnegative Radon measure μ on $T^*\mathbb{R} = \mathbb{R}_x \times \mathbb{R}_\xi$ such that

$$(\mathrm{Op}_{\varepsilon_n}(a)\psi_n, \psi_n)_{L^2} \to \langle \mu, a \rangle, \qquad \text{for all } a \in C_c^\infty(\mathbb{R} \times \mathbb{R}),$$

where we have again extended ψ_n by zero without changing names.

Writing $p(x, \xi) = \xi^2 + V(x)$, Proposition 2.4 gives that μ is a probability measure, supported by the compact set

$$p^{-1}(E_*) = \{(x, \xi) \in [0, L] \times \mathbb{R} \text{ such that } p(x, \xi) = E_*\},$$

and moreover invariant by the flow of the Hamiltonian vector field of p, namely $H_p = 2\xi \partial_x - V'(x) \partial_\xi$, locally in the interior of $(0, L)_x \times \mathbb{R}_\xi$. Note that, as already mentioned, we have slightly changed by a constant the notation for p with respect to Proposition 2.4 without changing the Hamiltonian flow.

We assume further in the proof that

$$V(L) < V(0). \tag{2-17}$$

The case $V(L) > V(0)$ is treated similarly. In the case $V(L) = V(0)$, there are actually less subcases to consider and the additional subcase $E_* = V(L) = V(0)$ is treated as in Subcase 2 below (glancing near both endpoints of the interval); the two closed trajectories at energy E_* are smooth and tangent to both boundaries

$x = 0$ and $x = L$. Given this additional assumption (2-17) on the shape of V, we only have to consider separately the following six subcases:

(1) $V(0) < E_* < +\infty$,

(2) $E_* = V(0)$,

(3) $V(L) < E_* < V(0)$,

(4) $E_* = V(L)$,

(5) $V(\boldsymbol{x}_0) < E_* < V(L)$,

(6) $E_* = V(\boldsymbol{x}_0)$.

Subcase 1: $V(0) < E_* < +\infty$. Both 0 and L belong to $K_{E_*} = \pi(p^{-1}(E_*))$ (where $\pi : \mathbb{R}_x \times \mathbb{R}_\xi \to \mathbb{R}_x$ is the canonical projection) and the set $p^{-1}(E_*)$ decomposes as $p^{-1}(E_*) = \mathcal{C}_+ \sqcup \mathcal{C}_- \sqcup \{(0, \pm\sqrt{E_* - V(0)})\} \sqcup \{(L, \pm\sqrt{E_* - V(L)})\}$ where $\mathcal{C}_\pm = \{(x, \pm\sqrt{E_* - V(x)}), x \in (0, L)\}$ are two disjoint bounded curves (that are both orbits of H_p in case V is regular enough). We may decompose accordingly the measure μ as

$$\mu = \mu\mathbb{1}_{\mathcal{C}_+} + \mu\mathbb{1}_{\mathcal{C}_-} + \mu\mathbb{1}_{\{(0,\sqrt{E_*-V(0)})\}} + \mu\mathbb{1}_{\{(0,-\sqrt{E_*-V(0)})\}} \\ + \mu\mathbb{1}_{\{(L,\sqrt{E_*-V(L)})\}} + \mu\mathbb{1}_{\{(L,-\sqrt{E_*-V(L)})\}}, \quad (2\text{-}18)$$

in the sense of measures, i.e., for F, E two Borel sets, $\mu\mathbb{1}_E(F) = \mu(E \cap F)$. In this decomposition, the four measures supported by points are proportional to Dirac masses. We define $\delta_{\mathcal{C}_\pm}$ as

$$\langle \delta_{\mathcal{C}_\pm}, \varphi \rangle = C_{E_*} \int_0^L \varphi(x, \pm\sqrt{E_* - V(x)}) \frac{dx}{\sqrt{E_* - V(x)}}, \\ \text{with } C_{E_*} = \left(\int_0^L (E_* - V(s))^{-1/2}\, ds \right)^{-1}, \quad (2\text{-}19)$$

for $\varphi \in C_c^0(\mathbb{R}_x \times \mathbb{R}_\xi)$ or, with a somewhat loose notation,

$$\delta_{\mathcal{C}_\pm} = C_{E_*} \frac{\mathbb{1}_{(0,L)}(x)dx}{\sqrt{E_* - V(x)}} \otimes \delta_{\xi = \pm\sqrt{E_* - V(x)}}.$$

Let us now prove, using invariance by H_p, that $\mu\mathbb{1}_{\mathcal{C}_\pm}$ is proportional to $\delta_{\mathcal{C}_\pm}$, that is, it is the unique invariant measure on $\mathcal{C}_\pm$. This would be straightforward if we would have $V' \in C^1$, as a consequence of the Cauchy–Lipschitz theorem, but we only assume $V' \in C^0$ here. We define a measure ν on $(0, L)$ by

$$\langle \nu, f \rangle_{M, C_c^0(0,L)} := \langle \mu\mathbb{1}_{\mathcal{C}_+}, \sqrt{E_* - V(x)}\, f \otimes 1 \rangle_{\mathcal{M}, C_c^0((0,L)\times\mathbb{R})}, \quad \text{with } f \otimes 1(x, \xi) = f(x).$$

Let us first prove that $\partial_x \nu = 0$ in the distributional sense: we have

$$\begin{aligned}\langle \nu, \partial_x f\rangle_{\mathcal{M},C_c^0(0,L)} &= \langle \mu \mathbb{1}_{\mathcal{C}_+}, \sqrt{E_*-V}\partial_x f \otimes 1\rangle_{\mathcal{M},C_c^0((0,L)\times\mathbb{R})}\\ &= \langle \mu \mathbb{1}_{\mathcal{C}_+}, \xi\partial_x f \otimes 1\rangle_{\mathcal{M},C_c^0((0,L)\times\mathbb{R})}\\ &= \langle \mu \mathbb{1}_{\mathcal{C}_+}, H_p(f \otimes 1)\rangle_{\mathcal{M},C_c^0((0,L)\times\mathbb{R})} = 0,\end{aligned}$$

where we have used that $\xi = \sqrt{E_* - V(x)}$ on $\mathcal{C}_+$ in the first equality and the invariance property of $\mu\mathbb{1}_{\mathcal{C}_+}$ in the last equality (the latter is a consequence of the invariance of μ and the fact that $\overline{\mathcal{C}}_+ \cap \overline{\mathcal{C}}_- = \varnothing$ in this subcase). This proves in particular that there exists a constant β so that $\nu = \beta\, dx$. In particular, for $\varphi \in C_c^0((0, L)\times\mathbb{R})$, we compute, using again that $\xi = \sqrt{E_* - V(x)}$ on $\mathcal{C}_+$,

$$\begin{aligned}\langle \mu \mathbb{1}_{\mathcal{C}_+}, \varphi(x,\xi)\rangle_{\mathcal{M},C_c^0((0,L)\times\mathbb{R})} &= \langle \mu \mathbb{1}_{\mathcal{C}_+}, \varphi(x, \sqrt{E_*-V(x)}) \otimes 1\rangle_{\mathcal{M},C_c^0((0,L)\times\mathbb{R})}\\ &= \left\langle \nu, \frac{\varphi(x,\sqrt{E_*-V(x)})}{\sqrt{E_*-V(x)}}\right\rangle_{\mathcal{M},C_c^0(0,L)}\\ &= \beta\int_0^L \frac{\varphi(x,\sqrt{E_*-V(x)})}{\sqrt{E_*-V(x)}}\,dx = \beta C_{E_*}^{-1}\langle \delta_{\mathcal{C}_+}, \varphi\rangle.\end{aligned}$$

Coming back to decomposition (2-18), we have now obtained

$$\begin{aligned}\mu = \theta_1\delta_{\mathcal{C}_+} + \theta_2\delta_{\mathcal{C}_-} + \theta_3\delta_{(0,\sqrt{E_*-V(0)})} + \theta_4\delta_{(0,-\sqrt{E_*-V(0)})}&\\ + \theta_5\delta_{(L,\sqrt{E_*-V(L)})} + \theta_6\delta_{(L,-\sqrt{E_*-V(L)})}, \quad \theta_j \in [0,1],\ \sum_j \theta_j = 1.&\end{aligned}$$

Note also that for any $\varphi \in C_c^1(\mathbb{R}^2)$,

$$\begin{aligned}&\langle \delta_{\mathcal{C}_\pm}, H_p\varphi\rangle\\ &\quad= C_{E_*}\int_0^L (\pm 2\sqrt{E_*-V(x)}\partial_x - V'(x)\partial_\xi)\varphi(x, \pm\sqrt{E_*-V(x)})\frac{dx}{\sqrt{E_*-V(x)}}\\ &\quad= \pm 2C_{E_*}\int_0^L \frac{d}{dx}[\varphi(x,\pm\sqrt{E_*-V(x)})]dx\\ &\quad= \pm 2C_{E_*}\varphi(L, \pm\sqrt{E_*-V(L)}) \mp 2C_{E_*}\varphi(0,\pm\sqrt{E_*-V(0)}).\end{aligned} \tag{2-20}$$

So that

$$H_p\delta_{\mathcal{C}_\pm} = \pm 2C_{E_*}\delta_{(0,\pm\sqrt{E_*-V(0)})} \mp 2C_{E_*}\delta_{(L,\pm\sqrt{E_*-V(L)})}. \tag{2-21}$$

Moreover, both boundary points are of hyperbolic type (as in the case $E_* = \infty$). Using Proposition 2.4, these measures satisfy the equation

$$\begin{aligned}H_p\mu = \frac{\ell_0}{2\sqrt{E_*-V(0)}}\delta_{x=0}\otimes(\delta_{\xi=\sqrt{E_*-V(0)}} - \delta_{\xi=-\sqrt{E_*-V(0)}})&\\ - \frac{\ell_L}{2\sqrt{E_*-V(L)}}\delta_{x=L}\otimes(\delta_{\xi=\sqrt{E_*-V(L)}} - \delta_{\xi=-\sqrt{E_*-V(L)}}),&\end{aligned} \tag{2-22}$$

where $H_p\mu$ is well-defined as a distribution according to Lemma 3.5 below (using that the coefficients of H_p are continuous and μ is a measure). We can thus conclude as in the case $E_* = \infty$ that $H_p\mu$ is a measure, and therefore $\theta_3 = \theta_4 = \theta_5 = \theta_6 = 0$. Again, comparing (2-21), (2-22) and $\mu = \theta_1\delta_{\mathcal{C}_+} + \theta_2\delta_{\mathcal{C}_-}$, we obtain $\theta_1 = \theta_2 = \ell_0/(4C_{E_*}\sqrt{E_* - V(0)}) = \ell_L/(4C_{E_*}\sqrt{E_* - V(L)})$. The fact that μ is a probability measure gives $\theta_1 = \theta_2 = \frac{1}{2}$. In particular, $\mu = \frac{1}{2}(\delta_{\mathcal{C}_+} + \delta_{\mathcal{C}_-})$, $\ell_0 = 2C_{E_*}\sqrt{E_* - V(0)}$ and $\ell_L = 2C_{E_*}\sqrt{E_* - V(L)}$, which gives the expected result for $\mathfrak{m}_{E_*} = \pi_*\mu$ and the limits of the boundary derivatives.

Subcase 2: $E_* = V(0)$. The set $p^{-1}(E_*)$ splits as

$$p^{-1}(E_*) = \mathcal{C}_+ \sqcup \mathcal{C}_- \sqcup \{(0,0), (L, \pm\sqrt{E_* - V(L)})\},$$

where again $\mathcal{C}_\pm = \{(x, \pm\sqrt{E_* - V(x)}), x \in (0, L)\}$ are two disjoint bounded curves (that are both orbits of H_p in case V is regular enough). As in the first subcase, we have accordingly

$$\mu = \theta_1\delta_{\mathcal{C}_+} + \theta_2\delta_{\mathcal{C}_-} + \theta_3\delta_{(0,0)} + \theta_4\delta_{(L,\sqrt{E_*-V(L)})} + \theta_5\delta_{(L,-\sqrt{E_*-V(L)})},$$
$$\theta_j \in [0,1], \quad \sum_j \theta_j = 1,$$

where $\delta_{\mathcal{C}_\pm}$ is the unique invariant measure carried by $\mathcal{C}_\pm$ and given by (2-19). Note that in the present situation, the right boundary $x = L$ is of hyperbolic type whereas the left boundary point $x = 0$ is of diffractive type. Now, the second part of Proposition 2.4 yields in this case the equation

$$(2\xi\partial_x - V'\partial_\xi)\mu = -\frac{\ell_L}{2\sqrt{E_* - V(L)}}\delta_{x=L} \otimes (\delta_{\xi=\sqrt{E_*-V(L)}} - \delta_{\xi=-\sqrt{E_*-V(L)}}), \quad (2\text{-}23)$$

in a neighborhood of $x = L$. In particular, the derivative $(2\xi\partial_x - V'\partial_\xi)\mu$ is a measure near $x = L$. This implies as in the above cases that $\theta_4 = \theta_5 = 0$, and thus

$$\mu = \theta_1\delta_{\mathcal{C}_+} + \theta_2\delta_{\mathcal{C}_-} + \theta_3\delta_{(0,0)}. \quad (2\text{-}24)$$

Near $x = L$, we are as in Case 1 previously, and differentiating this expression (i.e., away from $x = 0$) yields, using (2-21),

$$(2\xi\partial_x - V'\partial_\xi)\mu = -2C_{E_*}\delta_{x=L} \otimes (\theta_1\delta_{\xi=\sqrt{E_*-V(L)}} - \theta_2\delta_{\xi=-\sqrt{E_*-V(L)}}).$$

(See again Lemma 3.5 below for the meaning of the left-hand side.) Identifying the above two lines, we obtain again $\theta_1 = \theta_2 = \ell_L/(4C_{E_*}\sqrt{E_* - V(L)})$.

We now consider the diffractive boundary at $x = 0$ in (2-23). Assumption 1.2 and Proposition 2.4 imply that $H_p\mu = 0$ close to 0. A variant of (2-21) implies $H_p\delta_{\mathcal{C}_\pm} = 0$ close to 0, which combined with (2-24) gives $\theta_3 = 0$. As a consequence, $\theta_1 = \theta_2 = \frac{1}{2}$.

We have finally obtained that $\mu = \frac{1}{2}(\delta_{\mathcal{C}_+} + \delta_{\mathcal{C}_-})$ and $\ell_L = 2C_{E_*}\sqrt{E_* - V(L)}$. We can check that $\pi_*\mu$ gives the $\mathfrak{m}_{E_*}$ announced in Theorem 1.6. It only remains to check that the glancing case of Proposition 2.4 combined with the Assumption 1.2 (which implies $V'(0) < 0$) impose $\ell_0 = 0$. This is the expected result since $\sqrt{E_* - V(0)}\mathbb{1}_{V(0)<E_*} = 0$.

