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OBSERVABILITY OF THE HEAT EQUATION, GEOMETRIC CONSTANTS IN CONTROL THEORY, AND A CONJECTURE OF LUC MILLER

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We are concerned with the short-time observability constant of the heat equation from a subdomain ω of a bounded domain $\mathcal{M}$. The constant is of the form $e^{\mathfrak{K}/T}$, where $\mathfrak{K}$ depends only on the geometry of $\mathcal{M}$ and ω. Luc Miller (*J. Differential Equations* **204**:1 (2004), 202–226) conjectured that $\mathfrak{K}$ is (universally) proportional to the square of the maximal distance from ω to a point of $\mathcal{M}$. We show in particular geometries that $\mathfrak{K}$ may blow up like $|\log(r)|^2$ when ω is a ball of radius r, hence disproving the conjecture. We then prove in the general case the associated upper bound on this blowup. We also show that the conjecture is true for positive solutions of the heat equation.

The proofs rely on the study of the maximal vanishing rate of (sums of) eigenfunctions. They also yield lower and upper bounds for other geometric constants appearing as tunneling constants or approximate control costs.

As an intermediate step in the proofs, we provide a uniform Carleman estimate for Lipschitz metrics. The latter also implies uniform spectral inequalities and observability estimates for the heat equation in a bounded class of Lipschitz metrics, which are of independent interest.

1. Introduction and main results

We are interested in several constants appearing in the study of eigenfunctions concentration and control theory, and the links between them. In the whole paper, we are given a connected compact Riemannian manifold $(\mathcal{M}, g)$ with or without boundary $\partial\mathcal{M}$, we denote by Δ_g the (negative) Laplace–Beltrami operator on $\mathcal{M}$. In the case $\partial\mathcal{M} \neq \varnothing$, we denote by $\mathrm{Int}(\mathcal{M})$ the interior of $\mathcal{M}$, so that $\mathcal{M} = \partial\mathcal{M} \sqcup \mathrm{Int}(\mathcal{M})$; see, e.g., [Lee 2013, Chapter 1]. For readability, we first focus in the next section on results concerning the observability constant for the heat equation.

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1A. ***The control cost for the heat equation.*** Here, we study the so-called *cost of controllability* of the heat equation. It has been well known since the seminal papers [Lebeau and Robbiano 1995; Fursikov and Imanuvilov 1996] that for any time $T > 0$, the heat equation is controllable to zero. More precisely, by duality, the controllability problem is equivalent to the observability problem for solutions of the free heat equation (see, e.g., [Coron 2007, Section 2.5.2]): for any nonempty open set ω and $T > 0$, there exists $C_{T,\omega}$ such that we have

$$\|e^{T\Delta_g} u\|^2_{L^2(\mathcal{M})} \le C^2_{T,\omega} \int_0^T \|e^{t\Delta_g} u\|^2_{L^2(\omega)}\, dt \quad \text{for all } T > 0 \text{ and all } u \in L^2(\mathcal{M}). \tag{1}$$

Here, $(e^{t\Delta_g})_{t>0}$ denotes the semigroup generated by the *Dirichlet* Laplace operator on $\mathcal{M}$ (otherwise explicitly stated). The observability constant $C_{T,\omega}$ is then directly related to the cost of the control to zero and has been the object of several studies.

It has been proved in [Seidman 1984] in one dimension (in the closely related case of a boundary observation) and in [Fursikov and Imanuvilov 1996] in general (see also [Miller 2010] for obtaining this result via the Lebeau–Robbiano method) that the cost in small time blows up at most exponentially:

$$\omega \neq \varnothing \quad \Longrightarrow \quad \text{there are } C, \mathfrak{K} > 0 \text{ such that } C_{T,\omega} \le C e^{\frac{\mathfrak{K}}{T}} \text{ for all } T > 0. \tag{2}$$

Güichal [1985] in one dimension and Miller [2004a] in the general case proved that exponential blowup indeed occurs:

$$\bar{\omega} \neq \mathcal{M} \quad \Longrightarrow \quad \text{there is } c > 0 \text{ such that } C_{T,\omega} \ge c e^{\frac{c}{T}} \text{ for all } T > 0.$$

This suggests defining

$$\mathfrak{K}_{\text{heat}}(\omega) = \inf\{\mathfrak{K} > 0 : \text{there exists } C > 0 \text{ such that (1) holds with } C_{T,\omega} = C e^{\frac{\mathfrak{K}}{T}}\}, \tag{3}$$

which, according to the above-mentioned results satisfies $\mathfrak{K}_{\text{heat}}(\omega) < \infty$ as soon as $\omega \neq \varnothing$ and $\mathfrak{K}_{\text{heat}}(\omega) > 0$ as soon as $\bar{\omega} \neq \mathcal{M}$. This constant depends only on the geometry of the manifold $(\mathcal{M}, g)$ and the subset ω. It is expected to contain geometric features of short-time heat propagation and has thus received a lot of attention in the past fifteen years [Miller 2004a; 2004b; 2006b; 2010; Tenenbaum and Tucsnak 2007; 2011; Ervedoza and Zuazua 2011b; Bardos and Phung 2017; Dardé and Ervedoza 2019; Egidi and Veselić 2018; Nakić et al. 2018; Phung 2018].

In this direction, the result of [Miller 2004a] is actually more precise and provides a geometric lower bound: for all $(\mathcal{M}, g), \omega$, we have

$$\mathfrak{K}_{\text{heat}}(\omega) \ge \tfrac{1}{4}\mathcal{L}(\mathcal{M}, \omega)^2,$$

where, for $E \subset \mathcal{M}$, we write

$$\mathcal{L}(\mathcal{M}, E) = \sup_{x \in \mathcal{M}} \operatorname{dist}_g(x, E). \tag{4}$$

The proof relies on heat kernel estimates. Luc Miller [2004a; 2006a] also proved that in the case ω satisfies the geometric control condition in $(\mathcal{M}, g)$ (see [Bardos et al. 1992]) we have

$$\mathfrak{K}_{\text{heat}}(\omega) \le \alpha_* L^2_\omega,$$

where L_ω is the maximal length of a "ray of geometric optics" (i.e., geodesic curve in the case $\partial\mathcal{M} = \varnothing$) not intersecting ω, and $\alpha_* \leq 2$ is an absolute constant (independent of the geometry). Based on these results and the idea that the heat kernel provides the most concentrated solutions of the heat equation, he formulated the following conjecture [Miller 2004a, Section 2.1; 2006b, Section 3.1].

Conjecture 1.1 (Luc Miller). *For all $(\mathcal{M}, g)$ and $\omega \subset \mathcal{M}$ such that $\bar{\omega} \neq \mathcal{M}$, we have $\mathfrak{K}_{\text{heat}}(\omega) = \frac{1}{4}\mathcal{L}(\mathcal{M}, \omega)^2$.*

Note that it has been proved in [Lissy 2015] that, in the related context of the one-dimensional heat equation with a boundary observation, the factor $\frac{1}{4}$ might not be correct (and should be replaced by $\frac{1}{2}$, see Section 1D below). Our first result disproves Conjecture 1.1 in a stronger sense.

Theorem 1.2 (counterexamples). *Assume $(\mathcal{M}, g)$ is one of the following*:

(1) *$\mathcal{M} = \mathbb{S}^n \subset \mathbb{R}^{n+1}$ and g is the canonical metric (see Section 3A).*

(2) *$\mathcal{M} = \mathcal{S} \subset \mathbb{R}^3$ is a surface of revolution diffeomorphic to the sphere $\mathbb{S}^2$, and g is the metric induced by the Euclidean metric on $\mathbb{R}^3$ (with additional nondegeneracy conditions, see Section 3B).*

(3) *$\mathcal{M} = \mathbb{D} = \{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 \leq 1\} \subset \mathbb{R}^2$ is the unit disk, g is the Euclidean metric and Dirichlet conditions are taken on $\partial\mathcal{M}$ (see Section 3C).*

Then, for any $C > 0$, there exists $\omega \subset \mathcal{M}$ so that $\mathfrak{K}_{\text{heat}}(\omega) \geq C\,\mathcal{L}(\mathcal{M}, \omega)^2$ and $\mathfrak{K}_{\text{heat}}(\omega) \geq C$.

More precisely, assume that x_0 is either

(1) *any point in $\mathbb{S}^n$,*

(2) *one of the two points that intersect the axis of revolution of $\mathcal{S} \subset \mathbb{R}^3$,*

(3) *the center of $\mathbb{D}$.*

Then, there exists $C > 0$ and $r_0 > 0$ so that we have

$$\mathfrak{K}_{\text{heat}}(B_g(x_0, r)) \geq C\,|\log(r)|^2 \tag{5}$$

for any $0 < r \leq r_0$.

Here, $B_g(x_0, r)$ denotes the geodesic ball of $\mathcal{M}$ centered at x_0 of radius r. The results we obtain are slightly more precise. In particular, the constant C is an explicit geometric constant. The lower bounds are related to an appropriate Agmon distance associated to the problem. We refer to Corollary 1.10 below for more precise estimates.

Note also that this blowup of $\mathfrak{K}_{\text{heat}}(B(x_0, r))$ for small r does not always happen and is due here to a particular (de-)concentration phenomenon. For instance on $\mathcal{M} = \mathbb{T}^1$, the set $\omega = B(x_0, r)$ always satisfies the geometric control condition for any time $T > 1 - 2r$. Abstract results (see (15) below for more details) give $\mathfrak{K}_{\text{heat}}(B(x_0, r)) \leq \alpha_* \leq 2$ for any $r > 0$ and blowup does not occur.

Our next result shows that the blowup given by (5) is actually optimal as far as the asymptotics of $\mathfrak{K}_{\text{heat}}$ for small balls is concerned. We prove the following observability result from small balls (closely related to previous results of [Jerison and Lebeau 1999], see Section 1C2 below).

Theorem 1.3. *For all $x_0 \in \mathcal{M}$, there exist $C > 0$ such that for all $r > 0$ we have*

$$\mathfrak{K}_{\text{heat}}(B(x_0, r)) \le C|\log(r)|^2 + C.$$

Note that in [Bardos and Phung 2017; Phung 2018] it was recently proved independently that $\mathfrak{K}_{\text{heat}}(B(x_0, r)) \le C_\epsilon/r^\epsilon + C_\epsilon$ for all $\epsilon > 0$ in the case $\mathcal{M} \subset \mathbb{R}^n$ is star-shaped with respect to x_0.

These results seem to suggest that $\mathcal{L}(\mathcal{M}, \omega)$ is not the only appropriate parameter needed for estimating $\mathfrak{K}_{\text{heat}}(\omega)$. There are indeed some solutions of the heat equation concentrating more than the heat kernel for small times. Our last result concerning the heat equation goes actually in the opposite direction. It provides a large class of solutions of the heat equation, namely *positive* solutions, that do not concentrate more than the heat kernel, thus proving Conjecture 1.1 when restricted to this class of solutions. Recall that $\mathcal{L}(\mathcal{M}, E)$ is defined in (4).

Theorem 1.4. *Assume that $(\mathcal{M}, g)$ has geodesically convex boundary $\partial\mathcal{M}$. Then, for any nonempty open set $\omega \subset \mathcal{M}$ and $z_0 \in \mathcal{M}$, for any $\epsilon > 0$ there exist $C, D > 0$ so that for any $0 < T \le D$ we have*

$$\|u(T)\|^2_{L^2(\mathcal{M})} \le \frac{C}{T} e^{\frac{(1+\epsilon)(\mathcal{L}(M,\omega)+\epsilon)^2}{2T}} \int_0^T \|u(t,\cdot)\|^2_{L^2(\omega)}\, dt, \tag{6}$$

$$\|u(T)\|^2_{L^2(\mathcal{M})} \le \frac{C}{T} e^{\frac{(1+\epsilon)(\mathcal{L}(M,z_0)+\epsilon)^2}{2T}} \int_0^T u(t,z_0)^2\, dt \tag{7}$$

for all $u_0 \in L^2(\mathcal{M})$ such that $u_0 \ge 0$ a.e. on $\mathcal{M}$ and associated solution u to

$$(\partial_t - \Delta_g)u = 0 \quad \text{on } \mathbb{R}^+_* \times \operatorname{Int}(\mathcal{M}), \qquad u|_{t=0} = u_0 \quad \text{in } \operatorname{Int}(\mathcal{M}), \qquad \partial_\nu u = 0 \quad \text{on } \mathbb{R}^+ \times \partial\mathcal{M}.$$

Theorem 1.4 follows from classical Li–Yau estimates [1986]. Notice that here, Neumann boundary conditions are taken (ν denotes a unit vector field normal to $\partial\mathcal{M}$), and an additional geometric assumption is made (convexity of $\partial\mathcal{M}$). The result still holds without the convexity assumption up to replacing $(1+\epsilon)$ in the exponent by a geometric constant; see Remark 5.2. We also recall that for nonnegative initial data $u_0 \ge 0$, the solution of the heat equation remains nonnegative for all times. Of course, the counterexamples of Theorem 1.2 prevent these estimates from holding in general. Estimate (7) is particularly surprising (even without considering the value of the constants) and of course only true for positive solutions (otherwise just taking z_0 in a nodal set of an eigenfunction of Δ_g invalidates (7)). Finally, let us mention that the constants C and D are explicitly estimated by geometric quantities (see Remark 5.4).