Subcase 3: $V(L) < E_* < V(0)$. In this case, there is a single point $x_{E_*} \in (0, L)$ such that $V(x_{E_*}) = E_*$ (it is given by $x_{E_*} = x_-(E_*)$), and we have $x_{E_*} < \boldsymbol{x}_0$ and $V'(x_{E_*}) < 0$. The set $p^{-1}(E_*)$ splits as

$$p^{-1}(E_*) = \mathcal{C} \sqcup \{(L, \sqrt{E_* - V(L)})\} \sqcup \{(L, -\sqrt{E_* - V(L)})\}, \tag{2-25}$$

$$\text{where} \quad \mathcal{C} = \{(x, \pm\sqrt{E_* - V(x)}), x \in [x_{E_*}, L)\}. \tag{2-26}$$

We define the following probability measure on $\mathcal{C}$

$$\langle \delta_{\mathcal{C}}, \varphi \rangle = \tfrac{1}{2} C_{E_*} \sum_{\pm} \int_{x_{E_*}}^{L} \varphi(x, \pm\sqrt{E_* - V(x)}) \frac{dx}{\sqrt{E_* - V(x)}},$$
$$\text{with } C_{E_*} = \left(\int_{x_{E_*}}^{L} \frac{dx}{\sqrt{E_* - V(x)}} \right)^{-1},$$

and we now aim at proving that $\mu\mathbb{1}_{\mathcal{C}}$ is proportional to $\delta_{\mathcal{C}}$. Note that the difficulty in proving this comes again from the fact that V' is only continuous. Would we have $V' \in W^{1,\infty}$, then the Cauchy–Lipschitz theorem would apply to H_p and invariance of $\mu\mathbb{1}_{\mathcal{C}}$ would readily imply that it is proportional to $\delta_{\mathcal{C}}$.

We define as above

$$\langle \delta_{\mathcal{C}_\pm}, \varphi \rangle = C_{E_*} \int_{x_{E_*}}^{L} \varphi(x, \pm\sqrt{E_* - V(x)}) \frac{dx}{\sqrt{E_* - V(x)}},$$

and we decompose further $\mu\mathbb{1}_{\mathcal{C}} = \mu\mathbb{1}_{\mathcal{C}_+} + \mu\mathbb{1}_{\mathcal{C}_-} + \mu\mathbb{1}_{\{(x_{E_*},0)\}}$. We notice that these measures are all compactly supported; we may test them with any function in $C^0(\mathbb{R}^2)$. The same proof as in Subcase 2 implies that necessarily $\mu\mathbb{1}_{\mathcal{C}_\pm} = \alpha_\pm\delta_{\mathcal{C}_\pm}$ and $\mu\mathbb{1}_{\{(x_{E_*},0)\}} = \beta\delta_{(x_{E_*},0)}$. Invariance of μ reads $\langle \mu, H_p\phi \rangle = 0$ for all $\phi \in C^1(\mathbb{R}^2)$, $\operatorname{supp}(\phi) \subset (0, L) \times \mathbb{R}$. Applied to $\phi(x, \xi) = \widetilde{\chi}(x)\varphi(\xi)$ with $\widetilde{\chi} \in C^1_c(0, L)$ such that $\widetilde{\chi}(x_{E_*}) = 1$, we notice that $H_p\varphi = 2\xi\widetilde{\chi}'(x)\varphi(\xi) - V'(x)\widetilde{\chi}(x)\varphi'(\xi)$, and thus deduce

$$\langle \alpha_+\delta_{\mathcal{C}_+} + \alpha_-\delta_{\mathcal{C}_-} + \beta\delta_{(x_{E_*},0)}, 2\xi\widetilde{\chi}'(x)\varphi(\xi) - V'(x)\widetilde{\chi}(x)\varphi'(\xi) \rangle = 0.$$

Take $\chi \in C_c^\infty(\mathbb{R})$ with $\chi = 1$ in a neighborhood of zero and $\varphi_\epsilon(\xi) = \int_{-\infty}^{\xi/\epsilon} \chi(t)\,dt$. We obtain

$$0 = \langle \alpha_+\delta_{\mathcal{C}_+} + \alpha_-\delta_{\mathcal{C}_-} + \beta\delta_{(x_{E_*},0)}, 2\xi\widetilde{\chi}'(x)\varphi_\epsilon(\xi) - V'(x)\widetilde{\chi}(x)\frac{1}{\epsilon}\chi(\xi/\epsilon) \rangle,$$

whence multiplying by ϵ,

$$0=\epsilon\Big\langle\alpha_+\delta_{\mathcal{C}_+}+\alpha_-\delta_{\mathcal{C}_-},\,2\xi\widetilde{\chi}'(x)\varphi_\epsilon(\xi)-V'(x)\widetilde{\chi}(x)\frac{1}{\epsilon}\chi(\xi/\epsilon)\Big\rangle-\beta V'(x_{E_*}).$$

Letting $\epsilon\to 0$ and using dominated convergence, we deduce

$$\beta V'(x_{E_*})=-\langle\alpha_+\delta_{\mathcal{C}_+}+\alpha_-\delta_{\mathcal{C}_-},\,V'(x)\widetilde{\chi}(x)\mathbb{1}_{\{\xi=0\}}\rangle=0,$$

since $\mathcal{C}_\pm\cap\{\xi=0\}=\varnothing$. This implies $\beta V'(x_{E_*})=0$, and thus $\beta=0$ since $V'(x_{E_*})>0$.

Now we take any $\varphi\in C^1_c((0,L)\times\mathbb{R})$ and compute as in (2-20)

$$\langle\delta_{\mathcal{C}_\pm},H_p\varphi\rangle=C_{E_*}\int_{x_{E_*}}^{L}\pm\frac{d}{dx}\big[\varphi(x,\pm\sqrt{E_*-V(x)})\big]\,dx=\mp C_{E_*}\varphi(x_{E_*},0).$$

As a consequence, we obtain for all $\varphi\in C^1_c((0,L)\times\mathbb{R})$,

$$0=\langle\mu,H_p\varphi\rangle=\sum_\pm\alpha_\pm\langle\delta_{\mathcal{C}_\pm},H_p\varphi\rangle=-\alpha_+C_{E_*}\varphi(x_{E_*},0)+\alpha_-C_{E_*}\varphi(x_{E_*},0),$$

and thus $\alpha_+=\alpha_-$. This concludes the proof that $\mu\mathbb{1}_{\mathcal{C}}$ is proportional to $\delta_{\mathcal{C}}$.

We now come back to decomposition (2-25) and have obtained that

$$\mu=\theta_1\delta_{\mathcal{C}}+\theta_2\delta_{(L,\sqrt{E_*-V(L)})}+\theta_3\delta_{(L,-\sqrt{E_*-V(L)})},\quad \theta_j\in[0,1],\ \sum_j\theta_j=1.$$

The same computation as before gives

$$\begin{aligned}\langle\delta_{\mathcal{C}},H_p\varphi\rangle&=C_{E_*}\sum_\pm\int_{x_{E_*}}^{L}\pm\frac{d}{dx}\big[\varphi(x,\pm\sqrt{E_*-V(x)})\big]\,dx\\&=\sum_\pm\pm C_{E_*}\varphi(L,\pm\sqrt{E_*-V(L)}),\end{aligned}$$

so that $H_p\delta_{\mathcal{C}}=\sum_\pm\mp C_{E_*}\delta_{(L,\pm\sqrt{E_*-V(L)})}$. We thus argue as in the previous cases that $\theta_2=\theta_3=0$, hence $\theta_1=1$. As a consequence, $\mu=\delta_{\mathcal{C}}$. Concerning the boundary estimates at 0, we are in the elliptic case of Proposition 2.4 which implies $\ell_0=0$. This is the expected result since $\sqrt{E_*-V(0)}\mathbb{1}_{V(0)<E_*}=0$. At L, we are in the hyperbolic case, and we conclude as in the other subcases.

Subcase 4: $E_*=V(L)$. The set $p^{-1}(E_*)$ splits as $p^{-1}(E_*)=\mathcal{C}\sqcup\{(L,0)\}$, where $\mathcal{C}\subset(0,L)\times\mathbb{R}$ is defined as in (2-26) (and is an orbit of H_p in case V is regular). We have accordingly $\mu=\theta\delta_{\mathcal{C}}+(1-\theta)\delta_{(L,0)}$, with $\delta_{\mathcal{C}}$ the unique invariant measure carried by $\mathcal{C}$ (a proof of uniqueness of this measure under the sole regularity assumption $V'\in C^0$ follows as in the above two subcases). Moreover, the second part of Proposition 2.4 yields in this case the equation

$$(2\xi\partial_x-V'\partial_\xi)\mu=0.$$

The point $x = L$ is of diffractive type and the same analysis as in Subcase 2 yields $\mu(\{(L, 0)\}) = 0$, hence $\theta = 1$. This proves $\mu = \delta_{\mathcal{C}}$, and we can conclude as in all the previous cases. The proof that $\ell_0 = \ell_L = 0$ is performed as before for the respective elliptic and glancing cases (using that $V'(L) > 0$).

Subcase 5: $V(\boldsymbol{x}_0) < E_* < V(L)$. The set $p^{-1}(E_*)$ is a C^1 closed curve contained in $(0, L) \times \mathbb{R}$ and $dp|_{p^{-1}(E_*)}$ does not vanish. The measure μ is supported on this curve and invariant by the vector field H_p, being nondegenerate and tangent to $p^{-1}(E_*)$. Henceforth, μ is the unique probability measure carried by $p^{-1}(E_*)$ and invariant by H_p (again, uniqueness of this measure for $V' \in C^0$ follows as in the above subcases) defined by

$$\langle \mu, \varphi \rangle = \tfrac{1}{2} C_{E_*} \sum_{\pm} \int_{x_-(E_*)}^{x_+(E_*)} \varphi(x, \pm\sqrt{E_* - V(x)}) \frac{dx}{\sqrt{E_* - V(x)}},$$

$$\text{with } C_{E_*} = \left(\int_{x_-(E_*)}^{x_+(E_*)} \frac{dx}{\sqrt{E_* - V(x)}} \right)^{-1}.$$

The projection on x of μ gives the expected result. Moreover, we are in the elliptic case at both boundaries $x = 0$ and L so that the normal trace converges to zero.

Subcase 6: $E_* = V(\boldsymbol{x}_0)$. Note that the assumption $V(y_*) \le E_*$ implies that $V(y_*) = E_*$, and thus $y_* = \boldsymbol{x}_0$. We have $p^{-1}(E_*) = \{(y_*, 0)\}$, and the only probability measure carried by this set is $\mu = \delta_{(x,\xi)=(y_*,0)}$. We compute $\pi_* \mu = \delta_{\boldsymbol{x}_0}$, and we are again in the elliptic case at both points of the boundary.

This concludes the proof of the theorem. □

2C. ***Lower estimates in the classically forbidden region and near the turning points.*** Next, we define the following "semiclassical energy densities" of the eigenfunctions ψ: For $x \in [0, L]$,

$$\mathcal{E}(x) := \varepsilon^2 |\psi'|^2(x) + |\psi|^2(x),$$
$$\mathcal{E}^+(x) := \varepsilon^2 |\psi'|^2(x) + (V(x) - E)|\psi|^2(x).$$

The following lemma is a variant of [Allibert 1998, Lemma 12], see also [Laurent and Léautaud 2021b, Lemma 4.10], in which we keep track of the dependence with respect to the lower order terms:

Lemma 2.5 (tunneling into the classically forbidden region). *For all* $\alpha > 0$, *all solutions* E, ψ, ε *to* (1-2), *and all points* x, y *belonging to the same connected component of* $\{V - E \ge \alpha^2\}$, *we have*

$$\mathcal{E}^+(x) \le \exp\left(\frac{2}{\varepsilon} \left| \int_x^y \sqrt{V(s) - E}\, ds \right| + \frac{\|V'\|_\infty}{\alpha^2} L + \frac{\|q_\varepsilon\|_\infty}{\alpha \varepsilon} L \right) \mathcal{E}^+(y).$$

Proof of Lemma 2.5. We differentiate the function $\mathcal{E}^+$, yielding

$$(\mathcal{E}^+)' = 2\varepsilon^2 \operatorname{Re}(\overline{\psi}'\psi'') + V'|\psi|^2 + 2(V-E)\operatorname{Re}(\psi\overline{\psi}').$$

We recall from the definition in (1-1) and (1-2) that we have

$$E\psi = P_\varepsilon\psi = -\varepsilon^2\psi'' + V\psi + q_\varepsilon\psi.$$

This implies that

$$\begin{aligned}(\mathcal{E}^+)' &= 2(V-E+q_\varepsilon)\operatorname{Re}(\psi\overline{\psi}') + V'|\psi|^2 + 2(V-E)\operatorname{Re}(\psi\overline{\psi}')\\ &= (4(V-E)+2q_\varepsilon)\operatorname{Re}(\psi\overline{\psi}') + V'|\psi|^2. \end{aligned} \tag{2-27}$$

We now estimate each of the terms in the right-hand side of (2-27) on the set $\{V-E \geq \alpha^2\}$. We first have the pointwise estimate

$$\begin{aligned}|4(V-E)\operatorname{Re}(\psi\overline{\psi}')| &= 4\varepsilon^{-1}\sqrt{V-E}(\varepsilon|\psi'|)(\sqrt{V-E}|\psi|)\\ &\leq 2\varepsilon^{-1}\sqrt{V-E}(\varepsilon^2|\psi'|^2 + (V-E)|\psi|^2) = 2\varepsilon^{-1}\sqrt{V-E}\,\mathcal{E}^+.\end{aligned}$$

Second, we have the pointwise estimate

$$|V'|\psi|^2| = \frac{|V'|}{V-E}(V-E)|\psi|^2 \leq \frac{\|V'\|_\infty}{\alpha^2}\mathcal{E}^+, \qquad \text{on } \{V-E\geq\alpha^2\}.$$

Third, we have on $\{V-E\geq\alpha^2\}$,

$$\begin{aligned}|2q_\varepsilon \operatorname{Re}(\psi\overline{\psi}')| &\leq \frac{\|q_\varepsilon\|_\infty}{\varepsilon}\left(\frac{\varepsilon^2}{\alpha}|\psi'|^2 + \alpha|\psi^2|\right)\\ &\leq \frac{\|q_\varepsilon\|_\infty}{\varepsilon}\left(\frac{\varepsilon^2}{\alpha}|\psi'|^2 + \alpha\frac{V-E}{\alpha^2}|\psi^2|\right)\frac{\|q_\varepsilon\|_\infty}{\alpha\varepsilon}\mathcal{E}^+.\end{aligned}$$

Combining the last three estimates in (2-27) yields for all $t\in\{V-E\geq\alpha^2\}$,

$$|(\mathcal{E}^+)'(t)| \leq \left(\frac{2}{\varepsilon}\sqrt{V(t)-E} + \frac{\|V'\|_\infty}{\alpha^2} + \frac{\|q_\varepsilon\|_\infty}{\alpha\varepsilon}\right)\mathcal{E}^+(t).$$

Two applications of the Gronwall lemma imply that for all $z<x$ such that $[z,x]\subset\{V-E\geq\alpha^2\}$, we have

$$e^{-\mu(x,z)}\mathcal{E}^+(z) \leq \mathcal{E}^+(x) \leq e^{\mu(x,z)}\mathcal{E}^+(z),$$

for $\mu(x,z) = (2/\varepsilon)\int_z^x \sqrt{V(t)-E}\,dt + (\|V'\|_\infty/\alpha^2 + \|q_\varepsilon\|_\infty/\alpha\varepsilon)(x-z)$. This yields the sought result. ☐

Note that the estimate involving $\|V'\|_\infty$ could be slightly refined using a sign assumption on V'.

The following Lemma is an analogue of [Allibert 1998, Lemma 11], see also [Laurent and Léautaud 2021b, Lemma 4.11], and gives a rough Gronwall-type estimate for the energy $\mathcal{E}$, without precise constants. The interest of this less precise result is that it remains true uniformly for all $x \in [0, L]$. This allows one in particular to compensate for the fact that Lemma 2.5 is not uniform when x is close to the boundary of the set $\{V - E > 0\}$.

Lemma 2.6 (rough Gronwall estimate). *For all $E \in \mathbb{R}$, $\psi \in H^2([0, L]) \cap H^1_0([0, L])$, all $\varepsilon > 0$ such that $P_\varepsilon \psi = E\psi$ and all $x, y \in [0, L]$, we have*

$$\mathcal{E}(x) \leq \exp\Big(\frac{1}{\varepsilon}|x-y|(\|V-E+1\|_{L^\infty(I_{x,y})} + \|q_\varepsilon\|_\infty)\Big)\mathcal{E}(y),$$

where $I_{x,y}$ is the interval between x and y.

Proof. The proof is very close to that of Lemma 2.5. We write similarly

$$(\mathcal{E})' = 2\varepsilon^2 \operatorname{Re}(\overline{\psi}'\psi'') + 2\operatorname{Re}(\psi\overline{\psi}') = 2(V - E + q_\varepsilon + 1)\operatorname{Re}(\psi\overline{\psi}').$$

This implies on the interval $I_{x,y}$

$$|(\mathcal{E})'| \leq \frac{1}{\varepsilon}(\|V-E+1\|_{L^\infty(I_{x,y})} + \|q_\varepsilon\|_\infty)\mathcal{E},$$

and we conclude the proof with a Gronwall argument on $I_{x,y}$ as in Lemma 2.5. $\square$

2D. ***End of the proof of Theorem 1.4.*** With the three previous lemmata at hand, we are now in position to prove Theorem 1.4. We first prove the following intermediate result:

Lemma 2.7 (lower bounds on eigenfunctions). *Suppose that the functions V and V_ε satisfy Assumption 1.2. Then, there is a constant $D > 0$ such that for any $\boldsymbol{y}_0 \in [0, L]$ and any $\delta > 0$, there is $\varepsilon_0 > 0$ such that for all $E \in \mathbb{R}$, $0 < \varepsilon < \varepsilon_0$ and solutions ψ to* (1-2), *we have*

$$\|\psi\|_{L^2(U)} \geq e^{-(d_{A,E}(\boldsymbol{y}_0)+D\delta)/\varepsilon}, \quad U = (\boldsymbol{y}_0 - \delta, \boldsymbol{y}_0 + \delta) \cap [0, L], \tag{2-28}$$

$$\frac{\varepsilon}{\sqrt{|E|+1}}|\psi'(0)| \geq e^{-(d_{A,E}(0)+\delta)/\varepsilon}, \qquad \frac{\varepsilon}{\sqrt{|E|+1}}|\psi'(L)| \geq e^{-(d_{A,E}(L)+\delta)/\varepsilon}. \tag{2-29}$$

Note that in this statement (as well as in all statements of the article), δ is thought of as a small parameter.

Proof that Lemma 2.7 implies Theorem 1.4. Notice first that according to Remark 2.2, it suffices to consider $E \geq E_0$. Then, the only difference between the two statements concerns the internal observation. We write $U = [z_1, z_2]$ with $z_1, z_2 \in [0, L]$. We treat the case for which $z_1 \geq \boldsymbol{x}_0$; the case $z_2 \leq \boldsymbol{x}_0$ is treated similarly. Concerning the case $z_1 < \boldsymbol{x}_0 < z_2$, we take $\boldsymbol{y}_0 = \boldsymbol{x}_0$ and choose $\delta > 0$ small enough so that $(\boldsymbol{x}_0 - \delta, \boldsymbol{x}_0 + \delta) \subset (z_1, z_2)$, and Lemma 2.7 yields the result since in this case $\inf_{x\in U} d_{A,E}(x) = d_{A,E}(\boldsymbol{x}_0)$ for all $E \geq E_0$.

Since we assume now $z_1 \geq \boldsymbol{x}_0$, we have $\inf_{x\in U} d_{A,E}(x) = d_{A,E}(z_1)$. According to Lemma 2.1, $d_{A,E}$ is uniformly Lipschitz, so there is $\tilde{\delta} > 0$ small enough and uniform in $E \geq E_0$ so that $|d_{A,E}(z) - d_{A,E}(z_1)| \leq \delta$ for $|z - z_1| \leq \tilde{\delta}$. We can also assume $\tilde{\delta} < (z_2 - z_1)/2$ and $\tilde{\delta} \leq \delta$. Applying Lemma 2.7 with $\boldsymbol{y}_0 = z_1 + \tilde{\delta}$ and δ replaced by $\tilde{\delta}$, we obtain $\|\psi\|_{L^2((\boldsymbol{y}_0-\tilde{\delta}, \boldsymbol{y}_0+\tilde{\delta})\cap[0,L])} \geq e^{-(d_{A,E}(\boldsymbol{y}_0)+D\tilde{\delta})/\varepsilon}$. Since $(\boldsymbol{y}_0 - \tilde{\delta}, \boldsymbol{y}_0 + \delta) \cap [0, L] \subset U$, using the previous estimates gives $\|\psi\|_{L^2(U)} \geq e^{-(d_{A,E}(z_1)+(D+1)\delta)/\varepsilon}$, which is the expected result up to changing δ. □

We now prove Lemma 2.7, as a consequence of Theorem 1.6 and Lemmata 2.5 and 2.6.

Proof of Lemma 2.7. We first prove the internal observation inequality (2-28). We distinguish different cases according to the respective location of the points $\boldsymbol{y}_0$ and $\boldsymbol{x}_0$.

Consider first the case where $\boldsymbol{x}_0 \in (\boldsymbol{y}_0 - \delta, \boldsymbol{y}_0 + \delta)$. Then, Proposition 2.3 with ν small enough so that $(\boldsymbol{y}_0 - \nu, \boldsymbol{y}_0 + \nu) \subset U$, yields

$$\|\psi\|_{L^2(U)} \geq \|\psi\|_{L^2(\boldsymbol{x}_0-\nu,\boldsymbol{x}_0+\nu)} \geq C_0,$$

uniformly for $E \in \mathbb{R}$, which implies (2-28) in this case.

We now consider the case where $\boldsymbol{x}_0 \notin (\boldsymbol{y}_0 - \delta, \boldsymbol{y}_0 + \delta)$, and assume further in what follows that $\boldsymbol{x}_0 \leq \boldsymbol{y}_0 - \delta$. The case $\boldsymbol{x}_0 \geq \boldsymbol{y}_0 + \delta$ is proved similarly (by symmetry). In particular, this implies $V'(\boldsymbol{y}_0) > 0$ and $V(\boldsymbol{y}_0) > \min_{[0,L]} V$.

For this δ, Proposition 2.3 yields the existence of $C_0, \varepsilon_0 > 0$ such that for all $z \in [0, L]$, all $\varepsilon \in (0, \varepsilon_0]$, all $E \in \mathbb{R}$ and solutions ψ to (1-2), we have

$$E \geq V(z) \Longrightarrow \|\psi\|_{L^2((z-\delta/2,z+\delta/2)\cap[0,L])} \geq C_0. \tag{2-30}$$

Thanks to a variant of Remark 2.2, we can assume from now on that $E \geq E_0$.

Case 1: $x_+(E) \geq \boldsymbol{y}_0 - \frac{1}{2}\delta$. In this case, either $x_+(E) \geq \boldsymbol{y}_0$ (hence $E \geq V(\boldsymbol{y}_0)$), so that (2-30) with $z = \boldsymbol{y}_0$ yields

$$\|\psi\|_{L^2((\boldsymbol{y}_0-\delta,\boldsymbol{y}_0+\delta)\cap[0,L])} \geq \|\psi\|_{L^2((\boldsymbol{y}_0-\delta/2,\boldsymbol{y}_0+\delta/2)\cap[0,L])} \geq C_0,$$

which concludes the proof in that case; otherwise $x_+(E) \leq \boldsymbol{y}_0 \leq x_+(E) + \frac{1}{2}\delta$, so that (2-30) with $z = x_+(E)$ yields

$$\|\psi\|_{L^2((\boldsymbol{y}_0-\delta,\boldsymbol{y}_0+\delta)\cap[0,L])} \geq \|\psi\|_{L^2((x_+(E)-\delta/2,x_+(E)+\delta/2)\cap[0,L])} \geq C_0,$$

which concludes the proof in that case.

Case 2: $x_+(E) < \boldsymbol{y}_0 - \frac{1}{2}\delta$. Lemma 2.1 (uniform continuity of V^{-1} on the compact interval $[\boldsymbol{x}_0, L]$) implies the existence of $\alpha > 0$ such that for all $x, y \in [0, L]$, $E \in \mathbb{R}$,

$$x, y \in \{z \in [\boldsymbol{x}_0, L] : E - \alpha^2 \leq V(z) \leq E + \alpha^2\} \Longrightarrow |x - y| \leq \tfrac{1}{4}\delta. \tag{2-31}$$

In this case, $V(x_+(E)) = E$ together with (2-31) implies that

$$\boldsymbol{y}_0 \notin \{z \in [\boldsymbol{x}_0, L],\ E - \alpha^2 \leq V(z) \leq E + \alpha^2\}.$$

Since $x_+(E) < \boldsymbol{y}_0$ in this case, this implies necessarily that $V(\boldsymbol{y}_0) > E + \alpha^2$. Estimate (2-30) with $z = x_+(E)$ implies

$$C_0 \leq \|\psi\|_{L^2((x_+(E)-\delta/2, x_+(E)+\delta/2)\cap[0,L])}. \tag{2-32}$$

Lemma 2.6 together with $\|q_\varepsilon\|_\infty \leq 1$ yields

$$|\psi|^2(x) \leq \exp\Big(\frac{1}{\varepsilon}|x-y|(2+2\|V\|_\infty)\Big)\mathcal{E}(y), \quad x, y \in [0, L]. \tag{2-33}$$

Integrating over $x \in \big(x_+(E) - \frac{1}{2}\delta, x_+(E) + \frac{1}{2}\delta\big) \cap [0, L]$ implies

$$\|\psi\|^2_{L^2(x_+(E)-\delta/2, x_+(E)+\delta/2)\cap[0,L]} \leq \delta \exp\Big(\frac{\delta}{\varepsilon}(2+2\|V\|_\infty)\Big)\mathcal{E}\Big(x_+(E) + \frac{\delta}{2}\Big), \tag{2-34}$$

for $y = x_+(E) + \frac{1}{2}\delta < \boldsymbol{y}_0 \leq L$. Now, notice that (2-31) implies

$$0 < x_+(E+\alpha^2) - x_+(E) \leq \tfrac{1}{4}\delta.$$

The point $y = x_+(E) + \frac{1}{2}\delta \in \{z;\ V(z) - E \geq \alpha^2\}$ is chosen for Lemma 2.5. Note first that on the set $\{z;\ V(z) - E \geq \alpha^2\}$ and for $|\alpha| < 1$ (which we may assume), we have $\mathcal{E} \leq \alpha^{-2}\mathcal{E}^+$ and that $z \geq x_+(E) + \frac{1}{4}\delta \Longrightarrow z \in \{V - E \geq \alpha^2\}$ (this is the case for $z = \boldsymbol{y}_0$). Lemma 2.5 now implies, for all $z \geq x_+(E) + \frac{1}{4}\delta$,

$$\begin{aligned}
&\alpha^2\mathcal{E}\big(x_+(E) + \tfrac{1}{4}\delta\big)\\
&\quad\leq \mathcal{E}^+\big(x_+(E) + \tfrac{1}{4}\delta\big)\\
&\quad\leq \exp\left(\frac{2}{\varepsilon}\left|\int_{x_+(E)+\delta/4}^{z}\sqrt{V(s)-E}\,ds\right| + \frac{\|q_\varepsilon\|_\infty}{\alpha\varepsilon}L + \frac{\|V'\|_\infty}{\alpha^2}L\right)\mathcal{E}^+(z).
\end{aligned}\tag{2-35}$$

Integrating in $z \in \big(\boldsymbol{y}_0 - \frac{1}{4}\delta, \boldsymbol{y}_0 - \frac{1}{8}\delta\big)$ (which implies $z \geq x_+(E) + \frac{1}{4}\delta$ according to the assumption $x_+(E) < \boldsymbol{y}_0 - \frac{1}{2}\delta$) yields

$$\begin{aligned}
&\tfrac{1}{8}\delta\alpha^2\mathcal{E}\big(x_+(E) + \tfrac{1}{2}\delta\big)\\
&\ \leq \exp\left(\frac{2}{\varepsilon}\left|\int_{x_+(E)+\delta/4}^{\boldsymbol{y}_0}\sqrt{V(s)-E}\,ds\right| + \frac{\|q_\varepsilon\|_\infty}{\alpha\varepsilon}L + \frac{\|V'\|_\infty}{\alpha^2}L\right)\int_{\boldsymbol{y}_0-\delta/4}^{\boldsymbol{y}_0-\delta/8}\mathcal{E}^+(s)\,ds.
\end{aligned}$$

An interpolation estimate together with $P_\varepsilon\psi = E\psi$ yields

$$\begin{aligned}
&\int_{\boldsymbol{y}_0-\delta/4}^{\boldsymbol{y}_0-\delta/8}\mathcal{E}^+(s)\,ds\\
&\qquad\qquad \leq C\delta^{-1}(\|\psi\|^2_{L^2(\boldsymbol{y}_0-\delta/4, \boldsymbol{y}_0-\delta/8)} + \|\psi\|_{L^2(\boldsymbol{y}_0-\delta/2, \boldsymbol{y}_0)}\|\varepsilon^2\psi''\|_{L^2(\boldsymbol{y}_0-\delta/2, \boldsymbol{y}_0)})\\
&\qquad\qquad \leq C\delta^{-1}\|\psi\|^2_{L^2(\boldsymbol{y}_0-\delta, \boldsymbol{y}_0+\delta)\cap[0,L]}.
\end{aligned}$$

Note that we have used $E \le \|V\|_\infty$, otherwise this zone is empty. Combining the above two estimates with (2-32) and (2-34) yields the existence of constants $C = C(V, \delta, L) > 0$ (recall that α depends on δ and V) independent of E, ε such that

$$1 \le C \exp\left\{\frac{2}{\varepsilon}\left(\left|\int_{x_+(E)+/4}^{y_0} \sqrt{V(s) - E}\, ds\right| + (2 + 2\|V\|_\infty)\delta + \frac{\|q_\varepsilon\|_\infty}{\alpha} L\right)\right\} \cdot \|\psi\|^2_{L^2(y_0-\delta,\, y_0+\delta)\cap[0,L]}.$$

We further assume that ε_0 is sufficiently small so that $(\|q_\varepsilon\|_\infty/\alpha)L \le \delta$ for all $\varepsilon \in (0, \varepsilon_0)$. This then concludes the proof in that case, and hence the proof of (2-28) in the theorem.

We now explain how this proof needs to be modified in the case of boundary observability (2-29), say, from the right boundary point L. In this case, the range of energy levels $E \in \mathbb{R}$ is again split in three different regimes. We fix again $\alpha > 0$ as in (2-31).

First, if $E \ge V(L) + 1$ then Proposition 2.3 estimate (2-8) (taken for $\nu = 1$) yields $(\varepsilon/\sqrt{|E| + 1})|\psi'(L)| \ge C$, which concludes the proof in that case.