Let us now put these results in a broader context, and introduce several related geometric constants appearing in tunneling estimates and control theory.

1B. ***Tunneling constants in control theory, and their links.*** The lower bounds of Theorem 1.2 are proved using very particular solutions to the heat equation arising from eigenfunctions (exhibiting a very strong concentration far from x_0 as well as a strong deconcentration near x_0). It is therefore natural to study related constants measuring such (de-)concentration properties. In this section, we introduce all geometric constants studied in the paper and collect known links between them.

We first introduce spectral subspaces of the Laplace operator Δ_g (with Dirichlet boundary conditions if $\partial\mathcal{M} \neq \varnothing$), which are at the core of most results presented here. Namely, for $\lambda \in \mathrm{Sp}(-\Delta_g)$, the space

$$E_\lambda := \mathrm{span}\{\psi \in L^2(\mathcal{M}) : -\Delta_g \psi = \lambda\psi\}$$

denotes the eigenspace associated to the eigenvalue λ and, for all $\lambda > 0$,

$$E_{\leq\lambda} := \mathrm{span}\{E_{\lambda_j} : \lambda_j \in \mathrm{Sp}(-\Delta_g),\ \lambda_j \leq \lambda\}$$

denotes the space of linear combinations of eigenfunctions associated to eigenvalues $\leq \lambda$.

Let us now introduce the constants studied in the article, other than that involved in (1)–(2). For any nonempty open subset $\omega \subset \mathcal{M}$, we recall the following results:

- Vanishing of eigenfunctions [Donnelly and Fefferman 1988; Lebeau and Robbiano 1995]: there exist $C, \mathfrak{K}$ such that we have

$$\|\psi\|_{L^2(\mathcal{M})} \leq Ce^{\mathfrak{K}\sqrt{\lambda}}\|\psi\|_{L^2(\omega)} \quad \text{for all } \lambda \in \mathrm{Sp}(-\Delta_g) \text{ and } \psi \in E_\lambda. \tag{8}$$

- Vanishing of sums of eigenfunctions (so-called Lebeau–Robbiano spectral inequality) [Lebeau and Robbiano 1995; Jerison and Lebeau 1999; Lebeau and Zuazua 1998]: there exist $C, \mathfrak{K}$ such that we have

$$\|u\|_{L^2(\mathcal{M})} \leq Ce^{\mathfrak{K}\sqrt{\lambda}}\|u\|_{L^2(\omega)} \quad \text{for all } \lambda > 0 \text{ and all } u \in E_{\leq\lambda}. \tag{9}$$

- Infinite-time observability of the heat equation [Fursikov and Imanuvilov 1996]: there exist $C, \mathfrak{K}$ such that we have

$$\int_{\mathbb{R}+} e^{-\frac{2\mathfrak{K}}{t}}\|e^{t\Delta_g}u\|^2_{L^2(\mathcal{M})}\,dt \leq C\int_{\mathbb{R}+}\|e^{t\Delta_g}u\|^2_{L^2(\omega)}\,dt \quad \text{for all } u \in L^2(\mathcal{M}). \tag{10}$$

- Approximate observability for the wave equation [Laurent and Léautaud 2019]:

$$(\partial_t^2 - \Delta_g)u = 0, \quad u|_{(0,T)\times\partial\mathcal{M}} = 0, \quad (u, \partial_t u)|_{t=0} = (u_0, u_1). \tag{11}$$

For all $T > 2\mathcal{L}(\mathcal{M}, \omega)$, there exist $C, \mathfrak{K}, \mu_0 > 0$ such that we have

$$\|(u_0,u_1)\|_{L^2(\mathcal{M})\times H^{-1}(\mathcal{M})} \leq Ce^{\mathfrak{K}\mu}\|u\|_{L^2((0,T)\times\omega)} + \frac{1}{\mu}\|(u_0,u_1)\|_{H^1_0(\mathcal{M})\times L^2(\mathcal{M})}$$
$$\text{for all } \mu \geq \mu_0 \text{ and all } (u_0,u_1) \in H^1_0(\mathcal{M}) \times L^2(\mathcal{M}), \text{ and } u \text{ a solution to (11)}. \tag{12}$$

Recall the definition of $\mathcal{L}(\mathcal{M}, \omega)$ in (4). Note that this last estimate is equivalent to (see [Laurent and Léautaud 2019] or Corollary 2.2 below)

$$\|(u_0,u_1)\|_{H^1_0(\mathcal{M})\times L^2(\mathcal{M})} \leq C'e^{\mathfrak{K}'\Lambda}\|u\|_{L^2((0,T)\times\omega)}, \quad \Lambda = \frac{\|(u_0,u_1)\|_{H^1_0(\mathcal{M})\times L^2(\mathcal{M})}}{\|(u_0,u_1)\|_{L^2(\mathcal{M})\times H^{-1}(\mathcal{M})}}$$
$$\text{for all } (u_0,u_1) \in H^1_0(\mathcal{M}) \times L^2(\mathcal{M}), \text{ and } u \text{ a solution to (11)}. \tag{13}$$

Note that in [Laurent and Léautaud 2019], the observation term in the right-hand side of these inequalities is $\|u\|_{L^2(0,T;H^1(\omega))}$ instead of $\|u\|_{L^2((0,T)\times\omega)}$. That the stronger inequalities above hold is proved in [Laurent and Léautaud 2017, Section 5.3]; see also [Laurent and Léautaud $\geq$ 2021].

In all these inequalities, we are interested in the "best constant $\mathfrak{K}$" such that the estimate holds for some C. More precisely, we are interested in the way it depends on the geometry of $(\mathcal{M}, g)$ and ω (and, in the case of (12), the time T). Let us first formulate the precise definitions of these constants. These are the analogues to that of $\mathfrak{K}_{\mathrm{heat}}(\omega)$ given in (3).

Definition 1.5. Given $\omega \subset \mathcal{M}$ an open set, we define $\mathfrak{K}_{\mathrm{eig}}(\omega)$, $\mathfrak{K}_{\Sigma}(\omega)$, $\mathfrak{K}_{\infty}(\omega)$, $\mathfrak{K}_{\mathrm{wave}}(\omega, T)$ to be the best exponents in the above estimates (8)–(12), namely,

$$\begin{aligned}
\mathfrak{K}_{\mathrm{eig}}(\omega) &= \inf\{\mathfrak{K} > 0 : \text{there exists } C > 0 \text{ such that (8) holds}\},\\
\mathfrak{K}_{\Sigma}(\omega) &= \inf\{\mathfrak{K} > 0 : \text{there exists } C > 0 \text{ such that (9) holds}\},\\
\mathfrak{K}_{\infty}(\omega) &= \inf\{\mathfrak{K} > 0 : \text{there exists } C > 0 \text{ such that (10) holds}\},\\
\mathfrak{K}_{\mathrm{wave}}(\omega, T) &= \inf\{\mathfrak{K} > 0 : \text{there exist } C > 0,\ \mu_0 > 0 \text{ such that (12) holds}\}\\
&= \inf\{\mathfrak{K}' > 0 : \text{there exists } C' > 0 \text{ such that (13) holds}\}.
\end{aligned} \tag{14}$$

A proof of the equality in (14) is given in Corollary 2.2 below. Note that we may write $\mathfrak{K}_{\mathrm{wave}}(\omega, T) = +\infty$ if $T < 2\mathcal{L}(\mathcal{M}, \omega)$ since (12)–(13) are known not to hold; see the discussion in [Laurent and Léautaud 2019]. However, $\mathfrak{K}_{\mathrm{wave}}(\omega, T) < +\infty$ as soon as $T > 2\mathcal{L}(\mathcal{M}, \omega)$, by virtue of (12)–(13).

Let us now collect some known facts concerning these constants, in addition to the already-discussed bound $\mathfrak{K}_{\mathrm{heat}}(\omega) \geq \frac{1}{4}\mathcal{L}(\mathcal{M}, \omega)^2$ [Miller 2004a]. A first trivial (but useful) fact is that $\mathfrak{K}_{\mathrm{eig}}(\omega) \leq \mathfrak{K}_{\Sigma}(\omega)$. The following properties can also be found in the literature:

(1) For all $(\mathcal{M}, g)$, ω such that $\bar{\omega} \neq \mathcal{M}$, we have $\mathfrak{K}_{\Sigma}(\omega) \geq \frac{1}{2}\mathcal{L}(\mathcal{M}, \omega)$; see [Miller 2010, Theorem 5.3] (that $\mathfrak{K}_{\Sigma}(\omega) > 0$ had already been proved in [Jerison and Lebeau 1999]).

(2) $\mathfrak{K}_{\infty}(\omega) \leq \mathfrak{K}_{\mathrm{heat}}(\omega)$ [Miller 2006b, Theorem 1].

(3) For all $(\mathcal{M}, g)$, ω, we have $\mathfrak{K}_{\infty}(\omega) \geq \frac{1}{4}d_1(\omega)^2$, with

$$d_1(\omega) = \sup\{r > 0 : \text{there exists } x \in \mathcal{M} \text{ such that } B(x, r) \subset \mathcal{M} \setminus \bar{\omega}\};$$

see [Fernández-Cara and Zuazua 2000; Zuazua 2001, Section 4.1].

(4) Assume ω satisfies the geometric control condition in $(\mathcal{M}, g)$ and denote by L_ω the maximal length of a ray of geodesic optics not intersecting ω. Then, we have

$$\mathfrak{K}_{\mathrm{heat}}(\omega) \leq \alpha_* L_\omega^2 \tag{15}$$

with $\alpha_* = 2\big(\frac{36}{37}\big)^2$; see [Miller 2004a; 2006a] (improved to $\alpha_* = \frac{3}{4}$ in [Tenenbaum and Tucsnak 2007] and to 0.6966 in [Dardé and Ervedoza 2019]).

(5) Assume ω satisfies the geometric control condition in $(\mathcal{M}, g)$ and denote by L_ω the maximal length of a ray of geometric optics not intersecting ω. Then, we have $\mathfrak{K}_{\infty}(\omega) \leq \frac{1}{16}L_\omega^2$; see [Ervedoza and Zuazua 2011b, Theorem 1.1].

(6) $\mathfrak{K}_{\mathrm{heat}}(\omega) \leq 4\mathfrak{K}_{\Sigma}(\omega)^2$; see [Miller 2010, Corollary 1 and Section 2.4] (see also [Seidman 2008] for a proof of $\mathfrak{K}_{\mathrm{heat}}(\omega) \leq 8\mathfrak{K}_{\Sigma}(\omega)^2$).

(7) If (ω, T) satisfy the geometric control condition [Bardos et al. 1992], then $\mathfrak{K}_{\text{wave}}(\omega, T) = 0$ (more precisely, (12)–(13) hold with $\mathfrak{K} = \mathfrak{K}' = 0$). Conversely, if $(\mathcal{M}, g)$ is real-analytic and $(\bar{\omega}, T)$ does not satisfy the geometric control condition (for a ray that only intersects $\partial\mathcal{M}$ transversally), then $\mathfrak{K}_{\text{wave}}(\omega, T) > 0$; see [Lebeau 1992a].

Notice that in all these statements, the constants $\mathfrak{K}_{\text{heat}}$ and $\mathfrak{K}_\infty$ (heat equation) are homogeneous to a square of a distance (as for the heat kernel), whereas the other ones are homogeneous to a distance (as for the wave kernel).

Note also that every comparison statement above follows, in the associated reference, from a proper inequality (the above statements being only weak forms of those).

Also notice that the converse inequality $\mathfrak{K}_\Sigma(\omega)^2 \leq C\mathfrak{K}_{\text{heat}}(\omega)$ for a universal constant C does not seem to hold in general. For instance, in the related situation of boundary control on an interval $(0, 1)$ (see Section 1D), $\mathfrak{K}_{\text{heat}}(\{0\})$ is finite, while a dimensional analysis shows that no spectral inequality holds true; i.e., $\mathfrak{K}_\Sigma(\{0\})$ is infinite.

We first complete the above list of comparison results by the following proposition.

Proposition 1.6 (other links between the constants). *We have*

$$\tfrac{1}{4}\mathfrak{K}_{\text{eig}}(\omega)^2 \leq \mathfrak{K}_{\text{heat}}(\omega), \qquad \tfrac{1}{4}\mathfrak{K}_{\text{eig}}(\omega)^2 \leq \mathfrak{K}_\infty(\omega).$$

Also for all $T > 0$, we have $\mathfrak{K}_{\text{eig}}(\omega) \leq \mathfrak{K}_{\text{wave}}(\omega, T)$.

Note that the last statement is empty if $T < 2\mathcal{L}(\mathcal{M}, \omega)$, since (12)–(13) are known not to hold (see the discussion in [Laurent and Léautaud 2019]), but is nonempty if we have $\mathfrak{K}_{\text{wave}}(\omega, T) < \infty$, that is, if $T > 2\mathcal{L}(\mathcal{M}, \omega)$ by virtue of (12)–(13).