Second, we consider the case $V(L) - \alpha^2 \le E \le V(L) + 1$. We remark that we have again, by the definition of α and x_+,

$$L - \tfrac{1}{2}\delta \le x_+(V(L) - \alpha^2) \le x_+(V(L)) = L.$$

Hence, $\big(x_+(V(L)-\alpha^2) - \tfrac{1}{2}\delta,\, x_+(V(L)-\alpha^2) + \tfrac{1}{2}\delta\big) \cap [0, L] \subset (L-\delta, L]$. Applying estimate (2-30) for $z = x_+(V(L) - \alpha^2)$ and using $V(L) - \alpha^2 \le E$, yields

$$\|\psi\|_{L^2(L-\delta,L)} \ge C_0.$$

Using (2-33) integrated in $x \in (L - \delta, L)$ and with $y = L$ implies

$$C_0^2 \le \|\psi\|^2_{L^2(L-\delta,L)} \le C \exp\left(\frac{\delta}{\varepsilon}(2 + 2\|V\|_\infty)\right) \mathcal{E}(L),$$

where $\mathcal{E}(L) = \varepsilon^2 |\psi'(L)|^2$ on account of the Dirichlet boundary condition. This concludes the proof in that case.

Third, if $E \le V(L) - \alpha^2$, the proof follows exactly as in Case 2 above for the proof of (2-28), except that the proof is finished when writing estimate (2-35), at the point $z = L$, together with noticing that $\mathcal{E}^+(L) = \varepsilon^2 |\psi'(L)|^2$, on account of the Dirichlet boundary condition.

This concludes the proof of (2-29) at the right boundary point L, and the proof is the same at the left boundary point 0. □

3. Semiclassical measures for one-dimensional boundary-value problems

The objective of this section is to make precise different properties of semiclassical measures in the presence of boundary (and in dimension one only). The combination of all results proved in this section constitutes a proof of Proposition 2.4. The proof relies only on standard facts of semiclassical analysis for which we refer, e.g., to [Robert 1987; Dimassi and Sjöstrand 1999; Zworski 2012] and semiclassical measures [Gérard 1991; Gérard and Leichtnam 1993; Gérard et al. 1997; Zworski 2012]. Concerning the boundary value problem, we essentially follow [Gérard and Leichtnam 1993] with several major simplifications (due to absence of geometry of the boundary) and some minor complications (due to the family of limited regularity potentials converging in C^1). We thus present a self-contained proof except for usual results from semiclassical analysis and semiclassical measures in 1D. The latter material can be found in [Zworski 2012, Chapters 4 and 5] for instance.

To make the reading easier, we divide the proof into several lemmata.

3A. ***Regularity and traces.*** We begin with standard regularity estimates (see for instance [Gérard and Leichtnam 1993, Lemma 2.1]).

Lemma 3.1. *There is $C > 0$ such that for all $h \in (0, 1)$, $r \in L^2(0, L)$, $\mathcal{V} \in L^\infty(0, L)$ and $\psi \in H^2(0, L) \subset C^1([0, L])$ such that*

$$\psi(0) = \psi(L) = 0, \quad -h^2\psi'' + \mathcal{V}\psi = r \quad \textit{in } \mathcal{D}'((0, L)),$$

we have

$$h^2\|\psi'\|^2_{L^2(0,L)} \le \|\mathcal{V}\|_{L^\infty(0,L)}\|\psi\|^2_{L^2(0,L)} + \|r\|_{L^2(0,L)}\|\psi\|_{L^2(0,L)}, \tag{3-1}$$

and if moreover $\mathcal{V} = \mathcal{V}_1 + \mathcal{V}_2$ with $\mathcal{V}_2 \in C^1([0, L])$ and $h \in (0, 1)$,

$$\begin{aligned} &h^2|\psi'|^2(0^+) + h^2|\psi'|^2(L^-) \\ &\quad \le C(h^{-2}\|\mathcal{V}_1\|^2_{L^\infty(0,L)} + \|\mathcal{V}_2\|_{C^1(0,L)} + 1)\|\psi\|^2_{L^2(0,L)} + Ch^{-2}\|r\|^2_{L^2(0,L)}. \end{aligned} \tag{3-2}$$

Note that all along Section 3, we have $\mathcal{V}_2 = \mathcal{V}_2 \in C^1([0, L])$.

Proof. Multiplying the equation by $\overline{\psi}$, integrating on $(0, L)$ and using an integration by parts, we obtain

$$h^2\int_{(0,L)}|\psi'|^2\,dx + \int_{(0,L)}\mathcal{V}(x)|\psi|^2\,dx = \int_{(0,L)} r\overline{\psi}\,dx.$$

The Cauchy–Schwarz inequality yields (3-1). To prove the second inequality, multiply the equation by $\chi(x)\overline{\psi}'$ with $\chi \in C^\infty_c(\mathbb{R}; [0, 1])$ equal to -1 near 0 and equal to 1 near L. Integrating, we obtain

$$0 = h^2\,\mathrm{Re}\int_{(0,L)}\psi''\chi\overline{\psi}'\,dx - \mathrm{Re}\int_{(0,L)}\mathcal{V}(x)\psi\chi\overline{\psi}'\,dx + \mathrm{Re}\int_{(0,L)} r\chi\overline{\psi}'\,dx \tag{3-3}$$

Next, integrating by parts, we obtain for the first term of (3-3)

$$h^2 \operatorname{Re} \int_{(0,L)} \psi'' \chi \overline{\psi}' \, dx = \tfrac{1}{2} h^2 \int_{(0,L)} \chi \tfrac{1}{dx} d|\psi'|^2 \, dx$$
$$= \tfrac{1}{2} h^2 [|\psi'|^2(0^+) + |\psi'|^2(L^-)] - \tfrac{1}{2} h^2 \int_{(0,L)} \chi' |\psi'|^2 \, dx.$$

Concerning the last term of (3-3), we simply write

$$\left| \operatorname{Re} \int_{(0,L)} r \chi \overline{\psi}' \, dx \right| \le \|r\|_{L^2(0,L)} \|\psi'\|_{L^2(0,L)} \le h^2 \|\psi'\|^2_{L^2(0,L)} + h^{-2} \|r\|^2_{L^2(0,L)}.$$

We may estimate the second term of (3-3) with $\mathcal{V} = \mathcal{V}_1 + \mathcal{V}_2$, $\mathcal{V}_1 \in L^\infty$, $\mathcal{V}_2 \in C^1$ as

$$\left| \operatorname{Re} \int_{(0,L)} \mathcal{V}_1(x) \psi \chi \overline{\psi}' \, dx \right| \le \|\mathcal{V}_1\|_{L^\infty} \|\psi\|_{L^2} \|\psi'\|_{L^2} \le h^{-2} \|\mathcal{V}_1\|^2_{L^\infty} \|\psi\|^2_{L^2} + h^2 \|\psi'\|^2_{L^2},$$

and, integrating by parts, using $\psi(0) = \psi(L) = 0$,

$$\left| \operatorname{Re} \int_{(0,L)} \mathcal{V}_2(x) \psi \chi \overline{\psi}' \, dx \right| = \left| \frac{1}{2} \int_{(0,L)} \mathcal{V}_2(x) \chi \frac{d}{dx} |\psi|^2 \, dx \right|$$
$$= \left| \frac{1}{2} \int_{(0,L)} (\mathcal{V}_2 \chi)' |\psi|^2 \, dx \right| \le C \|\mathcal{V}_2\|_{C^1(0,L)} \|\psi\|^2_{L^2(0,L)}.$$

Combining the above four lines in (3-3) implies

$$h^2 |\psi'|^2(0^+) + h^2 |\psi'|^2(L^-) \le C h^2 \|\psi'\|^2_{L^2(0,L)} + C h^{-2} \|\mathcal{V}_1\|^2_{L^\infty} \|\psi\|^2_{L^2}$$
$$+ C \|\mathcal{V}_2\|_{C^1(0,L)} \|\psi\|^2_{L^2(0,L)} + C h^{-2} \|r\|^2_{L^2(0,L)}.$$

The sought estimate (3-2) then follows from (3-1) and $h \le 1$. ☐

We now extend the potentials $\mathcal{V}_n$, $\mathcal{V}$ as $\mathcal{V}_n, \mathcal{V} \in C^1_c((-1, L+1); \mathbb{R})$ (abusing notation slightly) such that $\|\mathcal{V}_n - \mathcal{V}\|_{C^1(-1,L+1)} \to 0$. We define the operator

$$P_n = -h_n^2 \frac{d^2}{dx^2} + \mathcal{V}_n, \qquad \text{acting on } L^2(\mathbb{R}).$$

Note that P_n is symmetric on $C^\infty_c(\mathbb{R})$ since $\mathcal{V}_n$ are real-valued. The equation in (2-12) together with the jump formula imply that

$$P_n \underline{\psi}_n = -h_n^2 (\psi_n'(0^+) \delta_0 - \psi_n'(L^-) \delta_L) + \underline{r}_n, \qquad \text{in } \mathcal{D}'(\mathbb{R}). \tag{3-4}$$

Corollary 3.2. *Assume* (2-12). *Then*:

(1) *If* $r_n = \mathcal{O}_{L^2(0,L)}(1)$ *and* $\mathcal{V}_n = \mathcal{O}_{L^\infty([0,L])}(1)$, *then* $h_n(\underline{\psi}_n)' = h_n(\underline{\psi}_n')$ *is a bounded sequence in* $L^2(\mathbb{R})$ *and in particular*,

$$\limsup_{n \to +\infty} \|\widehat{\underline{\psi}}_n\|_{L^2(|h_n \xi| \ge R)} \xrightarrow[R \to +\infty]{} 0 \tag{3-5}$$

(*where $\hat{u}$ denotes the classical Fourier transform of u*).

(2) *If $r_n = \mathcal{O}_{L^2(0,L)}(h_n)$ and $\mathcal{V}_n = \mathcal{O}_{C^1([0,L])}(1)$, then $h_n\psi_n'(0^+)$ and $h_n\psi_n'(L^-)$ are bounded sequences in $\mathbb{R}$, and up to a subsequence, there are $\ell_0 \geq 0$ and $\ell_L \geq 0$ so that*

$$|h_n\psi_n'(0^+)|^2 \to \ell_0, \quad |h_n\psi_n'(L^-)|^2 \to \ell_L. \tag{3-6}$$

Moreover, we have

$$\limsup_{n\to+\infty} \|h_n \widehat{\underline{\psi}'_n}\|_{L^2(|h_n\xi|\geq R)} \xrightarrow[R\to+\infty]{} 0. \tag{3-7}$$

Properties (3-5) and (3-7) say that sequences $\underline{\psi}_n$ and $h_n\underline{\psi_n'})$ are h_n-oscillating, respectively. This means that the scale h_n "captures the maximal oscillation rate of the sequence".

Proof. Using (3-1) (applied to ψ_n) together with the fact in (2-12) that ψ_n is normalized in L^2, and the assumption $r_n = \mathcal{O}_{L^2(0,L)}(1)$, we obtain that $h_n\psi_n'$ is bounded in $L^2(0, L)$, whence the first statement, since $(\underline{\psi}_n)' = (\underline{\psi_n'})$ thanks to the Dirichlet boundary condition. The Plancherel formula then implies that

$$\|\widehat{\underline{\psi}}_n\|_{L^2(|h_n\xi|\geq R)} \leq (2\pi)^{-1}R^{-1}\|h_n\underline{\psi_n'}\|_{L^2} \leq CR^{-1} \xrightarrow[R\to+\infty]{} 0.$$

The fact that $h_n\psi_n'(0^+)$ and $h_n\psi_n'(L^-)$ are bounded directly follows from (3-2) together with the fact that $h_n^{-1}\|r_n\|_{L^2(0,L)}$ and $\|\mathcal{V}_n\|_{C^1([0,L])}$ are bounded and ψ_n is normalized.

We finally consider the oscillation property for the sequence $h_n\underline{\psi_n'}$. Taking $\alpha \in (1/2, 1)$, using (3-4) and the Plancherel formula, we obtain

$$\begin{aligned}
\|\widehat{h_n\underline{\psi_n'}}\|_{L^2(|h_n\xi|\geq R)} &\leq R^{-1+\alpha}\| |h_n\xi|^{-\alpha} h_n^2 (\widehat{\underline{\psi}}_n)''\|_{L^2(|h_n\xi|\geq R)} \\
&\leq 2R^{-1+\alpha}h_n^2\Big(|\psi_n'(0^+)| + |\psi_n'(L^-)|\Big)\| |h_n\xi|^{-\alpha}\hat{\delta}_0\|_{L^2(|h_n\xi|\geq R)} \\
&\qquad + (2\pi)^{-1}R^{-1}\|\mathcal{V}_n\underline{\psi}_n - r_n\|_{L^2}.
\end{aligned}$$

Since

$$\begin{aligned}
\| |h_n\xi|^{-\alpha}\hat{\delta}_0\|_{L^2(|h_n\xi|\geq R)} &= \left(\int_{|h_n\xi|\geq R} |h_n\xi|^{-2\alpha}\, d\xi\right)^{1/2} \\
&= h_n^{-1/2}\sqrt{2}\left(\int_R^\infty |\eta|^{-2\alpha}\, d\eta\right)^{1/2} = C_\alpha h_n^{-1/2} R^{-\alpha+1/2},
\end{aligned}$$

we then deduce (3-7) from the fact that $h_n(|\psi_n'(0^+)| + |\psi_n'(L^-)|)$ and $\|\mathcal{V}_n\underline{\psi}_n - r_n\|_{L^2}$ are bounded. ☐

3B. ***Localization in the characteristic set.*** The existence of semiclassical measures μ associated to $(\underline{\psi}_n, h_n)_{n\in\mathbb{N}}$ as in (2-13) is classical, see for example [Zworski 2012, Theorem 5.2]. In this section, we explain how the fact that ψ_n solves (2-12) (or rather $\underline{\psi}_n$ solves (3-4)) relates the associated limit measures μ to the classical Hamiltonian

$$p(x,\xi) = \xi^2 + \mathcal{V}(x).$$

Note that in the case $\mathcal{V} \in C^\infty(\mathbb{R})$ and $\mathcal{V}_n = \mathcal{V}$, the function $p(x,\xi)$ is the semiclassical principal symbol of the operator P_n. We also denote by $H_p(x,\xi) := 2\xi\,\partial_x - \mathcal{V}'(x)\partial_\xi$ the Hamiltonian flow of p. Localization and flow invariance properties for measures μ *away from the boundary* are proved, e.g., in [Zworski 2012, Theorem 5.5] assuming C^∞ regularity. Limited regularity is considered in [Burq 1997a]. Here, we make precise these proofs in the case of the Dirichlet boundary condition and of a family of potentials converging in C^1 regularity.

Lemma 3.3. *Assume* (2-12) *with* $r_n = \mathcal{O}_{L^2(0,L)}(1)$ *and* $\mathcal{V}_n = \mathcal{O}_{L^\infty([0,L])}(1)$. *Then, the measure* μ *in* (2-13) *is a probability measure supported in the set* $[0, L] \times \mathbb{R}_\xi$.

Proof. To prove that μ is a probability measure, we take $\chi, \chi_L \in C_c^\infty(\mathbb{R}; [0,1])$ such that $\chi = 1$ in a neighborhood of 0, $\chi_L = 1$ in a neighborhood of $[0, L]$, and write (using $\operatorname{supp}(\underline{\psi}_n) \subset [0, L]$)

$$\begin{aligned} 1 = \|\underline{\psi}_n\|_{L^2(\mathbb{R})} &= \|\chi_L \underline{\psi}_n\|_{L^2(\mathbb{R})} \\ &\le \|\chi(h_n D/R)\chi_L\underline{\psi}_n\|_{L^2(\mathbb{R})} + \|(1-\chi(h_n D/R))\underline{\psi}_n\|_{L^2(\mathbb{R})}. \end{aligned}$$

Using item (1) of Corollary 3.2, we have

$$\limsup_{n\to+\infty} \|(1-\chi(h_n D/R))\underline{\psi}_n\|_{L^2(\mathbb{R})} \xrightarrow[R\to+\infty]{} 0,$$

and pseudodifferential calculus yields

$$\|\chi(h_n D/R)\chi_L\underline{\psi}_n\|^2_{L^2(\mathbb{R})} \xrightarrow[n\to+\infty]{} \langle \mu, \chi_L^2 \otimes \chi^2(\cdot/R)\rangle.$$

We deduce from the above two lines that

$$1 \le \langle \mu, \chi_L^2 \otimes \chi^2(\cdot/R)\rangle + o_{R\to\infty}(1),$$

and hence, $1 \le \langle \mu, \chi_L^2 \otimes 1\rangle \le 1$ by dominated convergence. This proves both that μ is a probability measure, and that $\operatorname{supp}(\mu) \subset [0, L] \times \mathbb{R}_\xi$. ☐

Lemma 3.4. *Assume condition* (2-12) *with* $r_n = \mathcal{O}_{L^2(0,L)}(h_n)$, $\mathcal{V}_n = \mathcal{O}_{C^1([0,L])}(1)$ *and* $\|\mathcal{V}_n - \mathcal{V}\|_{C^0(-1,L+1)} \to 0$. *Then, the measure* μ *in* (2-13) *is a probability measure supported in the set* $\{p(x,\xi) = 0\} \cap [0, L] \times \mathbb{R}_\xi$. *Moreover, for all* $a \in C_c^\infty(\mathbb{R}^2; \mathbb{R})$ *such that* $a = 1$ *in a neighborhood of* $\{p(x,\xi) = 0\} \cap [0, L] \times \mathbb{R}_\xi$, *we have* $\|\mathrm{Op}_{h_n}(1-a)\underline{\psi}_n\|_{L^2(\mathbb{R})} \to 0$ *as* $n \to +\infty$.

Note that the compactness of the set $\{p(x,\xi)=0\}\cap[0,L]\times\mathbb{R}_\xi\subset[0,L]\times[-A,A]$, with $A=\sqrt{-\min_{[0,L]}\mathcal{V}}$, thus implies that $\mu\in\mathcal{E}'(\mathbb{R}^2)$, i.e., has compact support.

Note also that the assumption that $r_n=\mathcal{O}_{L^2(0,L)}(h_n)$ can be weakened to $r_n=\mathcal{O}_{L^2(0,L)}(h_n^{1/2+\varepsilon})$ for any $\varepsilon>0$ for the same proof to work (using directly (3-2) instead of Corollary 3.2(2)). We did not try to optimize the proof in this respect.

Proof. Let $a\in C_c^\infty(\mathbb{R}_x\times\mathbb{R}_\xi)$. Applying $A=\mathrm{Op}_{h_n}(a)$ to (3-4) and taking the inner product with $\underline{\psi}_n$, we obtain (after having noticed that $\mathrm{Op}_{h_n}(a)$ is a smoothing operator),

$$(AP_n\underline{\psi}_n,\underline{\psi}_n)_{L^2(\mathbb{R})}=-h_n^2\big(A(\psi_n'(0^+)\delta_0-\psi_n'(L^-)\delta_L),\underline{\psi}_n\big)_{L^2(\mathbb{R})}+\boldsymbol{o}(1).$$

Corollary 3.2(2) (that is, boundedness of $h_n|\psi_n'(0)|$) and continuity of the trace $H^{1/2+\varepsilon}(\mathbb{R})\to\mathbb{C}$, $u\mapsto u(0)$, gives

$$\begin{aligned}h_n^2\big|\big(A(\psi_n'(0)\delta_0),\underline{\psi}_n\big)_{L^2}\big|&=h_n^2|\psi_n'(0)|\,|(A(\delta_0),\underline{\psi}_n)_{L^2}i|\\&=h_n^2|\psi_n'(0)|\,|\langle\delta_0,{}^tA\overline{\underline{\psi}}_n\rangle_{\mathcal{S}'(\mathbb{R}),\mathcal{S}(\mathbb{R})}|\\&\le Ch_n|(A^*\underline{\psi}_n)(0)|\le C_\varepsilon h_n\|A^*\underline{\psi}_n\|_{H^{1/2+\varepsilon}(\mathbb{R})}\\&\le C_\varepsilon h_n\|\underline{\psi}_n\|_{H^{1/2+\varepsilon}(\mathbb{R})},\end{aligned}$$

after having used uniform boundedness of A^* on $H^s(\mathbb{R})$ (classical Sobolev spaces). The last term is of order $\mathcal{O}_\varepsilon(h_n^{1/2-\varepsilon})$ by interpolation in Corollary 3.2 between L^2 and H^1, and hence converges to zero for $\varepsilon<1/2$. The same convergence to zero holds for $h_n^2\big|\big(A(\psi_n'(L)\delta_L),\underline{\psi}_n\big)_{L^2}\big|$, and we have thus proved that $(AP_n\underline{\psi}_n,\underline{\psi}_n)_{L^2(\mathbb{R})}\to 0$ for all $a\in C_c^\infty(\mathbb{R}^2)$.

For $\epsilon>0$, let $\rho_\epsilon(x)=(1/\epsilon)\rho(x/\epsilon)$ be an approximation of identity ($\rho\in C_c^\infty(\mathbb{R})$, $\rho\ge 0$, $\int_{\mathbb{R}}\rho=1$). We define $\mathcal{V}^\epsilon:=\rho_\epsilon*\mathcal{V}$ and $\mathcal{V}_n^\epsilon:=\rho_\epsilon*\mathcal{V}_n$. We notice that for any $\epsilon>0$, we have (under the assumptions of the lemma) that $\mathcal{V}_n^\epsilon=\mathcal{O}_{\epsilon,C^1([0,L])}(1)$ and $\|\mathcal{V}_n^\epsilon-\mathcal{V}^\epsilon\|_{C^0(-1,L+1)}\to 0$ as $n\to+\infty$. Moreover, $\|\mathcal{V}^\epsilon-\mathcal{V}\|_{C^0(-1,L+1)}\to 0$ as $\epsilon\to 0$. We now write

$$\begin{aligned}&(A(h_n^2D_x^2+\mathcal{V}_n^\epsilon)\underline{\psi}_n,\underline{\psi}_n)_{L^2(\mathbb{R})}\\&\qquad=(AP_n\underline{\psi}_n,\underline{\psi}_n)_{L^2(\mathbb{R})}+(A(\mathcal{V}_n^\epsilon-\mathcal{V}_n)\underline{\psi}_n,\underline{\psi}_n)_{L^2(\mathbb{R})}.\end{aligned}\tag{3-8}$$

The first term in the right hand-side converges to zero, whereas the second term is bounded by

$$\|A\|_{\mathcal{L}(L^2)}\|\mathcal{V}_n^\epsilon-\mathcal{V}_n\|_{C^0}\to\|A\|_{\mathcal{L}(L^2)}\|\mathcal{V}^\epsilon-\mathcal{V}\|_{C^0},\qquad\text{as } n\to+\infty.$$

Pseudodifferential calculus (composition rule) in the left hand-side of (3-8), together with the fact that $\|\mathcal{V}_n^\epsilon-\mathcal{V}^\epsilon\|_{C^0(-1,L+1)}\to 0$ as $n\to+\infty$ and the definition of μ imply that it converges towards $\langle\mu,(|\xi|^2+\mathcal{V}^\epsilon)a\rangle$. We have thus obtained that

$$|\langle\mu,(|\xi|^2+\mathcal{V}^\epsilon)a\rangle|\le\|A\|_{\mathcal{L}(L^2)}\|\mathcal{V}^\epsilon-\mathcal{V}\|_{C^0}\to 0,\qquad\text{as }\epsilon\to 0^+.$$

Since $\langle \mu, (|\xi|^2+\mathcal{V}^\epsilon)a\rangle \underset{\epsilon\to 0^+}{\longrightarrow} \langle \mu, (|\xi|^2+\mathcal{V})a\rangle$, we have obtained $\langle \mu, pa\rangle = 0$ for all $a \in C_c^\infty(\mathbb{R}^2)$. This implies that $\operatorname{supp}(\mu) \subset p^{-1}(\{0\})$, and concludes the proof of first statement of the lemma.

Concerning the second statement, using pseudodifferential calculus and the normalization of the ψ_n, we have

$$\begin{aligned}\|\mathrm{Op}_{h_n}(1-a)\underline{\psi}_n\|_{L^2(\mathbb{R})}^2 &= \big(\mathrm{Op}_{h_n}((1-a)^2)\underline{\psi}_n, \underline{\psi}_n\big)_{L^2(\mathbb{R})} + \mathcal{O}(h_n)\\ &= \|\underline{\psi}_n\|_{L^2(\mathbb{R})}^2 + (\mathrm{Op}_{h_n}(-2a+a^2)\underline{\psi}_n, \underline{\psi}_n)_{L^2(\mathbb{R})} + \mathcal{O}(h_n)\\ &\to 1 + \langle \mu, -2a+a^2\rangle.\end{aligned}$$

Recalling that $a = 1$ in a neighborhood of $\operatorname{supp}(\mu)$ and that μ is a probability measure, we have $\langle \mu, -2a+a^2\rangle = \langle \mu, -1\rangle = -1$, whence the sought result. □

3C. ***Propagation of the measure.*** We next want to investigate propagation properties for the measure μ, and start with a propagation statement "away from the boundary".

Lemma 3.5. *Under the assumptions of Lemma 3.4, with $\mathcal{V} \in C_c^1([-1, L+1])$, the distribution $H_p\mu$ defined by*

$$\langle H_p\mu, a\rangle_{\mathcal{D}'(\mathbb{R}^2),\mathcal{D}(\mathbb{R}^2)} := -\langle \mu, (2\xi\partial_x - \mathcal{V}'\partial_\xi)a\rangle_{\mathcal{M}(\mathbb{R}^2),C^0(\mathbb{R}^2)}, \quad a \in C_c^\infty(\mathbb{R}^2), \tag{3-9}$$

is of order at most 1. *If moreover $r_n = o_{L^2(0,L)}(h_n)$ and $\|\mathcal{V}_n - \mathcal{V}\|_{C^1(-1,L+1)} \to 0$, then*

$$\operatorname{supp}(H_p\mu) \subset \{\xi^2 + \mathcal{V}(x) = 0\} \cap (\{0, L\} \times \mathbb{R}_\xi). \tag{3-10}$$

The support statement (3-10) in Lemma 3.5 says that the measure μ is H_p-invariant "away from the boundary" of the interval $[0, L]$. The proof is classical in case $\mathcal{V}_n = \mathcal{V}$ is smooth, but requires some care in the present limited regularity setting.

Proof. Since $\mathcal{V} \in C^1(\mathbb{R})$, we have from definition (3-9) that $|\langle H_p\mu, a\rangle_{\mathcal{D}'(\mathbb{R}^2),\mathcal{D}(\mathbb{R}^2)}| \leq C_K\|a\|_{C^1(\mathbb{R}^2)}$ for all $a \in C_c^\infty(K)$, $K \subset \mathbb{R}^2$ compact. Hence, $H_p\mu$ is a distribution of order 1.

Let us now turn to the support property (3-10). Lemma 3.4 first implies that $H_p\mu$ is supported in the set $\{\xi^2 + \mathcal{V}(x) = 0\} \cap ([0, L] \times \mathbb{R}_\xi)$. It is therefore sufficient to prove that

$$\langle H_p\mu, a\rangle_{\mathcal{D}'((0,L)\times\mathbb{R}_\xi),\mathcal{D}((0,L)\times\mathbb{R}_\xi)} = 0 \tag{3-11}$$

for any $a \in C_c^\infty((0, L) \times \mathbb{R}_\xi)$ (with $H_p\mu$ defined in the sense of (3-9)). By the density of the vector spaces spanned by tensor products of smooth functions in $C_c^1((0, L) \times \mathbb{R}_\xi)$, it is enough to prove (3-11) for test functions a under the form $a(x, \xi) = \chi_1(x)\chi_2(\xi)$ with $\chi_1 \in C_c^\infty(0, L)$ and $\chi_2 \in C_c^\infty(\mathbb{R}_\xi)$.

As in the proof of Lemma 3.4, for $\epsilon > 0$, we let $\rho_\epsilon(x) = (1/\epsilon)\rho(x/\epsilon)$ be an approximation of identity and define $\mathcal{V}^\epsilon := \rho_\epsilon * \mathcal{V}$ and $\mathcal{V}_n^\epsilon := \rho_\epsilon * \mathcal{V}_n$. We notice that for any $\epsilon > 0$, we have $\|\mathcal{V}_n^\epsilon - \mathcal{V}^\epsilon\|_{C^1(-1,L+1)} \to 0$ as $n \to +\infty$ and $\|\mathcal{V}^\epsilon - \mathcal{V}\|_{C^1(-1,L+1)} \to 0$ as $\epsilon \to 0$. We also set $A = \chi_2(h_n D_x)\chi_1(x)$. The proof of (3-11) consists of computing in two different ways the limit of the quantity

$$L_A(h_n) := \frac{1}{h_n}\langle A P_n \underline{\psi}_n, \overline{\underline{\psi}}_n\rangle_{\mathcal{S}(\mathbb{R}),\mathcal{S}'(\mathbb{R})} - \frac{1}{h_n}\langle A \underline{\psi}_n, P_n \overline{\underline{\psi}}_n\rangle_{\mathcal{S}(\mathbb{R}),\mathcal{S}'(\mathbb{R})}, \tag{3-12}$$

which makes sense since $A\underline{\psi}_n \in \mathcal{S}(\mathbb{R})$ and $AP_n\underline{\psi}_n \in \mathcal{S}(\mathbb{R})$. Using that P_n is formally selfadjoint together with pseudodifferential rules, we have on the one hand

$$\begin{aligned} L_A(h_n) &= \frac{1}{h_n}([A, P_n]\underline{\psi}_n, \underline{\psi}_n)_{L^2} \\ &= \frac{1}{h_n}([A, P_n^\epsilon]\underline{\psi}_n, \underline{\psi}_n)_{L^2} + \frac{1}{h_n}([A, P_n - P_n^\epsilon]\underline{\psi}_n, \underline{\psi}_n)_{L^2}, \end{aligned} \tag{3-13}$$

where $P_n^\epsilon = -h_n^2 d^2/dx^2 + \mathcal{V}_n^\epsilon$. We first study the first term in (3-13). For fixed $\epsilon > 0$, $[A, P_n^\epsilon]$ is a semiclassical operator, so we can write

$$\begin{aligned} \frac{1}{h_n}([A, P_n^\epsilon]\underline{\psi}_n, \underline{\psi}_n)_{L^2} &\underset{n\to+\infty}{\longrightarrow} \left\langle \mu, \frac{1}{i}\{a, p^\epsilon\}\right\rangle \\ &= -\frac{1}{i}\langle \mu, H_{p^\epsilon} a\rangle \underset{\epsilon\to 0^+}{\longrightarrow} -\frac{1}{i}\langle \mu, H_p a\rangle. \end{aligned} \tag{3-14}$$