Hence, in order to produce lower bounds for $\mathfrak{K}_\Sigma(\omega)$, $\mathfrak{K}_{\text{heat}}(\omega)$, $\mathfrak{K}_\infty(\omega)$, $\mathfrak{K}_{\text{wave}}(\omega, T)$, we shall produce lower bounds for $\mathfrak{K}_{\text{eig}}(\omega)$, i.e., construct sequences of eigenfunctions having a maximal vanishing rate on ω. Note also that, summarizing the inequalities so far, we have

$$\tfrac{1}{4}\mathfrak{K}_{\text{eig}}(\omega)^2 \leq \mathfrak{K}_\infty(\omega) \leq \mathfrak{K}_{\text{heat}}(\omega) \leq 4\mathfrak{K}_\Sigma(\omega)^2, \tag{16}$$

so that the understanding of concentration properties for eigenfunctions and sums of eigenfunctions essentially contains those of the heat equation. Therefore, our main focus in the following is to produce

- maximally vanishing eigenfunctions in particular geometries to yield a lower bound for $\mathfrak{K}_{\text{eig}}$,
- a uniform Lebeau–Robbiano spectral inequality on small balls to yield an upper bound for $\mathfrak{K}_\Sigma$.

Note that reducing our attention to $\mathfrak{K}_{\text{eig}}$ in the search for lower bounds is already very restrictive! Indeed, as soon as the Schrödinger equation on $(\mathcal{M}, g)$ is observable from ω in finite time (in particular if ω satisfies the geometric control condition, see [Bardos et al. 1992; Lebeau 1992b]), then $\mathfrak{K}_{\text{eig}}(\omega) = 0$ (more precisely, (8) holds with $\mathfrak{K} = 0$).

Before starting to state these lower/upper bounds, let us give a link between $\mathfrak{K}_{\text{heat}}(\omega)$ and $\mathfrak{K}_{\text{wave}}(\omega, T)$, a consequence of a result of [Ervedoza and Zuazua 2011a] (weak observability with exponential cost for the wave equation implies observability of the heat equation).

Proposition 1.7. *There exist universal constants $\alpha_1, \alpha_2 > 0$ so that for any $S > 0$, we have*

$$\mathfrak{K}_{\mathrm{heat}}(\omega) \le \alpha_1 S^2 + \alpha_2 \mathfrak{K}_{\mathrm{wave}}(\omega, S)^2.$$

The proof of this result in Section 2C is a little more precise about this estimate. In particular, several values of (α_1, α_2) can be deduced from it. The value of α_1 is thought to be related to the cost of the boundary control of the one-dimensional heat equation. Note that, as in (16), this yields

$$\tfrac{1}{4}\mathfrak{K}_{\mathrm{eig}}(\omega)^2 \le \mathfrak{K}_{\infty}(\omega) \le \mathfrak{K}_{\mathrm{heat}}(\omega) \le \alpha_1 S^2 + \alpha_2 \mathfrak{K}_{\mathrm{wave}}(\omega, S)^2 \quad \text{for all } S > 0.$$

However, this upper bound seems for the moment less useful than that of (16), since the proof of (12)–(13) in [Laurent and Léautaud 2019] is more technically involved than that of (9) in [Lebeau and Robbiano 1995; Jerison and Lebeau 1999; Lebeau and Zuazua 1998]. The computation of $\mathfrak{K}_{\mathrm{wave}}(\omega, S)$ seems thus more intricate than that of $\mathfrak{K}_{\Sigma}(\omega)$.

1C. *Main results.*

1C1. *Constructing maximally vanishing eigenfunctions*: *lower bounds for* $\mathfrak{K}_{\mathrm{eig}}$. In this section, we provide lower bounds for $\mathfrak{K}_{\mathrm{eig}}$ in three different geometries. This then proves Theorem 1.2 as a direct corollary of Proposition 1.6.

The sphere. We first state the results we obtain on the two-dimensional sphere $\mathbb{S}^2$, since they are particularly simple. The higher-dimensional case $\mathbb{S}^n$ is completely similar. The sphere $\mathbb{S}^2$ is parametrized by $(s, \theta) \in (0, \pi) \times \mathbb{S}^1$. We denote by N the north pole described by $s = 0$ and by S the south pole described by $s = \pi$, and remark that s is the geodesic distance to the point N.

Theorem 1.8. *For $k \in \mathbb{N}$, the function*

$$\psi_k(s, \theta) = c_k \sin(s)^k e^{ik\theta}, \qquad c_k = \frac{k^{1/4}}{2^{1/2}\pi^{3/4}}\Big(1 + O\Big(\frac{1}{k}\Big)\Big) \quad \text{as } k \to +\infty$$

satisfies

$$-\Delta_g \psi_k = k(k+1)\psi_k \quad \text{on } \mathbb{S}^2, \qquad \psi_k \in C^\infty(\mathbb{S}^2), \qquad \|\psi_k\|_{L^2(\mathbb{S}^2)} = 1,$$

$$|\psi_k(s, \theta)| = c_k \sin(s)^k \le c_k s^k \quad \text{for } s \in [0, \pi],\ k \in \mathbb{N},$$

$$\|\psi_k\|^2_{L^2(B(N,r))} = \frac{c_k^2 \pi}{k+1}\frac{\sin(r)^{2k+2}}{\cos(r)}(1 + R), \qquad |R| \le \frac{\tan(r)^2}{2k+2} \quad \text{for } r \in \big[0, \tfrac{\pi}{2}\big),\ k \in \mathbb{N}.$$

This result is a much more explicit, more precise (and simpler to prove) version of the general results we obtain on surfaces of revolution. We turn to the general case and shall explain at the end of the section the links with Theorem 1.8.

Surfaces of revolutions. The precise description of the geometry of the surfaces we consider is given in Section 3B and we only give here the features required to state the result. We consider $\mathcal{M} = \mathcal{S} \subset \mathbb{R}^3$ a smooth compact surface diffeomorphic to the sphere $\mathbb{S}^2$. We assume moreover that it has revolution invariance around an axis, that intersects $\mathcal{S}$ in two points, the north and the south poles, respectively $N, S \in \mathcal{S}$. These points are the only invariant points of the revolution symmetry. The surface is then endowed with the metric g inherited from the Euclidean metric on $\mathbb{R}^3$, which itself enjoys the rotation invariance. Then, we describe (almost all) the surface by two coordinates, namely $s = \operatorname{dist}_g(\cdot, N)$, the geodesic distance to the north pole and θ, the angle of rotation. The variable s is in $(0, L)$ where $L = \operatorname{dist}_g(N, S)$. The surface is characterized by the function $R(s)$ associating to s the Euclidean distance in $\mathbb{R}^3$ to the symmetry axis, which, by definition, is rotationally invariant, and satisfies $R(0) = 0 = R(L)$. This function R is the "profile" of the revolution surface $\mathcal{S}$.

We shall now assume that R reaches at s_0 a global maximum, and introduce the relevant Agmon distance to the "equator" $s = s_0$, defined by the eikonal equation

$$\big(d_A'(s)\big)^2 - \left(\frac{1}{R(s)^2} - \frac{1}{R(s_0)^2}\right) = 0, \quad d_A(s_0) = 0, \quad \operatorname{sgn}(d_A'(s_0)) = \operatorname{sgn}(s - s_0), \tag{17}$$

or, more explicitly, for $s \in (0, L)$, by

$$d_A(s) = \left| \int_{s_0}^{s} \sqrt{\frac{1}{R(y)^2} - \frac{1}{R(s_0)^2}}\, dy \right|. \tag{18}$$

A more intrinsic definition of d_A is given in Remark 3.3 below (and requires additional notation).

Theorem 1.9. *Assume that $s \mapsto R(s)$ admits a **nondegenerate strict global** maximum at $s_0 \in (0, L)$. Then, for all $k \in \mathbb{N}$, there exists $\psi_k \in C^\infty(\mathcal{S})$ and $\lambda_k \geq 0$ such that*

$$\lambda_k = \frac{k^2}{R(s_0)^2} + k\sqrt{\frac{|R''(s_0)|}{R^3(s_0)}} + O(k^{\frac{1}{2}}), \quad \|\psi_k\|_{L^2(\mathcal{S})} = 1, \quad -\Delta_g \psi_k = \lambda_k \psi_k.$$

Moreover, there exist $C, C_*, C_0, k_0 > 0$ such that, for all $k \in \mathbb{N}$, $k \geq k_0$ and all $0 \leq r \leq s_0$, we have the estimate

$$\|\psi_k\|_{L^2(B(N,r))} \leq C \lambda_k^{C_0} e^{-d_A(r)(R(s_0)\sqrt{\lambda_k} - C_*)}.$$

Note that one can choose any $C_* > \frac{1}{2}\sqrt{|R''(s_0)| R(s_0)}$ in this result. This statement has to be completed by the asymptotic behavior of d_A (proved in Lemma 3.8) when $s \to 0$, namely

$$d_A(s) = -\log(s) + O(1) \quad \text{as } s \to 0^+. \tag{19}$$

That is to say that the equator and the poles are infinitely distant to each other for the Agmon distance d_A (as opposed to the geodesic distance dist_g). Note that at first order, d_A does not depend on the geometry of the surface $\mathcal{S}$ close to the north pole N $(s = 0)$. A similar statement holds close to the south pole S $(s = L)$.

This, together with Definition 1.5 and Proposition 1.6, yields the following direct corollary.

Corollary 1.10. *Under the assumptions of Theorem 1.9, for all $0 \leq r \leq s_0$, we have the estimate*

$$\mathfrak{K}_{\mathrm{eig}}(B_g(N, r)) \geq d_A(r) R(s_0).$$

This yields also

$$\mathfrak{K}_\Sigma(B_g(N, r)) \geq d_A(r) R(s_0), \qquad \mathfrak{K}_{\mathrm{wave}}(B_g(N, r), T) \geq d_A(r) R(s_0) \quad \textit{for any } T > 0,$$
$$\mathfrak{K}_\infty(B_g(N, r)) \geq \tfrac{1}{4}(d_A(r) R(s_0))^2, \qquad \mathfrak{K}_{\mathrm{heat}}(B_g(N, r)) \geq \tfrac{1}{4}(d_A(r) R(s_0))^2.$$

Note also that Theorem 1.9, combined with the explicit asymptotic expansion (19) of the Agmon distance d_A implies the following result.

Corollary 1.11 (rate of vanishing). *With (λ_k, ψ_k) as in Theorem 1.9, there exist $C, C_*, C_0, k_0 > 0$ such that, for all $k \in \mathbb{N}$, $k \geq k_0$ and all $r \geq 0$, we have*

$$\|\psi_k\|_{L^2(B(N,r))} \leq C e^{C\sqrt{\lambda_k}} r^{R(s_0)\sqrt{\lambda_k} - C_*},$$

and, in any local chart centered at N, we have $\partial^\alpha \psi_k(N) = 0$ for all $|\alpha| < R(s_0)\sqrt{\lambda_k} - C_* - \frac{n}{2}$.

As on the sphere, these eigenfunctions saturate the maximal vanishing rate predicted by the Donnelly–Fefferman theorem [1988].

Note that in these estimates, $R(s_0)\sqrt{\lambda_k} \sim k$ does not depend on the geometry.

The proofs rely on classical semiclassical decay estimates for eigenfunctions [Simon 1983; Helffer and Sjöstrand 1984]. We refer to the monographs [Helffer 1988; Dimassi and Sjöstrand 1999] for the historical background and more references. An additional difficulty here is linked to the degeneracy of the function R close to the north and south poles.

Note also that, to our knowledge, the idea of constructing such examples on surfaces of revolution is due to Lebeau [1996] and Allibert [1998].

The disk. Recall that $\mathbb{D} = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$. Our results on the disk are quite similar to the previous results on revolution surfaces. They are proved in Section 3C. Note the construction is more explicit there since it involves Bessel functions. As in the above example, the concentration is related to an Agmon distance to the maximum of the radius r, which corresponds to the boundary $\partial\mathbb{D}$ here.

Theorem 1.12 (whispering galleries on the disk). *Define, for $r \in (0, 1]$,*

$$d_A(r) = -(\tanh(\alpha(r)) - \alpha(r)), \quad \textit{with } \alpha(r) = \cosh^{-1}\left(\tfrac{1}{r}\right). \tag{20}$$

Then, for all $k \in \mathbb{N}$, there exists $\psi_k \in C^\infty(\mathbb{D}) \cap H_0^1(\mathbb{D})$ and $\lambda_k \geq 0$ such that

$$\lambda_k = k^2 + O(k^{\frac{4}{3}}), \quad \|\psi_k\|_{L^2(S)} = 1, \quad -\Delta_g \psi_k = \lambda_k \psi_k.$$

Moreover, there exist $C, \beta, k_0 > 0$ such that for all $k \geq k_0$ and $0 < r \leq 1 - \beta\lambda_k^{-1/3}$, we have

$$\|\psi_k\|_{L^\infty(B(0,r))} \leq \exp(-(\sqrt{\lambda_k} - C\lambda_k^{\frac{1}{6}}) d_A(r) + C\lambda_k^{\frac{1}{6}}).$$

That d_A indeed represents an Agmon distance in the present context is justified in the next paragraph. Note that d_A still satisfies $d_A(r) \sim_{r \to 0^+} \log\left(\frac{1}{r}\right)$ here, so that the analogues of Corollaries 1.10 and 1.11 still hold in this setting.

Remarks on the Agmon distance. In this paragraph we now compare the three geometries discussed above. In particular, we stress the fact that the results obtained on the sphere are refinements of those on general surfaces of revolution, and explain the similarities in the case of the disk.

Remark 1.13 (Agmon distance on the sphere). Note that the coordinates (s,θ) introduced on the unit sphere are the same as those defining general surfaces of revolution, with $L=\pi$, $s\in(0,\pi)$, $R(s)=\sin(s)$ and the maximum of R is reached at $s_0=\frac{\pi}{2}$. In particular, recalling the definition of the Agmon distance in (18), we obtain, for $s\in(0,\pi)$,

$$d_A(s)=\left|\int_{s_0}^{s}\sqrt{\frac{1}{R(y)^2}-\frac{1}{R(s_0)^2}}\,dy\right|=\left|\int_{\frac{\pi}{2}}^{s}\sqrt{\frac{1}{\sin(y)^2}-1}\,dy\right|=\left|\int_{\frac{\pi}{2}}^{s}\frac{\cos(y)}{\sin(y)}\,dy\right|=|\log(\sin(s))|.$$

This can be rewritten intrinsically as

$$d_A(m)=-\log(\sin(\mathrm{dist}_g(m,N))),\quad m\in\mathbb{S}^2\quad(\text{recall } \mathrm{dist}_g(m,N)+\mathrm{dist}_g(m,S)=\pi).$$

In view of this identity for the sphere, the estimates on the eigenfunctions ψ_k of Theorem 1.8 can be reformulated as ($\lambda_k=k(k+1)$)

$$|\psi_k(s,\theta)|=c_k e^{-kd_A(s)}\quad\text{for } s\in[0,\pi],\ k\in\mathbb{N},$$

$$\|\psi_k\|^2_{L^2(B(N,r))}=\frac{c_k^2\pi}{k+1}\frac{e^{-(2k+2)d_A(r)}}{\cos(r)}(1+R),\qquad |R|\le\frac{\tan(r)^2}{2k+2}\quad\text{for } r\in\left[0,\tfrac{\pi}{2}\right),\ k\in\mathbb{N}.$$

These two statements (pointwise estimate and fine asymptotics of the L^2-norm) are much more precise than those of Theorem 1.9 on general surfaces of revolution.

Note that one can put the disk in a general setting of surfaces of revolution with boundary. In this context, one can give a proof of (a slightly weaker version of) Theorem 1.12 following that of Theorem 1.9 (and only relying on Agmon estimates); see [Laurent and Léautaud 2021]. As opposed to the proof of Theorem 1.12, the latter proof does not make use of the explicit knowledge of eigenfunctions on the disk and properties of Bessel functions.

Remark 1.14 (Agmon distance in the disk). Recalling the definition of d_A in (20), we have

$$\alpha'(r)=-\frac{1}{r^2}\frac{1}{\sqrt{1/r^2-1}},$$

so that

$$(d_A'(r))^2=\alpha'(r)^2\left(\frac{1}{\cosh^2(\alpha(r))}-1\right)^2=\frac{1}{r^2}\frac{1}{1-r^2}(r^2-1)^2=\frac{1}{r^2}-1\quad\text{and}\quad d_A(1)=0.$$

As a consequence, d_A is exactly the Agmon distance to the boundary $r=1$, and we have

$$d_A'(r)=-\sqrt{\frac{1}{r^2}-1},\quad r\in(0,1].$$

Note again that $d_A(r)\sim_{r\to0^+}\log\left(\frac{1}{r}\right)$ and, in particular, the center of the disk is at infinite Agmon distance to the boundary: $d_A(0)=+\infty$.

1C2. *Uniform Lebeau–Robbiano spectral inequalities: upper bounds for* $\mathfrak{K}_\Sigma$. The counterpart of Corollary 1.11 is due to [Donnelly and Fefferman 1988] and roughly states that eigenfunctions vanish at most like $r^{C\sqrt{\lambda}+C}$ on balls of radius r (λ is the eigenvalue). It has been generalized in some sense to sums of eigenfunctions in [Jerison and Lebeau 1999]. We prove here a variant of this result under the form of a uniform Lebeau–Robbiano spectral inequality with observation on small balls.

Theorem 1.15 (uniform Lebeau–Robbiano spectral inequality with observation on small balls). *Let $(\mathcal{M}, g)$ be a compact Riemannian manifold with (or without) boundary $\partial\mathcal{M}$. For all $x_0 \in \mathcal{M}$, there exist constants $C_1, C_2 > 0$ such that for all $r > 0$, $\lambda \geq 0$ and $\psi \in E_{\leq\lambda}$ we have*

$$\|\psi\|_{L^2(\mathcal{M})} \leq e^{(C_1\sqrt{\lambda}+C_2)(1+\log(\frac{1}{r}))}\|\psi\|_{L^2(B(x_0,r))}.$$

Note that a careful inspection of the proofs (of all Carleman estimates used, that are stable by small perturbations) shows that the constants C_1, C_2 can actually be taken independent of the point x_0. Note that we prove the result in the case of Neumann boundary conditions as well. This uniform Lebeau–Robbiano spectral inequality directly implies Theorem 1.3 using [Miller 2010, Corollary 1] (recalled in Lemma 2.6 below).

One of the tools we develop for the proof of Theorem 1.3 also yields a uniform Lebeau–Robbiano inequality in a class of Lipschitz metrics. Even though it is not completely related to the main results of the paper, we choose to state it here since we believe it is of independent interest.

On the manifold $\mathcal{M}$, we denote here by $\mathfrak{g}$ a metric and $(\lambda_j^{\mathfrak{g}})_{j\in\mathbb{N}}$ the spectrum of the associated Laplace–Beltrami operator $-\Delta_{\mathfrak{g}}$ (with Dirichlet boundary condition if $\partial\mathcal{M} \neq \varnothing$) and by $(\psi_{\lambda_j}^{\mathfrak{g}})_{j\in\mathbb{N}}$ an associated Hilbert basis of eigenfunctions, in order to stress the dependence with respect to the metric. We also write

$$E_{\leq\lambda}^{\mathfrak{g}} = \operatorname{span}\{\psi_{\lambda_j}^{\mathfrak{g}} : \lambda_j^{\mathfrak{g}} \leq \lambda\},$$

which of course depends on the metric $\mathfrak{g}$. Now, given a reference Lipschitz metric $\mathfrak{g}_0$, we define

$$\Gamma_{\epsilon,D}(\mathcal{M}, \mathfrak{g}_0) = \{\mathfrak{g} \text{ Lipschitz continuous metric on } \mathcal{M} : \|\mathfrak{g}\|_{W^{1,\infty}(\mathcal{M})} \leq D,\ \epsilon\mathfrak{g}_0 \leq \mathfrak{g} \leq D\mathfrak{g}_0\}.$$

Theorem 1.16 (uniform Lebeau–Robbiano spectral inequality in a class of metrics). *Let $\mathcal{M}$ be a compact Riemannian manifold with (or without) boundary $\partial\mathcal{M}$, $\mathfrak{g}_0$ be a Lipschitz continuous Riemannian metric on $\mathcal{M}$, and $\omega \subset \mathcal{M}$ a nonempty open set. Then, for all $D \geq \epsilon > 0$, there exist constants $C, c > 0$ such that for all $\mathfrak{g} \in \Gamma_{\epsilon,D}(\mathcal{M}, \mathfrak{g}_0)$, $\lambda \geq 0$ and $w \in E_{\leq\lambda}^{\mathfrak{g}}$, we have*

$$\|w\|_{L^2(\mathcal{M})} \leq Ce^{c\sqrt{\lambda}}\|w\|_{L^2(\omega)}. \tag{21}$$

Note that the above estimate is valid whatever the choice of L^2-norm (i.e., with respect to $\mathfrak{g}$ or $\mathfrak{g}_0$) since all these norms are uniformly equivalent for metrics $\mathfrak{g}$ in the class $\Gamma_{\epsilon,D}(\mathcal{M}, \mathfrak{g}_0)$. This result could be reformulated by saying that (21) holds for all $w \in \bigcup_{\mathfrak{g}\in\Gamma_{\epsilon,D}(\mathcal{M},\mathfrak{g}_0)} E_{\leq\lambda}^{\mathfrak{g}}$.

This uniform Lebeau–Robbiano spectral inequality directly implies the following uniform estimate on the cost of the heat equation, using [Miller 2010, Corollary 1], recalled in Lemma 2.6 below (in which the constants are explicitly computed in terms of the constants in the spectral inequality).

Corollary 1.17. *Let $\mathcal{M}$ be a compact Riemannian manifold with (or without) boundary $\partial\mathcal{M}$, $\mathfrak{g}_0$ be a Lipschitz continuous Riemannian metric on $\mathcal{M}$, and $\omega \subset \mathcal{M}$ be a nonempty open set. Then, for all*

$D \geq \epsilon > 0$, *there exist constants* $C, \mathfrak{K} > 0$ *such that for all* $\mathfrak{g} \in \Gamma_{\epsilon,D}(\mathcal{M}, \mathfrak{g}_0)$ *we have*

$$\|e^{T\Delta_{\mathfrak{g}}}u\|^2_{L^2(\mathcal{M})} \leq Ce^{\frac{2\mathfrak{K}}{T}} \int_0^T \|e^{t\Delta_{\mathfrak{g}}}u\|^2_{L^2(\omega)}\,dt \quad \textit{for all } T > 0 \textit{ and all } u \in L^2(\mathcal{M}).$$

Note that the proofs of Theorem 1.16 and Corollary 1.17 are completely constructive, and, as such, provide explicitly computable constants.

1C3. *The case of a barrel*: *upper bounds for* $\mathfrak{K}_{\mathrm{wave}}$ *and* $\mathfrak{K}_{\mathrm{heat}}$. To conclude with the upper bounds on the constant, we present in this section some applications of results obtained in [Allibert 1998]. In the case of a "barrel-type surface" with boundary (a geometric setting close to that of the surfaces of revolution described above), Allibert estimates the attainable space for the controlled wave equation. As corollaries, we deduce from this result estimates of $\mathfrak{K}_{\mathrm{wave}}$ and, in view of Proposition 1.7, of $\mathfrak{K}_{\mathrm{heat}}$.

We first present the geometric context (which is very close to that of surfaces of revolution described above). In this section, $\mathcal{M} = \mathcal{S}$ is a surface of revolution of $\mathbb{R}^3$ with boundary, parametrized by the equation

$$\mathcal{S} = \{(x, y, z) \in \mathbb{R}^3 : z \in [0, \mathsf{L}],\ x^2 + y^2 = \mathsf{R}(z)\},$$

where R is a strictly positive smooth function on $[0, \mathsf{L}]$ that admits at the point $z_0 \in (0, \mathsf{L})$ a unique local (and therefore global) nondegenerate maximum (i.e., $\mathsf{R}''(z_0) < 0$). Observation takes place at the boundary, only on the bottom side, that is, $\Gamma = \{(x, y, 0) \in \mathbb{R}^3 : x^2 + y^2 = \mathsf{R}(0)\}$. We may also describe $\mathcal{S}$ by (z, θ), with $(x, y) = (\mathsf{R}(z)\cos\theta, \mathsf{R}(z)\sin\theta)$.

We refer to Remark 3.4 to explain the link between the two parametrizations of revolution surfaces by s and z (and in particular, that we may write $z = z(s)$ and $R(s) = \mathsf{R}(z(s))$).

As above, we define the Agmon distance to the point z_0, which in this z-parametrization can be written (note that it is almost the same as (18) but in different coordinates)

$$d_A(z) = \left| \int_{z_0}^{z} \sqrt{1 + \mathsf{R}'^2(y)} \sqrt{\frac{1}{\mathsf{R}(y)^2} - \frac{1}{\mathsf{R}(c)^2}}\, dy \right|.$$

We also need the following definition of a critical time T_1 (see [Allibert 1998] for more details), which, roughly speaking, represents the smallest period of the geodesic flow, modulo rotation. More precisely, the principal symbol of the wave operator on $\mathbb{R} \times \mathcal{S}$ is given by

$$p(t, z, \theta, \tau, \zeta, \eta) = \frac{\zeta^2}{1 + \mathsf{R}'^2(z)} + \frac{\eta^2}{\mathsf{R}^2(z)} - \tau^2,$$

where (τ, ζ, η) denotes the dual variables to (t, z, θ). For any (generalized) bicharacteristic curve γ of p, bouncing on the boundary according to the reflection law $\zeta \to -\zeta$, we denote by $T(\gamma)$ the smallest period of the function $\Pi_z(\gamma)$, where Π_z is the projection on the component z.

Then, T_1 is defined by

$$T_1 = \sup\{T(\gamma) : \gamma \text{ bicharacteristic curve of } p\},$$

and we have $T_1 \geq 2\mathcal{L}(\mathcal{M}, \Gamma)$ (this critical time is larger than the time of unique continuation from Γ).

In this context, we define similarly $\mathfrak{K}_{\mathrm{heat}}(\Gamma)$ and $\mathfrak{K}_{\mathrm{wave}}(\Gamma, T)$ with exactly the same definition as in (2) and Definition 1.5 with $\|u\|_{L^2([0,T]\times\omega)}$ replaced by $\|\partial_\nu u\|_{L^2([0,T]\times\Gamma)}$ in (1) and (12). Note that $\partial_\nu u$ is in $L^2([0,T]\times\Gamma)$ for initial data in L^2 (resp. $H_0^1\times L^2$) for the heat (resp. wave) equation thanks to hidden regularity. We deduce from [Allibert 1998] the following result.