Concerning the second term in (3-13), we have $(1/h_n)([A, P_n - P_n^\epsilon]\underline{\psi}_n, \underline{\psi}_n)_{L^2} = (1/h_n)([A, \mathcal{V}_n - \mathcal{V}_n^\epsilon]\underline{\psi}_n, \underline{\psi}_n)_{L^2}$ where, using the product form of A,

$$\begin{aligned} [A, \mathcal{V}_n - \mathcal{V}_n^\epsilon] &= \chi_2(h_n D_x)[\chi_1(x), \mathcal{V}_n - \mathcal{V}_n^\epsilon] + [\chi_2(h_n D_x), \mathcal{V}_n - \mathcal{V}_n^\epsilon]\chi_1(x) \\ &= [\chi_2(h_n D_x), \mathcal{V}_n - \mathcal{V}_n^\epsilon]\chi_1(x). \end{aligned}$$

As a consequence, recalling that $\underline{\psi}_n$ is normalized in $L^2(\mathbb{R})$, we obtain

$$\begin{aligned} &\left|\frac{1}{h_n}([A, P_n - P_n^\epsilon]\underline{\psi}_n, \underline{\psi}_n)_{L^2}\right| \\ &\qquad \le \frac{1}{h_n}\|[\chi_2(h_n D_x), \mathcal{V}_n - \mathcal{V}_n^\epsilon]\|_{\mathcal{L}(L^2)} \\ &\qquad \le C\|\partial_x(\mathcal{V}_n - \mathcal{V}_n^\epsilon)\|_{L^\infty} \underset{n\to+\infty}{\longrightarrow} C\|\partial_x(\mathcal{V} - \mathcal{V}^\epsilon)\|_{L^\infty} \underset{\epsilon\to 0^+}{\longrightarrow} 0, \end{aligned} \tag{3-15}$$

where we used Lemma 3.12 below with $\varepsilon = h_n$. Combining (3-13) with (3-14) and (3-15), and letting $n \to +\infty$ and then $\epsilon \to 0^+$ (recall that $L_A(h_n)$ is independent of ϵ), we have obtained for $a(x, \xi) = \chi_1(x)\chi_2(\xi)$ with $\chi_1 \in C_c^\infty(0, L)$ and $\chi_2 \in C_c^\infty(\mathbb{R}_\xi)$,

$$L_A(h_n) \underset{n\to+\infty}{\longrightarrow} -\frac{1}{i}\langle \mu, H_p a\rangle. \tag{3-16}$$

We now compute $L_A(h_n)$ in (3-12) using (3-4). Using moreover that A equals $\chi_2(h_n D_x)\chi_1(x)$ with $\mathrm{supp}(\chi_1) \subset (0, L)$, together with the pseudolocality of A^*, we obtain

$$L_A(h_n) = \frac{1}{h_n}(A\underline{r}_n, \underline{\psi}_n)_{L^2(\mathbb{R})} - \frac{1}{h_n}(\underline{\psi}_n, A^*\underline{r}_n)_{L^2(\mathbb{R})} \\ + \mathcal{O}(h_n^\infty)\|\underline{\psi}_n\|_{L^2(\mathbb{R})}(|\psi_n'(0)| + |\psi_n'(L)|).$$

Then, L^2 normalization of $\underline{\psi}_n$ together with L^2 boundedness of A and the assumption $r_n = o_{L^2(0,L)}(h_n)$ imply that the first two terms converge to zero as $n \to +\infty$. Item (2) of Corollary 3.2 implies that the last term converges to zero as well. Combined with (3-16), we have thus obtained (3-11) for all a in product form, and finally for all $a \in C_c^\infty((0, L) \times \mathbb{R}_\xi)$ by density. This concludes the proof of (3-10), and thus of the lemma. □

As a preliminary for propagation properties, we first prove that the convergence in (2-13) holds not only for compactly supported symbols a but also for symbols of order 2. For $m \in \mathbb{R}$, we shall say that $a \in S^m(\mathbb{R}^2)$ if $|\partial_x^\alpha \partial_\xi^\beta a(x, \xi, h)| \le C_{\alpha,\beta}\langle\xi\rangle^{m-\beta}$ for all $\alpha, \beta \in \mathbb{N}$, $(x, \xi) \in \mathbb{R}^2$, $h \in (0, 1]$ (note that such symbols depend implicitly on h, with uniform bounds).

Lemma 3.6. *Assume* (2-12) *with* $r_n = \mathcal{O}_{L^2(0,L)}(h_n)$, $\mathcal{V}_n = \mathcal{O}_{C^1([0,L])}(1)$ *and* $\|\mathcal{V}_n - \mathcal{V}\|_{C^0(-1,L+1)} \to 0$. *Then, for all* $a \in S^2(T^*\mathbb{R})$ *independent of* h, *we have*

$$\langle \mathrm{Op}_{h_n}(a)\underline{\psi}_n, \underline{\psi}_n\rangle_{H^{-1},H^1} \to \langle \mu, a\rangle_{\mathcal{E}'(\mathbb{R}^2),\mathcal{E}(\mathbb{R}^2)}. \tag{3-17}$$

For $s \in \mathbb{R}$, we denote by

$$\|u\|_{H_h^s}^2 = \int_{\mathbb{R}} (1+h^2|\xi|^2)^{s/2}|\hat{u}(\xi)|^2\,d\xi, \quad \text{where } \hat{u}(\xi) = \int_{\mathbb{R}} e^{-ix\xi}u(x)\,dx,\ u \in \mathcal{S}(\mathbb{R}),$$

the usual semiclassical Sobolev norm. Note that in expression (3-17), $\mathrm{Op}_{h_n}(a)\underline{\psi}_n$ is in H^{-1}, and is bounded uniformly in $H_{h_n}^{-1}$ since $\underline{\psi}_n$ and $h_n\underline{\psi}_n'$ are bounded in $L^2(\mathbb{R})$, see Corollary 3.2. In particular, one can replace $a \in S^2(T^*\mathbb{R})$ independent of h, in (3-17), by $a + \varepsilon(h)b$ with $b \in S^2(T^*\mathbb{R})$ possibly depending on h (with uniformly bounded seminorms in this class) and $\varepsilon(h) \to 0$.

Here and below, we take the convention that duality brackets between H^{-1} and H^1 or between H_h^{-1} and H_h^1 in (3-17) are $\mathbb{C}$-linear in the first variable and $\mathbb{C}$-antilinear in the second variable.

Before giving the proof of Lemma 3.6, we give the following corollary which is actually the last item of Proposition 2.4 (and is valid under less restrictive assumptions).

Corollary 3.7. *We have* $|\underline{\psi}_n(x)|^2\,dx \rightharpoonup \mathfrak{m}$ *where the nonnegative Radon measure* $\mathfrak{m}$ *on* $\mathbb{R}$ *is given by* $\mathfrak{m} = \pi^*\mu$.

Proof of Corollary 3.7 from Lemma 3.6. For any $\varphi \in C_c^\infty(\mathbb{R})$, we can apply Lemma 3.6 to $a = \varphi \circ \pi \in S^0(T^*\mathbb{R})$, to obtain

$$\int_{\mathbb{R}} \varphi(x)|\underline{\psi}_n(x)|^2\,dx \to \langle \mu, \varphi \circ \pi \rangle_{\mathcal{E}'(\mathbb{R}^2),\mathcal{E}(\mathbb{R}^2)} = \langle \mathfrak{m}, \varphi \rangle_{\mathcal{E}'(\mathbb{R}),\mathcal{E}(\mathbb{R})}$$

by the definition of $\mathfrak{m}$. By density of $C_c^\infty(\mathbb{R})$ in $C_c^0(\mathbb{R})$, this implies the result. □

Proof of Lemma 3.6. We choose $\phi \in C_c^\infty(\mathbb{R}^2)$ real-valued such that $\phi = 1$ in a neighborhood of $\operatorname{supp}(\mu)$. We decompose

$$\begin{aligned}\langle \mathrm{Op}_{h_n}(a)\underline{\psi}_n, \underline{\psi}_n \rangle_{H^{-1},H^1} \\ = \langle \mathrm{Op}_{h_n}(\phi)\,\mathrm{Op}_{h_n}(a)\underline{\psi}_n, \underline{\psi}_n \rangle_{H^{-1},H^1} + \langle (1-\mathrm{Op}_{h_n}(\phi))\,\mathrm{Op}_{h_n}(a)\underline{\psi}_n, \underline{\psi}_n \rangle_{H^{-1},H^1}.\end{aligned}$$

We first notice that we have on the one hand (for any $a \in S^m(\mathbb{R}^2)$ with principal part independent of h)

$$(\mathrm{Op}_{h_n}(\phi)\,\mathrm{Op}_{h_n}(a)\underline{\psi}_n, \underline{\psi}_n)_{L^2} = (\mathrm{Op}_{h_n}(a\phi)\underline{\psi}_n, \underline{\psi}_n)_{L^2} + \mathcal{O}(h_n) \xrightarrow[n\to+\infty]{} \langle \mu, \phi a \rangle = \langle \mu, a \rangle,$$

using pseudodifferential calculus and the support properties of ϕ.

To conclude the proof, it suffices to prove

$$\langle (1-\mathrm{Op}_{h_n}(\phi))\,\mathrm{Op}_{h_n}(a)\underline{\psi}_n, \underline{\psi}_n \rangle_{H^{-1},H^1} \to 0. \tag{3-18}$$

We first prove the intermediate statement that

$$\|(1-\mathrm{Op}_{h_n}(\phi))h_n\underline{\psi_n'}\|_{L^2} \xrightarrow[n\to+\infty]{} 0. \tag{3-19}$$

To prove (3-19), we decompose for $\chi \in C_c^\infty(\mathbb{R})$ equal to one near zero,

$$\begin{aligned}(1-\mathrm{Op}_{h_n}(\phi))h_n\underline{\psi_n'} \\ = (1-\mathrm{Op}_{h_n}(\phi))\chi(R^{-1}h_nD_x)h_n\underline{\psi_n'} + (1-\mathrm{Op}_{h_n}(\phi))(1-\chi(R^{-1}h_nD_x))h_n\underline{\psi_n'},\end{aligned}$$

for R large. For the second term, we have

$$\begin{aligned}\limsup_{n\to+\infty} \|(1-\mathrm{Op}_{h_n}(\phi))(1-\chi(R^{-1}h_nD_x))h_n\underline{\psi_n'}\|_{L^2} \\ \le C \limsup_{n\to+\infty} \|(1-\chi(R^{-1}h_nD_x))h_n\underline{\psi_n'}\|_{L^2} \xrightarrow[R\to+\infty]{} 0,\end{aligned}$$

using (3-7). As for the first term, using that $\underline{\psi}_n$ is supported in $[0, L]$, we write for any $R > 0$,

$$\|(1-\mathrm{Op}_{h_n}(\phi))\chi(R^{-1}h_nD_x)h_n\underline{\psi_n'}\|_{L^2} = (B_R\underline{\psi}_n, \underline{\psi}_n)_{L^2}$$

for $B_R = \chi(x/L)\chi(R^{-1}h_nD_x)(1-\mathrm{Op}_{h_n}(\phi)^*)(1-\mathrm{Op}_{h_n}(\phi))\chi(R^{-1}h_nD_x)$, that is a semiclassical pseudodifferential operator in $\mathrm{Op}_{h_n}(S^0)$. Writing

$$B_R = \mathrm{Op}_h(b_R) + h\,\mathrm{Op}_h(S^{-1})$$

with $b_R(x,\xi)=\chi(x/L)\chi(R^{-1}\xi)^2(1-\phi(x,\xi))^2$, we have obtained

$$(B_R\underline{\psi}_n,\underline{\psi}_n)_{L^2}\to\langle\mu,b_R\rangle=0,$$

since $\phi=1$ in a neighborhood of $\operatorname{supp}(\mu)$, for any $R>0$. Combining the above three lines, we have proved (3-19).

We finally prove (3-18) for $a\in S^2$. Denoting by ${}^t\mathrm{Op}_{h_n}(\phi)$ the transpose of $\mathrm{Op}_{h_n}(\phi)$ for the duality bracket between H_h^{-1} and H_h^1, we have

$$\begin{aligned}
&|\langle(1-\mathrm{Op}_{h_n}(\phi))\,\mathrm{Op}_{h_n}(a)\underline{\psi}_n,\underline{\psi}_n\rangle_{H^{-1},H^1}|\\
&\quad=|\langle\mathrm{Op}_{h_n}(a)\underline{\psi}_n,(1-{}^t\mathrm{Op}_{h_n}(\phi))\underline{\psi}_n\rangle_{H_h^{-1},H_h^1}|\\
&\quad\le\|\mathrm{Op}_{h_n}(a)\underline{\psi}_n\|_{H_h^{-1}}\|(1-{}^t\mathrm{Op}_{h_n}(\phi))\underline{\psi}_n\|_{H_h^1}\\
&\quad\le C\|\underline{\psi}_n\|_{H_h^1}\big(\|(1-{}^t\mathrm{Op}_{h_n}(\phi))h_n\underline{\psi}_n'\|_{L^2}+\|(1-{}^t\mathrm{Op}_{h_n}(\phi))\underline{\psi}_n\|_{L^2}\big)\xrightarrow[n\to+\infty]{}0,
\end{aligned}$$

where we have used (3-19), the fact that $\mathrm{Op}_h(a):H_h^1\to H_h^{-1}$ uniformly in h and Lemma 3.4 for the last convergence. This concludes the proof of the lemma. ☐

Note that the right hand-side of (3-17) makes sense for any $a\in C^\infty(\mathbb{R}^2)$, using that μ is compactly supported. Convergence in (3-17) however uses $a\in S^2(\mathbb{R}^2)$.

Lemma 3.8. *Assume* (2-12) *with* $r_n=o_{L^2(0,L)}(h_n)$ *and* $\|\mathcal{V}_n-\mathcal{V}\|_{C^1(-1,L+1)}\to 0$. *Then, for all* $a_0,a_1\in C_0^\infty(\mathbb{R}_x)$ *real valued and* $a(x,\xi)=a_0(x)+a_1(x)\xi$, *the measure* μ *in* (2-13) *satisfies*

$$\langle\mu,H_pa\rangle=-\ell_0a_1(0)+\ell_La_1(L).$$

In the following proofs we need a function χ such that

$$\chi\in C_c^\infty((-2,2);[0,1]),\quad \chi\text{ even},\quad \chi(s)=1\text{ for }|s|\le 1, \tag{3-20}$$

When $\varepsilon>0$, is given, we will denote by $\chi_\varepsilon(s)$ the function $\chi(\varepsilon s)$.

The proof of Lemma 3.8 follows the general scheme of that of Lemma 3.5, but we now need extra care to handle the boundary terms.

Proof. We set $A=\chi(h_n^3D_x)A_0$ with $A_0:=a_0(x)+a_1(x)h_nD_x$. As in the proofs of Lemmata 3.4 and 3.5, for $\epsilon>0$, we let $\rho_\epsilon(x)=(1/\epsilon)\rho(x/\epsilon)$ be an approximation of identity and define $\mathcal{V}^\epsilon:=\rho_\epsilon*\mathcal{V}$ and $\mathcal{V}_n^\epsilon:=\rho_\epsilon*\mathcal{V}_n$. We notice that for any $\epsilon>0$, we have $\|\mathcal{V}_n^\epsilon-\mathcal{V}^\epsilon\|_{C^1(-1,L+1)}\to 0$ as $n\to+\infty$ and $\|\mathcal{V}^\epsilon-\mathcal{V}\|_{C^1(-1,L+1)}\to 0$ as $\epsilon\to 0$.