Theorem 1.18. *Under the above geometric assumptions, we have the estimates*

$$\mathfrak{K}_{\mathrm{wave}}(\Gamma, T) \le \mathsf{R}(z_0) d_A(\Gamma) \quad \textit{for all } T > T_1, \tag{22}$$

$$\mathfrak{K}_{\mathrm{heat}}(\Gamma) \le \alpha(T_1(\Gamma)^2 + \mathsf{R}(z_0)^2 d_A(\Gamma)^2) \tag{23}$$

for some universal constant $\alpha > 0$.

The first estimate (22) follows simply from [Allibert 1998, Théorème 2] (see Proposition 2.7 below), which is stated in terms of analytic spaces with respect to the rotation variable θ. Then, (22) implies (23) thanks to Proposition 1.7. Note that (22) also proves an analogue of Theorem 1.9 in this geometry, so that in fact

$$\mathfrak{K}_{\mathrm{eig}}(\Gamma) = \mathsf{R}(z_0) d_A(\Gamma) \quad \text{and} \quad \mathfrak{K}_{\mathrm{wave}}(\Gamma, T) = \mathsf{R}(z_0) d_A(\Gamma) \quad \text{for all } T > T_1. \tag{24}$$

He also proves upper and lower estimates for $T \in (2\mathcal{L}(\mathcal{M}, \Gamma), T_1)$ (which do not coincide). This other set of estimates for $\mathfrak{K}_{\mathrm{wave}}(\Gamma, T)$ can also provide other estimates for $\mathfrak{K}_{\mathrm{heat}}(\Gamma)$ using the proof of Theorem 1.18 and Proposition 2.7. Note finally that in [Laurent and Léautaud 2021], using the methods of [Allibert 1998], we also prove that $\mathfrak{K}_{\mathrm{eig}}(B_g(N, r)) = d_A(r) R(s_0)$ in the context of Theorem 1.9 and Corollary 1.10.

1D. *Previous results.* Except for the bounds (24) (and that of $\mathfrak{K}_{\mathrm{eig}}$ in [Laurent and Léautaud 2021]) following from Allibert's result and the computation of $\mathfrak{K}_\infty(\{0\})$ on $\mathcal{M} = [0, L]$ in [Fattorini and Russell 1971], we are not aware of other situations in which the constants described in the previous paragraph are known exactly. We collect in this section previous results on the constants $\mathfrak{K}_{\mathrm{heat}}$ and $\mathfrak{K}_{\mathrm{wave}}$, which received a lot of attention in the past fifteen years.

Parabolic equations in one dimension. The most-studied case concerns the constant $\mathfrak{K}_{\mathrm{heat}}$, with observation/control at the boundary in the one-dimensional case, say $\mathcal{M} = [-1, 1]$. Yet, it seems that the constant $\mathfrak{K}_{\mathrm{heat}}(\{-1, 1\})$ is still unknown. Note that the latter has a particular importance since it has applications to higher dimensions (with geometric conditions) via the transmutation method of Luc Miller [2006a].

Here, we list previous results on $\mathcal{M} = [-1, 1]$ with Neumann trace observation (Dirichlet control) on both sides of the interval. Note also that each improvement of the constant was also the occasion of finding new techniques of proofs:

- $\mathfrak{K}_{\mathrm{heat}}(\{-1, 1\}) \le 2\left(\frac{36}{37}\right)^2$ [Miller 2006a], using the transmutation method.
- $\mathfrak{K}_{\mathrm{heat}}(\{-1, 1\}) \le \frac{3}{4}$ [Tenenbaum and Tucsnak 2007], using results of analytic number theory.
- $\mathfrak{K}_{\mathrm{heat}}(\{-1, 1\}) \ge \frac{1}{2}$ [Lissy 2015], using complex analysis arguments.
- $\mathfrak{K}_{\mathrm{heat}}(\{-1, 1\}) \le 0,7$ [Dardé and Ervedoza 2019], combining Carleman estimates and complex analysis.

Note that in this setting, the analogue of Conjecture 1.1 would be $\mathfrak{K}_{\text{heat}}(\{-1,1\}) = \frac{1}{4}$, which [Lissy 2015] disproved in this context (by a factor 2). However, this result does not in general prevent the existence of a universal constant $C > 0$ so that $\mathfrak{K}_{\text{heat}}(\omega) = C\mathcal{L}(\mathcal{M},\omega)^2$.

As noticed in [Ervedoza and Zuazua 2011b], the result in [Fattorini and Russell 1971] implies that on the interval $(0, L)$, we have $\mathfrak{K}_\infty(\{0\}) = \frac{1}{4}L^2$ (and [Ervedoza and Zuazua 2011b] even prove (10) for the critical $\mathfrak{K} = \frac{1}{4}L^2$).

Parabolic equations in higher dimensions. There are many papers concerning the controllability properties of the heat equation. We only mention those providing estimates on the constants studied in this paper.

The first computable estimates were obtained using the transmutation method to give estimates similar to (15). We can find several references improving the universal constant involved; see [Miller 2004a; 2006a; Tenenbaum and Tucsnak 2007; Dardé and Ervedoza 2019].

In [Tenenbaum and Tucsnak 2007], the authors prove $\mathfrak{K}_\Sigma(\omega^*) \le 3\log((4\pi e)^N/|\omega^*|)$, where $\mathcal{M} = (0,\pi)^N$ is a cubic domain and $|\omega^*|$ is the volume of the biggest rectangle included in ω. The proof of this result uses a number-theoretic argument of Turán concerning families of the complex exponential $(e^{ikx})_{k\in\mathbb{Z}}$ (which can be interpreted as an estimate of $\mathfrak{K}_\Sigma(I)$ for I a subinterval of $\mathbb{T}$). In this particular flat-torus geometry, we have no idea of what the right constant should be.

In [Bardos and Phung 2017], the authors prove $\mathfrak{K}_\Sigma(B(0,r)) \le C_\epsilon/r^\epsilon$ for all $\epsilon > 0$ in convex geometries. This has just been extended in [Phung 2018]. Our Theorem 1.3 improves this result. Note also that [Nakić et al. 2018] gave results related to this in a periodic setting, tracking uniformity with respect to several parameters.

In the Euclidean space $\mathbb{R}^n$ where Δ is the usual flat Laplacian, spectral estimates like (9) can be interpreted as a manifestation of the uncertainty principle. Several results relying on this fact have been recently stated. We refer for instance to [Egidi and Veselić 2018; Wang et al. 2019] and the references therein.

The wave equation. Lebeau [1992a] proved in the analytic setting a result close to the fact that $\mathfrak{K}_{\text{wave}}(\omega, T)$ is finite for any open set ω and in optimal time $T > 2\mathcal{L}(\mathcal{M},\omega)$. It was only very recently shown to be finite by the authors [Laurent and Léautaud 2019] in a general C^∞ context. We refer the reader to the introduction of that work for a detailed discussion of the literature on unique continuation for waves, and estimates like (12)–(13).

Estimates on analytic spaces of controllable data were computed by Allibert in the above-described examples. We refer to Section 2D for more details about why they have implications on the constant $\mathfrak{K}_{\text{wave}}$ (and therefore $\mathfrak{K}_{\text{heat}}$ by Proposition 1.7). In [Allibert 1998], he studied the example of the barrel as we describe it in Section 1C3. In [Allibert 1999], he studied the example of a cylinder $(0,\pi)\times\mathbb{S}^1$. The results he obtained in that paper imply $\mathfrak{K}_{\text{wave}}(\Gamma, T) \le C_\delta/T^{1-\delta}$, where $\Gamma = \{0\}\times\mathbb{S}^1$ and $T > 2\pi$. Notice finally that the blowup of the observability constant for the wave equation, when the time tends to the minimal geometric control time, has recently been investigated in [Laurent and Léautaud 2016].

1E. ***Plan of the paper.*** The paper is divided in four main parts. In Section 2, we give the links between the different constants, proving in particular Propositions 1.6 and 1.7. We also interpret the description of the reachable set as an upper bound on the constant $\mathfrak{K}_{\text{wave}}(\omega, T)$.

In Section 3, we construct the various counterexamples on rotationally invariant geometries, presented in Section 1C1. This proves in particular Theorem 1.2.

Section 4 is devoted to the proof of the uniform Lebeau–Robbiano inequality on small balls, stated in Theorem 1.15.

Finally, we prove in Section 5 the observability inequality of Theorem 1.4 concerning positive solutions of the heat equation.

The paper ends with two appendices; in the first, Appendix A, we prove a uniform Carleman estimate for bounded families of Lipschitz metrics. Such an estimate is used as an intermediate step in the proof of Theorem 1.15. The result also yields Theorem 1.16.

Note finally that in the companion paper [Laurent and Léautaud 2021], we apply similar techniques for the problem of uniform observability/controllability of transport equations in the vanishing viscosity limit.

2. Preliminaries: links between the different constants

2A. ***Different definitions of $\mathfrak{K}_{\mathrm{wave}}(\omega, T)$.*** Let us start by proving equality (14). This is a consequence of the following lemma.

Lemma 2.1. *Let $\mu_0 \geq 0$, $\mathfrak{K} \geq 0$ and assume that $\Lambda > 0$ and $X \geq 0$ satisfy*

$$\frac{1}{\Lambda} \leq e^{\mathfrak{K}\mu} X + \frac{1}{\mu} \quad \text{for all } \mu > \mu_0. \tag{25}$$

Then, for all $\alpha > 0$, we have

$$1 \leq \left(\mathbb{1}_{\Lambda+\alpha \leq \mu_0} \frac{\mu_0 - \alpha}{\alpha} e^{\mathfrak{K}\mu_0} + \mathbb{1}_{\Lambda+\alpha > \mu_0} \frac{e^{\mathfrak{K}\alpha}}{\alpha} \Lambda(\Lambda+\alpha) e^{\mathfrak{K}\Lambda} \right) X. \tag{26}$$

Let $F : \mathbb{R}^+ \to \mathbb{R}^+$ be a nondecreasing function and assume that $\Lambda > 0$ and $X \geq 0$ satisfy

$$\Lambda \geq 1 \quad \text{and} \quad 1 \leq F(\Lambda) X. \tag{27}$$

Then, we have

$$\frac{1}{\Lambda} \leq F(\mu) X + \frac{1}{\mu} \quad \text{for all } \mu > 0. \tag{28}$$

As a direct consequence of this lemma, we obtain the following corollary, clarifying the definition of $\mathfrak{K}_{\mathrm{wave}}(\omega, T)$.

Corollary 2.2. *Assume (12) with constants $\mathfrak{K}, C, \mu_0 > 0$. Then, there is $C'' > 0$ such that*

$$\|(u_0, u_1)\|_{H_0^1(\mathcal{M}) \times L^2(\mathcal{M})} \leq C'' \Lambda^2 e^{\mathfrak{K}\Lambda} \|u\|_{L^2((0,T)\times\omega)}, \qquad \Lambda = \frac{\|(u_0, u_1)\|_{H_0^1(\mathcal{M}) \times L^2(\mathcal{M})}}{\|(u_0, u_1)\|_{L^2(\mathcal{M}) \times H^{-1}(\mathcal{M})}}$$

for all $(u_0, u_1) \in H_0^1(\mathcal{M}) \times L^2(\mathcal{M})$ and u a solution to (11),

$$\|(u_0, u_1)\|_{L^2(\mathcal{M}) \times H^{-1}(\mathcal{M})} \leq C'' \mu^2 e^{\mathfrak{K}\mu} \|u\|_{L^2((0,T)\times\omega)} + \frac{1}{\mu} \|(u_0, u_1)\|_{H_0^1(\mathcal{M}) \times L^2(\mathcal{M})}$$

for all $\mu > 0$ and all $(u_0, u_1) \in H_0^1(\mathcal{M}) \times L^2(\mathcal{M})$, and u a solution to (11).

Reciprocally, if (13) *holds with constants* $\mathfrak{K}', C' > 0$, *then* (12) *holds with* $\mathfrak{K} = \mathfrak{K}'$, $C = C'$, *and* $\mu_0 = 0$ (*and for all* $\mu > 0$).

In particular, we have

$$\begin{aligned}\mathfrak{K}_{\text{wave}}(\omega, T) &= \inf\{\mathfrak{K} > 0 : \textit{there exist } C > 0,\ \mu_0 > 0 \textit{ such that (12) holds}\}\\ &= \inf\{\mathfrak{K}' > 0 : \textit{there exists } C' > 0 \textit{ such that (13) holds}\}\\ &= \inf\{\mathfrak{K} > 0 : \textit{there exists } C > 0 \textit{ such that (12) holds with } \mu_0 = 0 \textit{ (and all } \mu > 0)\}.\end{aligned}$$

Proof of Lemma 2.1. Let $\alpha > 0$. In the case $\Lambda + \alpha > \mu_0$, the assumption (25) with $\mu = \Lambda + \alpha > \mu_0$ yields

$$\frac{1}{\Lambda}\Big(1 - \frac{\Lambda}{\Lambda+\alpha}\Big) \le e^{\mathfrak{K}(\Lambda+\alpha)} X,$$

and hence

$$1 \le \frac{1}{\alpha} e^{\mathfrak{K}\alpha} \Lambda(\Lambda + \alpha) e^{\mathfrak{K}\Lambda} X. \tag{29}$$

If now $\Lambda + \alpha \le \mu_0$ (and, in particular, $\alpha < \mu_0$), that is $\frac{1}{\Lambda} \ge \frac{1}{\mu_0 - \alpha} > 0$, the assumption (25) implies

$$\frac{1}{\mu_0 - \alpha} \le \frac{1}{\Lambda} \le e^{\mathfrak{K}\mu} X + \frac{1}{\mu} \quad \text{for all } \mu \ge \mu_0.$$

This yields in particular

$$X \ge \Big(\frac{1}{\mu_0 - \alpha} - \frac{1}{\mu}\Big) e^{-\mathfrak{K}\mu} \quad \text{for all } \mu \ge \mu_0,$$

and hence

$$X \ge \max_{\mu \ge \mu_0} \Big(\frac{1}{\mu_0 - \alpha} - \frac{1}{\mu}\Big) e^{-\mathfrak{K}\mu} \ge \frac{\alpha}{\mu_0 - \alpha} e^{-\mathfrak{K}\mu_0} > 0.$$

With (29), this proves (26).