The proof consists of computing in two different ways the limit of the quantity

$$L_A(h_n):=\frac{1}{h_n}\langle AP_n\underline{\psi}_n,\overline{\underline{\psi}}_n\rangle_{\mathcal{S}(\mathbb{R}),\mathcal{S}'(\mathbb{R})}-\frac{1}{h_n}\langle A\underline{\psi}_n,P_n\overline{\underline{\psi}}_n\rangle_{\mathcal{S}(\mathbb{R}),\mathcal{S}'(\mathbb{R})}, \tag{3-21}$$

which makes sense since $A\underline{\psi}_n \in \mathcal{S}(\mathbb{R})$ and $AP_n\underline{\psi}_n \in \mathcal{S}(\mathbb{R})$. Using that P_n is formally selfadjoint together with pseudodifferential rules, we have on the one hand

$$\begin{aligned} L_A(h_n) &= \frac{1}{h_n}([A, P_n]\underline{\psi}_n, \underline{\psi}_n)_{L^2} \\ &= \frac{1}{h_n}([A, P_n^\epsilon]\underline{\psi}_n, \underline{\psi}_n)_{L^2} + \frac{1}{h_n}([A, P_n - P_n^\epsilon]\underline{\psi}_n, \underline{\psi}_n)_{L^2}, \end{aligned} \tag{3-22}$$

where $P_n^\epsilon = -h_n^2 d^2/dx^2 + \mathcal{V}_n^\epsilon$.

We first study the first term in (3-22). Recalling $A = \chi(h_n^3 D_x)A_0$, we decompose

$$[A, P_n^\epsilon] = \chi(h_n^3 D_x)[A_0, P_n^\epsilon] + [\chi(h_n^3 D_x), P_n^\epsilon]A_0.$$

On the one hand, we have $[\chi(h_n^3 D_x), P_n^\epsilon] = [\chi(h_n^3 D_x), \mathcal{V}_n^\epsilon] = \mathcal{O}_{\mathcal{L}(L^2)}(h_n^3)$ according to pseudodifferential calculus (or Lemma 3.12 below), and thus

$$\left| \frac{1}{h_n}([\chi(h_n^3 D_x), P_n^\epsilon]A_0\underline{\psi}_n, \underline{\psi}_n)_{L^2} \right| \le C_\epsilon h_n^2 \|A_0\underline{\psi}_n\|_{L^2} \|\underline{\psi}_n\|_{L^2} = \mathcal{O}_\epsilon(h_n^2),$$

according to Corollary 3.2 and the definition of A_0. On the other hand, we have

$$\frac{1}{h_n}\langle \chi(h_n^3 D_x)[A_0, P_n^\epsilon]\underline{\psi}_n, \underline{\psi}_n\rangle_{H^{-1},H^1} = \left\langle \frac{1}{h_n}[A_0, P_n^\epsilon]\underline{\psi}_n, \underline{\psi}_n \right\rangle_{H^{-1},H^1} + R_n^\epsilon,$$

with

$$\begin{aligned} |R_n^\epsilon| &= \left| \left\langle \frac{1}{h_n}[A_0, P_n^\epsilon]\underline{\psi}_n, (1 - \chi(h_n^3 D_x))\underline{\psi}_n \right\rangle_{H^{-1},H^1} \right| \\ &\le C_\varepsilon \|\underline{\psi}_n\|_{H^1_{h_n}} \|(1 - \chi(h_n^3 D_x))\underline{\psi}_n\|_{H^1_{h_n}}, \end{aligned}$$

using that $(1/h_n)[A_0, P_n^\epsilon] \in \mathrm{Op}_{h_n}(S^2)$ (actually, it is a semiclassical differential operator of order 2, that is a finite sum of terms of the form $c_{jk}(x)h_n^k D_x^j$, $k \ge j$, $0 \le j \le 2$). We conclude that R_n^ϵ converges to zero as $n \to \infty$ thanks to Corollary 3.2.

Combining the above lines and using Lemma 3.6 (and the remark thereafter), we have obtained that the first term in (3-22) satisfies

$$\begin{aligned} \frac{1}{h_n}([A, P_n^\epsilon]\underline{\psi}_n, \underline{\psi}_n)_{L^2} &= \left\langle \frac{1}{h_n}[A_0, P_n^\epsilon]\underline{\psi}_n, \underline{\psi}_n \right\rangle_{H^{-1},H^1} + o_\epsilon(1) \\ &\xrightarrow[n\to+\infty]{} \left\langle \mu, \frac{1}{i}\{a, p^\epsilon\} \right\rangle = -\frac{1}{i}\langle \mu, H_{p^\epsilon} a\rangle \xrightarrow[\epsilon\to 0^+]{} -\frac{1}{i}\langle \mu, H_p a\rangle. \end{aligned} \tag{3-23}$$

Concerning the second term in (3-22), we have $(1/h_n)([A, P_n - P_n^\epsilon]\underline{\psi}_n, \underline{\psi}_n)_{L^2} = (1/h_n)([A, \mathcal{V}_n - \mathcal{V}_n^\epsilon]\underline{\psi}_n, \underline{\psi}_n)_{L^2}$, where

$$\begin{aligned}[A, \mathcal{V}_n - \mathcal{V}_n^\epsilon] &= \chi(h_n^3 D_x)[(a_0(x) + a_1(x)h_n D_x), \mathcal{V}_n - \mathcal{V}_n^\epsilon] \\ &\qquad + [\chi(h_n^3 D_x), \mathcal{V}_n - \mathcal{V}_n^\epsilon](a_0(x) + a_1(x)h_n D_x) \\ &= \chi(h_n^3 D_x)a_1(x)\frac{h_n}{i}\partial_x(\mathcal{V}_n - \mathcal{V}_n^\epsilon) \\ &\qquad + [\chi(h_n^3 D_x), \mathcal{V}_n - \mathcal{V}_n^\epsilon](a_0(x) + a_1(x)h_n D_x).\end{aligned}$$

As a consequence, recalling that $\underline{\psi}_n$ and $h_n\underline{\psi}_n'$ are bounded in L^2, we obtain

$$\begin{aligned}&\left|\frac{1}{h_n}([A, P_n - P_n^\epsilon]\underline{\psi}_n, \underline{\psi}_n)_{L^2}\right| \\ &\quad = \left|\frac{1}{i}\big(a_1(x)\partial_x(\mathcal{V}_n - \mathcal{V}_n^\epsilon)\underline{\psi}_n, \chi(h_n^3 D_x)\underline{\psi}_n\big)_{L^2}\right. \\ &\qquad\qquad \left. - \frac{1}{h_n}\big((a_0(x) + a_1(x)h_n D_x)\underline{\psi}_n, [\chi(h_n^3 D_x), \mathcal{V}_n - \mathcal{V}_n^\epsilon]\underline{\psi}_n\big)_{L^2}\right| \\ &\quad \le C\|\partial_x(\mathcal{V}_n - \mathcal{V}_n^\epsilon)\|_{L^\infty} + \frac{C}{h_n}\|[\chi(h_n^3 D_x), \mathcal{V}_n - \mathcal{V}_n^\epsilon]\|_{\mathcal{L}(L^2)} \\ &\quad \le C\|\partial_x(\mathcal{V}_n - \mathcal{V}_n^\epsilon)\|_{L^\infty} + Ch_n^2\|\partial_x(\mathcal{V}_n - \mathcal{V}_n^\epsilon)\|_{L^\infty} \\ &\qquad\qquad \xrightarrow[n\to+\infty]{} C\|\partial_x(\mathcal{V} - \mathcal{V}^\epsilon)\|_{L^\infty},\end{aligned} \tag{3-24}$$

where we used Lemma 3.12 below with $\varepsilon = h_n^3$ in the last line. Combining now (3-22) with (3-23), (3-24), and the fact that $\|\partial_x(\mathcal{V}-\mathcal{V}^\epsilon)\|_{L^\infty} \to 0$ as $\epsilon \to 0$, we have obtained

$$L_A(h_n) \xrightarrow[n\to+\infty]{} -\frac{1}{i}\langle\mu, H_p a\rangle. \tag{3-25}$$

We now compute $L_A(h_n)$ defined in (3-21) in a different way using (3-4). We obtain

$$\begin{aligned}L_A(h_n) &= -h_n\langle A(\psi_n'(0^+)\delta_0 - \psi_n'(L^-)\delta_L), \overline{\underline{\psi}}_n\rangle_{\mathcal{S}(\mathbb{R}),\mathcal{S}'(\mathbb{R})} \\ &\qquad + h_n\langle A\underline{\psi}_n, (\overline{\psi}_n'(0^+)\delta_0 - \overline{\psi}_n'(L^-)\delta_L)\rangle_{\mathcal{S}(\mathbb{R}),\mathcal{S}'(\mathbb{R})} + \boldsymbol{o}(1) \\ &= h_n[-\psi_n'(0^+)(\overline{A^*\underline{\psi}_n})(0) + (A\underline{\psi}_n)(0)\overline{\psi}_n'(0^+)] \\ &\qquad + h_n[\psi_n'(L^-)(\overline{A^*\underline{\psi}_n})(L) - (A\underline{\psi}_n)(L)\overline{\psi}_n'(L^-)] + \boldsymbol{o}(1).\end{aligned} \tag{3-26}$$

We now only treat the boundary terms at 0; the boundary terms at L being handled similarly. Recalling the definition of A at the beginning of the proof, we have

$$A^* = \left(a_0(x) + \frac{h_n}{i}a_1'(x) + a_1(x)h_n D_x\right)\chi(h_n^3 D_x).$$

As a consequence, we have

$$(\overline{A^*\underline{\psi}_n})(0) = \Big(a_0(0) + \frac{h}{i}a_1'(0)\Big)[\chi(h_n^3 D_x)\overline{\underline{\psi}}_n](0) \\ + a_1(0)[\chi(h_n^3 D_x)h_n D_x\overline{\underline{\psi}}_n](0), \tag{3-27}$$

$$(A\underline{\psi}_n)(0) = [\chi(h_n^3 D_x)(a_0\underline{\psi}_n)](0) + [\chi(h_n^3 D_x)(a_1 h_n D_x\underline{\psi}_n)](0). \tag{3-28}$$

It is now possible to apply Lemma 3.9 below with $\varepsilon = h_n^3$ with $f = \underline{\psi}_n$, $h_n D_x\underline{\psi}_n$, $a_0(x)\underline{\psi}_n$ or $a_1(x)h_n D_x\underline{\psi}_n$. For instance, using that $\underline{\psi}_n(0) = 0$, we have

$$|[\chi(h_n^3 D_x)(a_0\underline{\psi}_n)](0)| \le C h_n^{3/2}(\|(a_0\underline{\psi}_n)'\|_{L^2} + \|a_0\underline{\psi}_n\|_{L^2}) \le C h_n^{1/2},$$

on account of Corollary 3.2. Similarly, according to Lemma 3.9, we have

$$[\chi(h_n^3 D_x)(a_1 h_n D_x\underline{\psi}_n)](0) = \tfrac{1}{2}a_1(0)h_n D_x\psi_n(0^+) + s_n, \\ \text{with } |s_n| \le C h_n^{5/2}(\|(a_1\psi_n')'\|_{L^2(0,L)} + \|a_1\psi_n'\|_{L^2(0,L)}) \le C h_n^{1/2},$$

where we used (2-12). Note that the power 3 in $\chi(h_n^3 D_x)$ has been chosen to handle the remainder terms. Collecting all terms in (3-26), (3-27), (3-28), we have obtained

$$L_A(h_n) = \frac{1}{i}a_1(0)|h_n\psi_n'(0^+)|^2 - \frac{1}{i}a_1(L)|h_n\psi_n'(L^-)|^2 + \mathcal{O}(h_n^{1/2}) \\ \xrightarrow[n\to+\infty]{} \frac{1}{i}(a_1(0)\ell_0 - a_1(L)\ell_L),$$

where we used (3-6) in the limit. This concludes the proof of the lemma when combined with (3-25). □

We have used the following lemma, which is a simpler 1D version of [Gérard and Leichtnam 1993, Lemma 3.8], and whose proof relies on the elementary lemmata 3.10 and 3.11 below, which sometimes use the specific properties (parity) for χ in (3-20).

Lemma 3.9. *Let $f \in L^2_{\mathrm{comp}}(\mathbb{R})$ be such that, in $\mathcal{D}'(\mathbb{R})$, we have*

$$f' = F + \alpha\delta_0 + \beta\delta_L, \quad \text{with } F \in L^2(\mathbb{R}),\ \alpha, \beta \in \mathbb{C}.$$

Then, with χ as in (3-20), *we have $f|_{(-\infty,0)} \in C^0([-\infty, 0])$, $f|_{(0,L)} \in C^0([0, L])$, $f|_{(L,\infty)} \in C^0([L, \infty])$, together with*

$$(\chi(\varepsilon D_x)f)(0) = \frac{f(0^+) + f(0^-)}{2} + r, \quad \text{with } |r| \le C\varepsilon^{1/2}(\|F\|_{L^2(\mathbb{R})} + \|f\|_{L^2(\mathbb{R})}).$$

Proof. The fact that f is piecewise continuous follows from the fact that f' is a Radon measure. Using Lemma 3.11 below and a partition of the unity, we are reduced to the case where f is supported in $(-L/2, L/2)$ and $\beta = 0$.

We define $g(x) = \frac{1}{2}(f(x) + f(-x))$. Then, g is $C^0_c(\mathbb{R})$ with $g' \in L^2(\mathbb{R})$ and $\|g'\|_{L^2(\mathbb{R})} \le \|F\|_{L^2(\mathbb{R})}$. We have $g(0) = \frac{1}{2}(f(0^+) + f(0^-))$. Using that χ is even and writing $\chi_\varepsilon(s) = \chi(\varepsilon s)$, we also have (denoting by $\check{\chi}_\varepsilon$ the inverse Fourier transform of χ_ε)

$$
\begin{aligned}
[\chi(\varepsilon D_x)g](0) &= \frac{1}{2}\int_{\mathbb{R}_y} [f(-y) + f(y)]\check{\chi}_\varepsilon(y)\,dy \\
&= \frac{1}{2}\int_{\mathbb{R}_y} [\check{\chi}_\varepsilon(y) + \check{\chi}_\varepsilon(-y)] f(y)\,dy = [\chi(\varepsilon D_x) f](0).
\end{aligned}
$$

We can conclude by applying Lemma 3.10 below to g. □

Lemma 3.10. *There is $C > 0$ such that for all $f \in C_c^0(\mathbb{R})$ with $f' \in L^2(\mathbb{R})$, we have*

$$
|(\chi(\varepsilon D_x) f)(0) - f(0)| \le C\varepsilon^{1/2}\|f'\|_{L^2(\mathbb{R})}, \quad \textit{for all } \varepsilon > 0.
$$

Proof. Denoting by $\check{\chi}_\varepsilon$ the inverse Fourier transform of χ_ε, we have $\check{\chi}_\varepsilon(\xi) = (1/\varepsilon)\check{\chi}(\varepsilon^{-1}\xi)$. Since $\chi(0) = \chi_\varepsilon(0) = 1$, we have $\int_{\mathbb{R}} \check{\chi}_\varepsilon(\xi)\,d\xi = 1$ so that

$$
\begin{aligned}
(\chi_\varepsilon(D_x) f)(0) - f(0) &= \int_{\mathbb{R}_y} [f(-y) - f(0)]\check{\chi}_\varepsilon(y)\,dy \\
&= -\int_{\mathbb{R}_y} y\check{\chi}_\varepsilon(y) \int_0^1 f'(-ty)\,dt\,dy \\
&= -\varepsilon \int_{\mathbb{R}_x} x\check{\chi}(x) \int_0^1 f'(-t\varepsilon x)\,dt\,dx.
\end{aligned}
$$

We have by the Cauchy–Schwarz inequality

$$
\begin{aligned}
|(\chi_\varepsilon(D_x) f)(0) - f(0)| &\le \varepsilon \int_0^1 \int_{\mathbb{R}_x} |x\check{\chi}(x)|\,|f'(-t\varepsilon x)|\,dt\,dx \\
&\le \varepsilon \|x\check{\chi}\|_{L^2} \int_0^1 \left(\int_{\mathbb{R}_x} |f'(-t\varepsilon x)|^2\,dx \right)^{1/2} dt \\
&\le C\varepsilon^{1/2} \int_0^1 t^{-1/2} \left(\int_{\mathbb{R}_s} |f'(s)|^2\,ds \right)^{1/2} dt \\
&\le C\varepsilon^{1/2}\|f'\|_{L^2(\mathbb{R})},
\end{aligned}
$$

which is the sought estimate. □

Lemma 3.11. *Let $c > 0$ and $N \in \mathbb{R}_+$, then there exists $C_{N,c} > 0$ so that for all $f \in L^2(\mathbb{R})$ so that $f = 0$ a.e. in $(-c, c)$, we have*

$$
|(\chi(\varepsilon D_x) f)(0)| \le C_{N,c}\varepsilon^N \|f\|_{L^2}.
$$

Proof. With the same notations as the proof of Lemma 3.10, we have

$$\begin{aligned}|(\chi(\varepsilon D_x)f)(0)| &= \left|\int_{\mathbb{R}} f(-y)\check{\chi}_\varepsilon(y)\,dy\right| \\ &\le \|y^{-N}f\|_{L^2}\|y^N\check{\chi}_\varepsilon(y)\|_{L^2} \le \varepsilon^{N-1/2}c^{-N}\|f\|_{L^2}\|x^N\check{\chi}(x)\|_{L^2},\end{aligned}$$

whence the sought result after having changed the value of N. □

We have also used the following lemma to handle "low-regularity" potentials.

Lemma 3.12. *Assume $V \in C^0(\mathbb{R})$ such that $V' \in L^\infty(\mathbb{R})$ and $\chi \in C_c^\infty(\mathbb{R})$. Then, we have*

$$[\chi(\varepsilon D), V(x)] \in \mathcal{L}(L^2(\mathbb{R})), \qquad \text{with } \|[\chi(\varepsilon D), V(x)]\|_{\mathcal{L}(L^2)} \le C_\chi\varepsilon\|V'\|_{L^\infty(\mathbb{R})}.$$

Proof. The operator $\chi(\varepsilon D)$ is the convolution by $(1/\varepsilon)\check{\chi}(\cdot/\varepsilon)$ where $\check{\chi}$ is the inverse Fourier transform of χ. Its kernel is therefore $(1/\varepsilon)\check{\chi}((x-y)/\varepsilon)$ and the kernel of $[\chi(\varepsilon D), V(x)]$ is therefore $K_\varepsilon(x, y) = (1/\varepsilon)\check{\chi}((x-y)/\varepsilon)(V(y)-V(x))$. The Schur lemma and symmetry of the kernel in (x, y) give

$$\begin{aligned}\|[\chi(\varepsilon D), V(x)]\|_{\mathcal{L}(L^2(\mathbb{R}))} &\le \max[\sup_{x\in\mathbb{R}}\|K_\varepsilon(x, y)\|_{L^1(\mathbb{R}_y)}, \sup_{y\in\mathbb{R}}\|K_\varepsilon(x, y)\|_{L^1(\mathbb{R}_x)}] \\ &\le \frac{1}{\varepsilon}\sup_{x\in\mathbb{R}}\int_{\mathbb{R}_y}\left|\check{\chi}\left(\frac{x-y}{\varepsilon}\right)\right||V(y)-V(x)|\,dy \\ &\le \sup_{s\in\mathbb{R}}\int_{\mathbb{R}_t}|\check{\chi}(s-t)||V(\varepsilon t)-V(\varepsilon s)| \\ &\le \varepsilon\|V'\|_{L^\infty(\mathbb{R})}\sup_{s\in\mathbb{R}}\int_{\mathbb{R}_t}|\check{\chi}(s-t)(s-t)|\,dt.\end{aligned}$$

This yields the expected result with $C_\chi = \|t\check{\chi}(t)\|_{L^1_t}$. □

3D. *Invariance properties near the boundary.* Now, we will state the propagation at the boundary. We only consider the boundary problem at $x = 0$, the problem near $x = L$ being handled similarly. The following is a 1D version of [Gérard and Leichtnam 1993, Theorem 2.3].

Lemma 3.13. *Under the assumptions of Proposition 2.4, with $\|r_n\|_{L^2} = \boldsymbol{o}(1)$, we have*:

- *Elliptic case*: *if $\mathcal{V}(0) > 0$, then $\ell_0 = 0$ and $\mu = 0$ for x close to* 0.
- *Glancing case*: *if $\mathcal{V}(0) = 0$, then $H_p\mu = -\ell_0\delta_{x=0}\otimes\delta'_{\xi=0}$ for x close to* 0.
- *Hyperbolic case*: *if $\mathcal{V}(0) < 0$, then*

$$H_p\mu = \frac{\ell_0}{2\sqrt{-\mathcal{V}(0)}}\delta_{x=0}\otimes(\delta_{\xi=\sqrt{-\mathcal{V}(0)}} - \delta_{\xi=-\sqrt{-\mathcal{V}(0)}})$$

for x close to 0.

Note that the simple 1D setting here allows one to avoid the use of the Malgrange preparation theorem and provides a self-contained elementary proof (as compared to [Gérard and Leichtnam 1993, Theorem 2.3]).

Remark 3.14. Note that one recovers the equation in the glancing case $\mathcal{V}(0) = 0$ by taking the limit $\sqrt{-\mathcal{V}(0)} \to 0$ in the equation obtained in the hyperbolic case.

Proof. In the first elliptic case, that $\mu = 0$ near $x = 0$ is a consequence of Lemma 3.4 together with $\{p = 0\} \cap (\{0\} \times \mathbb{R}) = \varnothing$ if $\mathcal{V}(0) > 0$. Applying Lemma 3.8 with $a_0 = 0$ and a_1 satisfying $a_1(0) = 1$ and $\operatorname{supp}(a_1) \times \mathbb{R} \cap \operatorname{supp}(\mu) = \varnothing$ yields $\ell_0 = 0$.

For the glancing case, we use Lemma 3.5 together with

$$\{\xi^2 + \mathcal{V}(x) = 0\} \cap \{0\} \times \mathbb{R}_\xi = \{(0, 0)\}.$$

Classical distribution theory implies that close to $x = 0$,

$$H_p\mu = q_{(0,0)}\delta_{(0,0)} + q_{(1,0)}\partial_x\delta_{(0,0)} + q_{(0,1)}\partial_\xi\delta_{(0,0)},$$

where $q_\alpha \in \mathbb{C}$. Lemma 3.8 gives, for every $a(x, \xi) = a_0(x) + \xi a_1(x)$,

$$\ell_0 a_1(0) = -\langle \mu, H_p a\rangle = q_{(0,0)}a_0(0) - q_{(1,0)}a_0'(0) - q_{(0,1)}a_1(0).$$

Since a_0 and a_1 are arbitrary smooth functions, we obtain $q_{(0,0)} = q_{(1,0)} = 0$ and $q_{(0,1)} = -\ell_0$, so that $H_p\mu = -\ell_0\delta_{x=0} \otimes \delta'_{\xi=0}$, which is the sought result.

For the hyperbolic case, Lemma 3.5 together with

$$\{\xi^2 + \mathcal{V}(x) = 0\} \cap \{0\} \times \mathbb{R}_\xi = \{(0_x, \sqrt{-\mathcal{V}(0)})\} \cup \{(0_x, -\sqrt{-\mathcal{V}(0)})\}$$

imply again that, close to $x = 0$,

$$\begin{aligned} H_p\mu = {} & q^+_{(0,0)}\delta_{(0,\sqrt{-\mathcal{V}(0)})} + q^+_{(1,0)}\partial_x\delta_{(0,\sqrt{-\mathcal{V}(0)})} + q^+_{(0,1)}\partial_\xi\delta_{(0,\sqrt{-\mathcal{V}(0)})} \\ & + q^-_{(0,0)}\delta_{(0,-\sqrt{-\mathcal{V}(0)})} + q^-_{(1,0)}\partial_x\delta_{(0,-\sqrt{-\mathcal{V}(0)})} + q^-_{(0,1)}\partial_\xi\delta_{(0,-\sqrt{-\mathcal{V}(0)})}. \end{aligned} \tag{3-29}$$

This time, Lemma 3.8 gives for every $a(x, \xi) = a_0(x)$,

$$0 = -\langle \mu, H_p a\rangle = \langle H_p\mu, a\rangle = (q^+_{(0,0)} + q^-_{(0,0)})a_0(0) - (q^+_{(1,0)} + q^-_{(1,0)})a_0'(0).$$

This implies

$$q_{(0,0)} := q^+_{(0,0)} = -q^-_{(0,0)} \quad \text{and} \quad q_{(1,0)} := q^+_{(1,0)} = -q^-_{(1,0)}. \tag{3-30}$$

Then, Lemma 3.8 gives for every $a(x, \xi) = a_1(x)\xi$,

$$\begin{aligned} \ell_0 a_1(0) &= -\langle \mu, H_p a\rangle \\ &= \sqrt{-\mathcal{V}(0)}[2a_1(0)q_{(0,0)} - 2a_1'(0)q_{(1,0)}] - (q^+_{(0,1)} + q^-_{(0,1)})a_1(0). \end{aligned}$$

This gives $q_{(1,0)} = 0$ and

$$\ell_0 = \sqrt{-\mathcal{V}(0)}2q_{(0,0)} - (q^+_{(0,1)} + q^-_{(0,1)}). \tag{3-31}$$

To finish, we now choose $a(x,\xi)=p(x,\xi)b(x,\xi)$ as a test function, for $b\in C_c^\infty(\mathbb{R}^2)$, and obtain

$$\langle \mu, H_p a\rangle = \langle \mu, pH_p b\rangle + \langle b\mu, H_p p\rangle = 0,$$

where we have used Lemma 3.4 for the first term and $H_p p = 0$ for the second. Applying again (3-29) to the function a and using the information we already have on the coefficients in (3-29), we obtain using $a(0, \pm\sqrt{-\mathcal{V}(0)}) = 0$ (recall that $p=\xi^2+\mathcal{V}(x)$) that

$$0 = -\langle \mu, H_p a\rangle = -q^+_{(0,1)}(\partial_\xi a)(0, \sqrt{-\mathcal{V}(0)}) + q^-_{(0,1)}(\partial_\xi a)(0, -\sqrt{-\mathcal{V}(0)}).$$

But now for $a(x,\xi) = p(x,\xi)b(x,\xi)$, on the set $p=0$, we have

$$(\partial_\xi a)(x,\xi) = (\partial_\xi p)(x,\xi)b(x,\xi) + (\partial_\xi b)(x,\xi)p(x,\xi) = 2\xi b(x,\xi).$$

So, we deduce

$$0 = -\sqrt{-\mathcal{V}(0)}q^+_{(0,1)}b(0,\sqrt{-\mathcal{V}(0)}) - \sqrt{-\mathcal{V}(0)}q^-_{(0,1)}b(0,-\sqrt{-\mathcal{V}(0)}).$$

Since b is arbitrary and $\sqrt{-\mathcal{V}(0)}\neq 0$, we obtain $q^+_{(0,1)} = q^-_{(0,1)} = 0$. This, together with (3-31) implies that $\ell_0 = 2\sqrt{-\mathcal{V}(0)}q_{(0,0)}$, which, combined with (3-29), (3-30) and $q_{(1,0)}=0$, gives the expected result in the hyperbolic case. □

We now specify the glancing and diffractive case at $x=0$.

Lemma 3.15. *If $\mathcal{V}(0)=0$ and $\mathcal{V}'(0)\leq 0$, then $\ell_0=0$. If moreover $\mathcal{V}'(0)<0$, then $\mu(\{(0,0)\})=0$.*

Proof. For this, we follow [Burq and Gérard 1997]. We take $\chi\in C_c^\infty(-1,1)$ with $\chi=1$ in a neighborhood of 0, $\chi\geq 0$ and $\int_{\mathbb{R}}\chi=1$. Define $\widetilde{\chi}(s)=\int_{-\infty}^s\chi\in C^\infty(\mathbb{R})$ and test the identity $H_p\mu = -\ell_0\delta_{x=0}\otimes\delta'_{\xi=0}$ obtained in Lemma 3.13 with the function $a(x,\xi)=\chi(x/\alpha)\widetilde{\chi}(\xi/\beta)\in C^\infty(\mathbb{R}^2)$ for $\alpha,\beta>0$. This yields (for α sufficiently small)

$$\left\langle \mu, -\frac{2\xi}{\alpha}\chi'(x/\alpha)\widetilde{\chi}(\xi/\beta)\right\rangle + \left\langle \mu, \frac{\mathcal{V}'(x)}{\beta}\chi(x/\alpha)\widetilde{\chi}'(\xi/\beta)\right\rangle = \frac{\ell_0}{\beta}\chi(0)\chi(0) = \frac{\ell_0}{\beta}.$$

Multiplying by β, choosing $\alpha=\sqrt{\beta}$, and using dominated convergence yields, in the limit $\beta\to 0^+$

$$\mathcal{O}(\sqrt{\beta}) + \langle \mu, \mathcal{V}'(x)\chi(x/\sqrt{\beta})\chi(\xi/\beta)\rangle = \ell_0.$$

Now, taking the limit $\beta\to 0^+$ and using again dominated convergence implies $\mathcal{V}'(0)\mu(\{(0,0)\})=\ell_0$. That $\mathcal{V}'(0)\leq 0$, $\mu\geq 0$ and $\ell_0\geq 0$ implies that $\ell_0=0$. If moreover $\mathcal{V}'(0)<0$, then we obtain $\mu(\{(0,0)\})=0$, which concludes the proof of the lemma. □

Acknowledgements

Léautaud is partially supported by the Agence Nationale de la Recherche under grants SALVE ANR-19-CE40-0004 and ADYCT (ANR-20-CE40-0017).

The authors would like to thank Bernard Helffer for his comments on a preliminary version of this work and for pointing out several references, and the referee for useful remarks, which helped to improve the quality of the article.

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Received 4 Mar 2022. Revised 22 Jul 2022.

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Tunisian Journal of Mathematics (ISSN 2576-7666 electronic, 2576-7658 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840 is published continuously online.

TJM peer review and production are managed by EditFlow® from MSP.

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Tunisian Journal of Mathematics

2023 vol. 5 no. 1