Let us now prove (28). If $\Lambda \ge \mu$, then $\frac{1}{\Lambda} \le \frac{1}{\mu}$ and (28) holds. If $\Lambda \le \mu$, then (27) gives $\frac{1}{\Lambda} \le 1 \le F(\Lambda) X \le F(\mu) X$ and (28) also holds in this case, concluding the proof. □

2B. ***The constant $\mathfrak{K}_{\text{eig}}(\omega)$ as a lower bound for $\mathfrak{K}_{\text{heat}}(\omega)$, $\mathfrak{K}_{\infty}(\omega)$, $\mathfrak{K}_{\text{wave}}(\omega, T)$: proof of Proposition 1.6.*** We prove a slightly more precise version of Proposition 1.6.

Lemma 2.3. *Assume that* (1) *holds with constants* $\mathfrak{K}, C > 0$. *Then, we have*

$$\|\psi\|_{L^2(\mathcal{M})} \le \sqrt{\frac{C}{2\lambda}} e^{2\sqrt{\mathfrak{K}\lambda}} \|\psi\|_{L^2(\omega)} \quad \textit{for all } \lambda \in \mathrm{Sp}(-\Delta_g) \setminus \{0\} \textit{ and } \psi \in E_\lambda. \tag{30}$$

In particular,

$$\tfrac{1}{4}\mathfrak{K}_{\text{eig}}(\omega)^2 \le \mathfrak{K}_{\text{heat}}(\omega). \tag{31}$$

Assume that (10) *holds with constants* $\mathfrak{K}, C > 0$. *Then, there exists* $C'' > 0$ *such that*

$$\|\psi\|_{L^2(\mathcal{M})} \le \frac{C''}{\lambda^{1/8}} e^{2\sqrt{\mathfrak{K}\lambda}} \|\psi\|_{L^2(\omega)} \quad \textit{for all } \lambda \in \mathrm{Sp}(-\Delta_g) \setminus \{0\} \textit{ and } \psi \in E_\lambda. \tag{32}$$

In particular

$$\tfrac{1}{4}\mathfrak{K}_{\text{eig}}(\omega)^2 \le \mathfrak{K}_{\infty}(\omega). \tag{33}$$

Assume that (13) *holds in time T with constants C', $\mathfrak{K}'$. Then, we have*

$$\|\psi\|_{L^2(\mathcal{M})} \le \sqrt{\frac{T}{\lambda}} C' e^{\mathfrak{K}'\sqrt{\lambda}} \|\psi\|_{L^2(\omega)} \quad \textit{for all } \lambda \in \operatorname{Sp}(-\Delta_g)\setminus\{0\} \textit{ and } \psi \in E_\lambda. \tag{34}$$

In particular, for all $T > 0$, we have $\mathfrak{K}_{\mathrm{eig}}(\omega) \le \mathfrak{K}_{\mathrm{wave}}(\omega, T)$.

Proof of Proposition 1.6. From (1), applied to $u(t,x) = e^{-t\lambda}\psi(x)$ with $\lambda \in \operatorname{Sp}(-\Delta_g)\setminus\{0\}$ and $\psi \in E_\lambda$, we have

$$e^{-2T\lambda}\|\psi\|^2_{L^2(\mathcal{M})} \le C e^{\frac{2\mathfrak{K}}{T}} \int_0^T e^{-2t\lambda}\|\psi\|^2_{L^2(\omega)}\,dt = C e^{\frac{2\mathfrak{K}}{T}} \frac{1-e^{-2T\lambda}}{2\lambda}\|\psi\|^2_{L^2(\omega)} \quad \text{for all } T > 0.$$

Taking $T = D/\sqrt{\lambda}$, with $D > 0$ to be chosen, this implies

$$\|\psi\|^2_{L^2(\mathcal{M})} \le C e^{2T\lambda} e^{\frac{2\mathfrak{K}}{T}} \frac{1}{2\lambda}\|\psi\|^2_{L^2(\omega)} = \frac{C}{2\lambda} e^{2\sqrt{\lambda}(D+\frac{\mathfrak{K}}{D})}\|\psi\|^2_{L^2(\omega)}.$$

Minimizing the exponent with respect to D leads to choosing $D = \sqrt{\mathfrak{K}}$, which implies (30) when taking the square root. From (30), inequality (31) follows directly when taking the infimum over all $\mathfrak{K}$.

Let us now prove the second statement of the proposition. From (10), again applied to $u(t,x) = e^{-t\lambda}\psi(x)$ with $\lambda \in \operatorname{Sp}(-\Delta_g)\setminus\{0\}$ and $\psi \in E_\lambda$, we have

$$\int_{\mathbb{R}^+} e^{-\frac{2\mathfrak{K}}{t}} e^{-2t\lambda}\|\psi\|^2_{L^2(\mathcal{M})}\,dt \le C \int_{\mathbb{R}^+} e^{-2t\lambda}\|\psi\|^2_{L^2(\omega)}\,dt = \frac{C}{2\lambda}\|\psi\|^2_{L^2(\omega)}. \tag{35}$$

The left-hand side may also be computed asymptotically for $\lambda \to +\infty$ using the Laplace method, setting $\mu = \sqrt{\lambda}$, as

$$\begin{aligned}
\int_{\mathbb{R}^+} e^{-\frac{2\mathfrak{K}}{t}} e^{-2\mu^2 t}\,dt &= \int_{\mathbb{R}^+} e^{-2\sqrt{\mathfrak{K}}\mu(\frac{1}{s}+s)} \frac{\sqrt{\mathfrak{K}}}{\mu}\,ds \\
&= (1+o(1))\frac{\sqrt{\mathfrak{K}}}{\mu}\int_{\mathbb{R}} e^{-2\sqrt{\mathfrak{K}}\mu(2+(s-1)^2)}\,ds \\
&= (1+o(1))\frac{\sqrt{\mathfrak{K}}}{\mu} e^{-4\sqrt{\mathfrak{K}}\mu}\sqrt{\frac{\pi}{2\sqrt{\mathfrak{K}}\mu}} = (1+o(1))\left(\frac{\pi\sqrt{\mathfrak{K}}}{2\mu^3}\right)^{\frac12} e^{-4\sqrt{\mathfrak{K}}\mu}.
\end{aligned}$$

From (35), we then obtain that, for any eigenfunction ψ associated to the eigenvalue μ^2 for $\mu \to \infty$, we have

$$(1+o(1))\left(\frac{\pi\sqrt{\mathfrak{K}}}{2\mu^3}\right)^{\frac12} e^{-4\sqrt{\mathfrak{K}}\mu}\|\psi\|^2_{L^2(\mathcal{M})} \le \frac{C}{2\mu^2}\|\psi\|^2_{L^2(\omega)}.$$

Coming back to $\lambda = \mu^2$, this implies the existence of $\widetilde{C}, \lambda_0 > 0$ such that for all $\lambda \ge \lambda_0$

$$\|\psi\|^2_{L^2(\mathcal{M})} \le \frac{\widetilde{C}}{\lambda^{1/4}} e^{4\sqrt{\mathfrak{K}\lambda}}\|\psi\|^2_{L^2(\omega)},$$

and hence the sought result of (32). That of (33) follows as above.

Let us now prove the last statement of the proposition. We want to apply (13) to the function $u(t,x)=\cos(t\sqrt{\lambda})\psi$ with $\lambda\in \mathrm{Sp}(-\Delta_g)\setminus\{0\}$ and $\psi\in E_\lambda$, which is a particular solution to (11). We have

$$\Lambda=\frac{\|(u|_{t=0},\partial_t u|_{t=0})\|_{H_0^1(\mathcal{M})\times L^2(\mathcal{M})}}{\|(u|_{t=0},\partial_t u|_{t=0})\|_{L^2(\mathcal{M})\times H^{-1}(\mathcal{M})}}=\frac{\|\psi\|_{H_0^1(\mathcal{M})}}{\|\psi\|_{L^2(\mathcal{M})}}=\sqrt{\lambda}$$

and (13) then yields

$$\sqrt{\lambda}\|\psi\|_{L^2(\mathcal{M})}=\|\psi\|_{H_0^1(\mathcal{M})}=\|u|_{t=0},\partial_t u|_{t=0})\|_{H_0^1(\mathcal{M})\times L^2(\mathcal{M})}\le C'e^{\mathfrak{K}'\Lambda}\|u\|_{L^2((0,T)\times\omega)},$$

where

$$\|u\|^2_{L^2((0,T)\times\omega)}=\int_0^T\cos^2(t\sqrt{\lambda})\|\psi\|^2_{L^2(\omega)}\,dt\le T\|\psi\|^2_{L^2(\omega)}.$$

This finally implies (34). The last result follows from Corollary 2.2. □

2C. *Link between $\mathfrak{K}_{\mathrm{heat}}(\omega)$ and $\mathfrak{K}_{\mathrm{wave}}(\omega,T)$: proof of Proposition 1.7.* The proof will follow very closely the method of [Ervedoza and Zuazua 2011a], but with a different assumption. Note that this strategy was applied to approximate controllability problems for parabolic equations in [Laurent and Léautaud 2017]. We first summarize the results we need from [Ervedoza and Zuazua 2011a; 2011b] in the next proposition for readability.

Proposition 2.4 [Ervedoza and Zuazua 2011a; 2011b]. *Let $T,S>0$ and $\alpha>2S^2$. Let $\mathcal{L}$ be a negative self adjoint operator. Then, there exists a kernel function $k_T(t,s)$ such that:*

- *If y is solution of the heat equation $\partial_s w-\mathcal{L}w=0$, then $w(s)=\int_0^T k_T(t,s)y(t)\,dt$ is solution of*

$$\begin{cases}\partial_s^2 w-\mathcal{L}w=0 & \textit{for } s\in(-S,S),\\ (w,\partial_s w)|_{s=0}=\left(0,\int_0^T\partial_s k_T(t,0)y(t)\,dt\right)=\left(0,\int_0^T e^{-\alpha\left(\frac{1}{t}+\frac{1}{T-t}\right)}y(t)\,dt\right).\end{cases}\tag{36}$$

- *For all $\delta\in(0,1)$ and all $(t,s)\in(0,T)\times(-S,S)$, we have*

$$|k_T(t,s)|\le|s|\exp\left(\frac{1}{\min\{t,T-t\}}\left(\frac{s^2}{\delta}-\frac{\alpha}{(1+\delta)}\right)\right).\tag{37}$$

Note that this last estimate is most useful for δ sufficiently close to 1 so that $\alpha\ge S^2\left(1+\frac{1}{\delta}\right)$.

We first prove from this proposition an observability inequality for data in $E_{\le\lambda}$ (i.e., at low frequency) as a consequence of the approximate observability result for waves (13) (coming from [Laurent and Léautaud 2019]), with a precise dependence on the cutoff frequency λ and the control time T. Combined with an argument of [Miller 2010], this allows us to prove observability for all data in $L^2(\mathcal{M})$ (still keeping track of the constants), and we finally conclude the proof of Proposition 1.7 at the end of the section.

Lemma 2.5. *Assume that (13) holds on the time interval $(0,2S)$ and with constant $\mathfrak{K}'$. Then, there are $C,\alpha_0>0$ such that we have*

$$\|e^{T\Delta_g}y_0\|^2_{L^2(\mathcal{M})}\le C(1+\lambda)e^{2\mathfrak{K}'(1+\lambda)^{1/2}}e^{\frac{18S^2}{T}}\int_0^T\|e^{t\Delta_g}y_0\|^2_{L^2(\omega)}\,dt$$

for all $0<T\le\alpha_0$, $\lambda>0$ and $y_0\in E_{\le\lambda}$.

Proof. For $\alpha > 2S^2$ (to be fixed later on), we use the kernel k_T described in Proposition 2.4. Let $w(s)$ be associated to y by $w(s) = \int_0^T k_T(t,s) y(t)\,dt$, where $y(t) = e^{t\Delta_g} y_0$ with $y_0 \in E_{\leq\lambda}$. Then, in (36), W_0 is of the particular form

$$W_0 = \left(0, \int_0^T e^{-\alpha\left(\frac{1}{t}+\frac{1}{T-t}\right)} y(t)\,dt\right),$$

so that a calculation (see [Ervedoza and Zuazua 2011a, equation (3.3)]) yields

$$\begin{aligned}\|W_0\|^2_{L^2\times\mathcal{H}_{\mathcal{L}}^{-1}} &\geq (1+\lambda)^{-1}\|W_0\|^2_{\mathcal{H}^1_{\mathcal{L}}\times L^2} = (1+\lambda)^{-1}\left\|\int_0^T e^{-\alpha\left(\frac{1}{t}+\frac{1}{T-t}\right)} y(t)\,dt\right\|{L^2}^2\\ &\geq (1+\lambda)^{-1}\sum_i |y_i|^2 e^{-2\lambda_i T}\left|\int_0^T e^{-\alpha\left(\frac{1}{t}+\frac{1}{T-t}\right)}\,dt\right|^2.\end{aligned}$$

The integral can be estimated by the Laplace method,

$$\int_0^T e^{-\alpha\left(\frac{1}{t}+\frac{1}{T-t}\right)}\,dt = T\int_0^1 e^{-\frac{\alpha}{T}\left(\frac{1}{s}+\frac{1}{1-s}\right)}\,ds \geq CT\left(\frac{T}{\alpha}\right)^{\frac{1}{2}} e^{-4\frac{\alpha}{T}} \quad \text{for } \frac{\alpha}{T}\geq 1,$$

since the nondegenerate minimum of $\frac{1}{s}+\frac{1}{1-s}$ is 4 reached at $s=\frac{1}{2}$ and the function is positive. We have thus obtained

$$\|W_0\|^2_{L^2\times\mathcal{H}_{\mathcal{L}}^{-1}} \geq C(1+\lambda)^{-1}T^3\alpha^{-1}e^{-\frac{8\alpha}{T}}\|y(T)\|^2_{L^2}. \tag{38}$$

Moreover, we have $W_0 \in E_{\leq\lambda}\times E_{\leq\lambda}$ so that

$$\frac{\|W_0\|_{H^1_0\times L^2}}{\|W_0\|_{L^2\times H^{-1}}} \leq (1+\lambda)^{\frac{1}{2}}.$$

As a consequence, (13) on the time interval $(-S,S)$ (which, by time-translation invariance, is the same as on $(0,2S)$) with constant $\mathfrak{K}'$ implies

$$\|W_0\|_{L^2\times\mathcal{H}_{\mathcal{L}}^{-1}} \leq Ce^{\mathfrak{K}'(1+\lambda)^{1/2}}\|w\|_{L^2((-S,S)\times\omega)}. \tag{39}$$

Using the Cauchy–Schwarz inequality, we have

$$\|w\|^2_{L^2((-S,S)\times\omega)} \leq \left(\int_{(0,T)\times(-S,S)} k_T(t,s)^2\,dt\,ds\right)\int_0^T\int_\omega |y(t,x)|^2\,dx\,dt. \tag{40}$$

Now, we use (37) with $\delta\in(0,1)$ fixed sufficiently close to 1 so that $\alpha \geq S^2\frac{1+\delta}{\delta}$ (which is possible since we have assumed $\alpha > 2S^2$). This yields

$$\begin{aligned}\int_{(0,T)\times(-S,S)} k_T(t,s)^2\,dt\,ds &\leq CS^2\int_{(0,T)\times(-S,S)} \exp\left(\frac{1}{\min\{t,T-t\}}\left(\frac{S^2}{\delta}-\frac{\alpha}{(1+\delta)}\right)\right)dt\,ds\\ &\leq CS^3T.\end{aligned} \tag{41}$$

Combining (38), (39), (40) and (41) then gives the sought result, since the estimates are true for any $\alpha > 2S^2$. $\square$

The following result, taken from [Miller 2010], deduces observability from low-frequency observability. The values of the constants are tracked precisely.

Lemma 2.6 [Miller 2010]. *Let $T_0, a, b, C > 0$ and assume*

$$\|e^{T\Delta_g} y_0\|^2_{L^2(\mathcal{M})} \le C e^{2a\lambda^{1/2} + \frac{2b}{T}} \int_0^T \|e^{t\Delta_g} y_0\|^2_{L^2(\omega)}\, dt \quad \textit{for all } 0 < T < T_0 \textit{ and all } y_0 \in E_{\le\lambda}.$$

Then, we have

$$\|e^{T\Delta_g} y_0\|^2_{L^2(\mathcal{M})} \le C' e^{2\frac{c_*}{T}} \int_0^T \|e^{t\Delta_g} y_0\|^2_{L^2(\omega)}\, dt \quad \textit{for all } 0 < T < T_0 \textit{ and all } y_0 \in L^2(\mathcal{M}),$$

with $c_* = (a + \sqrt{b} + \sqrt{a^2 + 2a\sqrt{b}})^2$ and C' a constant depending only on T_0, a, b, C.

Proof. The result is not stated exactly that way, but the author proves this as an intermediate result of [Miller 2010, Theorem 2.2]. More precisely, the assumptions of our lemma are exactly estimate (10) in [Miller 2010], with $\alpha = \frac{1}{2}$ and $\beta = 1$. It gives the result with $c_* = 4b^2(\sqrt{a + 2\sqrt{b}} - \sqrt{a})^{-4} = \frac{1}{4}(\sqrt{a + 2\sqrt{b}} + \sqrt{a})^4 = (a + \sqrt{b} + \sqrt{a^2 + 2a\sqrt{b}})^2$. □

With these two lemmas in hand, we now conclude the proof of Proposition 1.7.

Proof of Proposition 1.7. To simplify notation, we prove the existence of universal constants so that $\mathfrak{K}_{\mathrm{heat}}(\omega) \le \alpha_3 S^2 + \alpha_4 \mathfrak{K}_{\mathrm{wave}}(\omega, 2S)^2$ for all $S > 0$.

Let $\mathfrak{K}' > \mathfrak{K}_{\mathrm{wave}}(\omega, 2S)$ so that there exists $C > 0$ so that we have the estimate (see Corollary 2.2 for the equivalence)

$$\|(u_0, u_1)\|_{H^1_0(\mathcal{M}) \times L^2(\mathcal{M})} \le C e^{\mathfrak{K}'\Lambda} \|u\|_{L^2((-S,S)\times\omega)}, \qquad \Lambda = \frac{\|(u_0, u_1)\|_{H^1_0(\mathcal{M}) \times L^2(\mathcal{M})}}{\|(u_0, u_1)\|_{L^2(\mathcal{M}) \times H^{-1}(\mathcal{M})}}$$

$$\text{for all } (u_0, u_1) \in H^1_0(\mathcal{M}) \times L^2(\mathcal{M}), \text{ and } u \text{ a solution to (11)}. \tag{42}$$

Note that when compared to (12)–(13), we have changed the interval $(0, 2S)$ to $(-S, S)$, which gives the same result by conservation of energy. The proof is a direct consequence of above Lemmas 2.5 and 2.6. □

2D. *Link between $\mathfrak{K}_{\mathrm{wave}}(\omega, T)$ and analytic spaces.* As already mentioned, Theorem 1.18 is a corollary of observability estimates in spaces of ultradistributions (implying by duality that some spaces of analytic functions are attainable/controllable for the control problem) obtained in [Allibert 1998]. The following proposition explains (in the general setting of the paper) the link between such estimates and (12)–(13); see also [Lebeau 1992a].

Proposition 2.7. *Assume there are $C_0, C > 0$ such that for all $(u_0, u_1) \in H^1_0(\mathcal{M}) \times L^2(\mathcal{M})$ and associated u a solution of (11), we have*

$$\|e^{-C_0\sqrt{-\Delta_g}}(u_0, u_1)\|_{L^2(\mathcal{M}) \times H^{-1}(\mathcal{M})} \le C \|u\|_{L^2((0,T)\times\omega)} \quad (\textit{resp. } \le C\|\partial_\nu u\|_{L^2((0,T)\times\Gamma)}). \tag{43}$$

Then (12) is satisfied with constant $\mathfrak{K} = C_0$ and all $\mu > 0$. In particular, we have

$$\mathfrak{K}_{\mathrm{wave}}(\omega, T) \le C_0 \quad (\textit{resp. } \mathfrak{K}_{\mathrm{wave}}(\Gamma, T) \le C_0).$$

Again, in this statement, Δ_g denotes the Laplace operator with Dirichlet boundary conditions.

Proof. Given $\mu > 0$, we decompose the data (u_0, u_1) as $u_0 = \mathbb{1}_{\sqrt{-\Delta_g}\le\mu} u_0 + \mathbb{1}_{\sqrt{-\Delta_g}>\mu} u_0$ (and similarly for u_1). Here $\mathbb{1}_{\sqrt{-\Delta_g}\le\mu}$ denotes the orthogonal projector on the spectral space of $-\Delta_g$ associated to eigenfunctions λ_j with $\sqrt{\lambda_j} \le \mu$. Noting that

$$\begin{aligned}\|1_{\sqrt{-\Delta_g}>\mu}(u_0,u_1)\|_{L^2(\mathcal{M})\times H^{-1}(\mathcal{M})} &\le \frac{1}{\mu}\|\mathbb{1}_{\sqrt{-\Delta_g}>\mu}(u_0,u_1)\|_{H_0^1(\mathcal{M})\times L^2(\mathcal{M})}\\ &\le \frac{1}{\mu}\|(u_0,u_1)\|_{H_0^1(\mathcal{M})\times L^2(\mathcal{M})},\end{aligned}$$

we obtain

$$\begin{aligned}\|(u_0,u_1)\|_{L^2(\mathcal{M})\times H^{-1}(\mathcal{M})} &\le \|1_{\sqrt{-\Delta_g}\le\mu}(u_0,u_1)\|_{L^2(\mathcal{M})\times H^{-1}(\mathcal{M})} + \frac{1}{\mu}\|(u_0,u_1)\|_{H_0^1(\mathcal{M})\times L^2(\mathcal{M})}\\ &\le e^{C_0\mu}\|e^{-C_0\sqrt{-\Delta_g}}(u_0,u_1)\|_{L^2(\mathcal{M})\times H^{-1}(\mathcal{M})} + \frac{1}{\mu}\|(u_0,u_1)\|_{H_0^1(\mathcal{M})\times L^2(\mathcal{M})}\\ &\le Ce^{C_0\mu}\|u\|_{L^2((0,T)\times\omega)} + \frac{1}{\mu}\|(u_0,u_1)\|_{H_0^1(\mathcal{M})\times L^2(\mathcal{M})},\end{aligned}$$

where we used the assumption (43) in the last inequality. This concludes the proof of (12), and that of the proposition. □

We now extract an estimate like (43) on some surfaces of revolution from [Allibert 1998]. Indeed, a combination of several estimates in that work gives the following result on barrel-type surfaces.

Theorem 2.8 [Allibert 1998]. *Under the geometric assumptions of Section 1C3, for any $T > T_1$ and $C_0 > \mathsf{R}(z_0)d_A(\Gamma)$, there exists $C > 0$ so that*

$$\|e^{-C_0\sqrt{-\Delta_g}}(u_0,u_1)\|_{H_0^1\times L^2} \le C\|\partial_\nu u\|_{L^2((0,T)\times\Gamma)} \tag{44}$$

for any $(u_0,u_1) \in H_0^1(\mathcal{M}) \times L^2(\mathcal{M})$ and associated solution u of (11).

The result is not stated exactly this way in the article. It is also more precise since it involves analytic spaces only in the θ-variable. More precisely, denoting by E_0^k the spaces of functions in $H_0^1 \times L^2$ of the form $f(s)e^{ik\theta}$, the following estimate is proved in [Allibert 1998, Theorème 2, Définition 3 and Proposition 1]:

$$\|(u_0,u_1)\|_{H_0^1\times L^2} \le C(k)\|\partial_\nu u\|_{L^2((0,T)\times\Gamma)} \tag{45}$$

for any $(u_0,u_1) \in E_0^k$, where $C(k)$ satisfies

$$\limsup_{n\to+\infty} \frac{\ln C(k)}{k} = d_A(\Gamma).$$

In particular, for any $\delta > 0$, there is $k_0 \in \mathbb{N}$ such that $C(k) \le e^{k(d_A(\Gamma)+\delta)}$. Recalling that $\frac{1}{\mathsf{R}}$ has a unique minimum at $z = z_0$, together with the action of $\Delta_{z,\theta}$ on functions of the form $f(s)e^{ik\theta}$ (see (53) in Remark 3.4, or the formula of P_n in [Allibert 1998]), we see that

$$(-\Delta_{z,\theta}(f(s)e^{ik\theta}), f(s)e^{ik\theta})_{L^2(\mathcal{M})} \ge k^2\left\|\frac{f}{\mathsf{R}}\right\|_{L^2}^{2} \ge \frac{k^2}{\mathsf{R}(z_0)^2}\|f(s)e^{ik\theta}\|^2_{L^2(\mathcal{M})}$$

(and a similar formula in H_0^1). Denoting by $\lambda_{k,n}$ the n-th eigenvalue of the operator restricted to the space E_0^k, this yields $\lambda_{k,n} \geq k^2/\mathsf{R}(z_0)^2$ and thus $C(k) \leq e^{(d_A(\Gamma)+\delta)\mathsf{R}(z_0)\sqrt{\lambda_{k,n}}}$ for all $k \geq k_0$ and $n \in \mathbb{N}$. As a consequence of (45), we obtain for $k \geq k_0$

$$\|e^{-(d_A(\Gamma)+\delta)\mathsf{R}(z_0)\sqrt{-\Delta_g}}(u_0, u_1)\|_{H_0^1\times L^2} \leq \|\partial_\nu u\|_{L^2((0,T)\times\Gamma)} \quad \text{for all } (u_0, u_1) \in E_0^k.$$

This finally gives (44) for any $C_0 > \mathsf{R}(z_0)d_A$, when taking into account the orthogonality of the subspaces E_0^k for the norm of $H_0^1 \times L^2$ and the norm of the observation.

With Theorem 2.8 in hand, Theorem 1.18 is now a straightforward consequence of Propositions 2.7 and 1.7.

2E. *Reformulation of the definition of the constants in terms of localization functions.* This section is aimed at giving an alternative definition for the geometric constants $\mathfrak{K}_{\mathrm{eig}}(\omega)$, $\mathfrak{K}_\Sigma(\omega)$, $\mathfrak{K}_{\mathrm{heat}}(\omega)$ in terms of localization functions.

Definition 2.9. Let $\omega \subset \mathcal{M}$ be an open set. We set

$$\mathrm{Loc}_{\mathrm{eig}}(\omega, \lambda) = \inf\left\{\frac{\|\psi\|_{L^2(\omega)}}{\|\psi\|_{L^2(\mathcal{M})}} : \psi \in E_\lambda \setminus \{0\}\right\} \in [0, 1], \quad \lambda \in \mathrm{Sp}(-\Delta_g),$$
$$\mathrm{Loc}_\Sigma(\omega, \lambda) = \inf\left\{\frac{\|u\|_{L^2(\omega)}}{\|u\|_{L^2(\mathcal{M})}} : u \in E_{\leq\lambda} \setminus \{0\}\right\} \in [0, 1],$$
$$\mathrm{Loc}_{\mathrm{heat}}(\omega, T) = \inf\left\{\frac{\|e^{t\Delta}u_0\|_{L^2((0,T)\times\omega)}}{\|e^{T\Delta}u_0\|_{L^2(\mathcal{M})}} : u_0 \in L^2(\mathcal{M}) \setminus \{0\}\right\}.$$

Note that if the Schrödinger equation is observable from ω in finite time (in particular if ω satisfies the geometric control condition, see [Bardos et al. 1992; Lebeau 1992b]), then, there exists $C > 0$ so that $\mathrm{Loc}_{\mathrm{eig}}(\omega, \lambda) \geq C$ for all $\lambda \in \mathrm{Sp}(-\Delta_g)$. Under the sole assumption that $\omega \neq \varnothing$, we have $\mathrm{Loc}_{\mathrm{eig}}(\omega, \lambda) \geq C^{-1}e^{-C\sqrt{\lambda}}$ [Donnelly and Fefferman 1988; Lebeau and Robbiano 1995], $\mathrm{Loc}_\Sigma(\omega, \lambda) \geq C^{-1}e^{-C\sqrt{\lambda}}$ [Lebeau and Robbiano 1995; Jerison and Lebeau 1999; Lebeau and Zuazua 1998] and $\mathrm{Loc}_{\mathrm{heat}}(\omega, T) \geq C^{-1}e^{-C/T}$ [Fursikov and Imanuvilov 1996; Miller 2010].

Lemma 2.10. *We have*

$$\mathfrak{K}_{\mathrm{eig}}(\omega) = \limsup_{\lambda\to+\infty,\, \lambda\in\mathrm{Sp}(-\Delta_g)} \frac{-\log \mathrm{Loc}_{\mathrm{eig}}(\omega, \lambda)}{\sqrt{\lambda}},$$
$$\mathfrak{K}_\Sigma(\omega) = \limsup_{\lambda\to+\infty} \frac{-\log \mathrm{Loc}_\Sigma(\omega, \lambda)}{\sqrt{\lambda}},$$
$$\mathfrak{K}_{\mathrm{heat}}(\omega) = \limsup_{T\to 0^+} -T \log \mathrm{Loc}_{\mathrm{heat}}(\omega, T).$$

Note that we do not have a similar formulation for the constants $\mathfrak{K}_\infty(\omega)$ and $\mathfrak{K}_{\mathrm{wave}}(\omega, T)$ since they do not correspond to an asymptotic regime (like $T \to 0$ or $\lambda \to +\infty$).

Proof. We only prove the second statement, the other proofs being similar. Setting

$$\mathfrak{C}_\Sigma(\omega) = \limsup_{\lambda\to+\infty} \frac{-\log \mathrm{Loc}_\Sigma(\omega,\lambda)}{\sqrt{\lambda}},$$

we want to prove that $\mathfrak{C}_\Sigma(\omega) = \mathfrak{K}_\Sigma(\omega)$. Assume $\mathfrak{K}, C$ satisfy (9); then we have

$$\mathrm{Loc}_\Sigma(\omega,\lambda) \geq \frac{1}{C} e^{-\mathfrak{K}\sqrt{\lambda}},$$

and hence

$$\frac{-\log \mathrm{Loc}_\Sigma(\omega,\lambda)}{\sqrt{\lambda}} \leq \frac{\mathfrak{K}\sqrt{\lambda}+\log(C)}{\sqrt{\lambda}}.$$

Taking the $\limsup_{\lambda\to+\infty}$, this implies $\mathfrak{C}_\Sigma(\omega) \leq \mathfrak{K}$. Taking the infimum over all such $\mathfrak{K}$ and recalling Definition 1.5, we obtain $\mathfrak{C}_\Sigma(\omega) \leq \mathfrak{K}_\Sigma(\omega)$.

We now prove the converse inequality. The definition of $\mathfrak{C}_\Sigma(\omega)$ implies that for all ϵ there exists $\lambda_0(\epsilon)$ such that for all $\lambda \geq \lambda_0(\epsilon)$

$$\frac{-\log \mathrm{Loc}_\Sigma(\omega,\lambda)}{\sqrt{\lambda}} \leq \mathfrak{C}_\Sigma(\omega)+\epsilon,$$

that is, $\mathrm{Loc}_\Sigma(\omega,\lambda) \geq e^{-(\mathfrak{C}_\Sigma(\omega)+\epsilon)\sqrt{\lambda}}$. This, together with the fact that $\mathrm{Loc}_\Sigma(\omega,\lambda) > 0$ does not vanish on $[0,\lambda_0(\epsilon)]$, implies the existence of a constant $C(\epsilon) > 1$ such that

$$\mathrm{Loc}_\Sigma(\omega,\lambda) \geq \frac{1}{C(\epsilon)} e^{-(\mathfrak{C}_\Sigma(\omega)+\epsilon)\sqrt{\lambda}}$$

for all $\lambda \geq 0$. This is precisely estimate (9) with $\mathfrak{K} = \mathfrak{C}_\Sigma(\omega)+\epsilon$ and $C = C(\epsilon)$. Taking the infimum over all such $\mathfrak{K}$ and recalling Definition 1.5, we obtain $\mathfrak{K}_\Sigma(\omega) \leq \mathfrak{C}_\Sigma(\omega)+\epsilon$ for all $\epsilon > 0$, and hence $\mathfrak{K}_\Sigma(\omega) \leq \mathfrak{C}_\Sigma(\omega)$, which concludes the proof. □

3. Construction of maximally vanishing eigenfunctions

3A. *The sphere.* In this section, we consider the simplest case of our results that is, the unit sphere in $\mathbb{R}^3$:

$$\mathbb{S}^2 = \{(x_1,x_2,x_3) \in \mathbb{R}^3 : x_1^2+x_2^2+x_3^2 = 1\} = \{x \in \mathbb{R}^3 : |x| = 1\}.$$

Eigenfunctions and eigenvalues of the Laplace–Beltrami operator on $\mathbb{S}^2$ are well-understood: eigenfunctions are restrictions to $\mathbb{S}^2$ of *harmonic homogeneous polynomials* of $\mathbb{R}^3$, associated to the eigenvalue $k(k+1)$, where k is the degree of the polynomial. We are particularly interested in so-called equatorial spherical harmonics, given by

$$u_k = P_k|_{\mathbb{S}^2} \in C^\infty(\mathbb{S}^2), \quad P_k(x_1,x_2,x_3) = (x_1+ix_2)^k,$$

known to concentrate exponentially on the equator given by $x_3 = 0$.

Since it can be written $P_k = z^k$, where $z = x_1+ix_2 \in \mathbb{C}$, it is easy to check that P_k is holomorphic as a function of z, and hence harmonic as a function of $(x_1,x_2,x_3) \in \mathbb{R}^3$. Moreover, P_k is homogeneous of degree k. Therefore, see, e.g., [Shubin 1987, Proposition 22.2 p. 169], the function u_k is an eigenfunction of the Laplace–Beltrami operator on $\mathbb{S}^2$:

$$-\Delta_{\mathbb{S}^2} u_k = \lambda_k u_k, \quad \text{with } \lambda_k = k(k+1)$$

(this fact can also be checked directly with the expression in (46)). Note that we have

$$|u_k(\omega)|^2 = (x_1^2 + x_2^2)^k = (1 - x_3^2)^k, \quad \omega = \frac{x}{|x|}.$$

We denote by $N = (0,0,1)$ and $S = (0,0,-1)$ the north and south poles, and have coordinates

$$\begin{aligned} (0,\pi) \times \mathbb{S}^1 &\to \mathbb{S}^2 \setminus \{N, S\}, \\ (s,\theta) &\mapsto (\sin s \cos\theta, \sin s \sin\theta, \cos s). \end{aligned}$$

Note that $s(x) = \operatorname{dist}_g(x, N)$ for $x \in \mathbb{S}^2$. In these coordinates, the metric is given by $ds^2 + (\sin s)^2 d\theta^2$, the Riemannian volume element is $d\omega = \sin s\, ds\, d\theta$, and the sequence u_k is defined by

$$u_k(s,\theta) = \sin(s)^k e^{ik\theta}. \tag{46}$$

Remark 3.1. The construction works equally well in the unit sphere $\mathbb{S}^n \subset \mathbb{R}^{n+1}$, $n \geq 2$. The coordinates have to be changed by

$$\begin{aligned} (0,\pi) \times \mathbb{S}^1 \times \mathbb{S}^{n-2} &\to \mathbb{S}^n \setminus \{N, S\}, \\ (s,\theta,t) &\mapsto (\sin s \cos\theta, \sin s \sin\theta, t \cos s), \end{aligned}$$

and we can still consider the eigenfunction $u_k = (x_1 + i x_2)^k|_{\mathbb{S}^n}$ with $-\Delta_{\mathbb{S}^n} u_k = \lambda_k u_k$ and $\lambda_k = k(k+n-1)$.

With the above choice of the eigenfunction u_k, we have

$$|u_k(x)|^2 = (1 - x_3^2)^k = (\sin s)^{2k} = |\sin \operatorname{dist}_g(x,N)|^{2k} = e^{-2k d_A(x)}, \quad d_A(x) = -\log \sin \operatorname{dist}_g(x,N).$$

Note that d_A is actually the Agmon distance to the equator $\left(s = \frac{\pi}{2}\right)$, where $\mathbb{S}^2$ is seen as a surface of revolution; see Remark 1.13.

Also, given $f \in L^1(\mathbb{S}^2)$, we have

$$\int_{\mathbb{S}^2} f(\omega)|u_k(\omega)|^2\, d\omega = \int_{(0,\pi)\times\mathbb{S}^1} f(s,\theta)(\sin s)^{2k+1}\, ds\, d\theta = 2\pi \int_{(0,\pi)} F(s)(\sin s)^{2k+1}\, ds,$$

where

$$F(s) = \frac{1}{2\pi} \int_{\mathbb{S}^1} f(s,\theta) d\theta.$$

In the case $f = 1$, this yields the asymptotics of the norm of u_k, given by the Laplace method (see, e.g., [Erdélyi 1956; Copson 1965]):

$$\begin{aligned} \frac{1}{2\pi}\|u_k\|^2_{L^2(\mathbb{S}^2)} &= \frac{1}{2\pi}\int_{\mathbb{S}^2} |u_k(\omega)|^2\, d\omega = \int_{-1}^{1} (1 - x_3^2)^k\, dx_3 = \int_{-1}^{1} e^{k\log(1-x_3^2)}\, dx_3 \\ &= \left(1 + O\left(\frac{1}{k}\right)\right)\int_{\mathbb{R}} e^{-kx_3^2}\, dx_3 = \sqrt{\frac{\pi}{k}}\left(1 + O\left(\frac{1}{k}\right)\right), \end{aligned}$$

and hence $\|u_k\|_{L^2(\mathbb{S}^2)} \sim 2^{1/2}\pi^{3/4}k^{-1/4}$ as $k \to +\infty$.

We have the elementary estimate

$$\|u_k\|^2_{L^2(B(N,r))} = 2\pi \int_0^r (\sin s)^{2k+1}\, ds \leq \frac{\pi}{k+1} r^{2k+2}.$$

This can be slightly refined, e.g., by writing

$$\begin{aligned}\left| \|u_k\|^2_{L^2(B(N,r))} - \frac{\pi}{k+1}(\sin r)^{2k+2} \right| &= \left| \|u_k\|^2_{L^2(B(N,r))} - 2\pi \int_0^r \cos s (\sin s)^{2k+1}\, ds \right| \\ &= 2\pi \int_0^r (1-\cos s)(\sin s)^{2k+1}\, ds \\ &\leq \tfrac{1}{2} r^2 2\pi \int_0^r (\sin s)^{2k+1}\, ds = \tfrac{1}{2} r^2 \|u_k\|^2_{L^2(B(N,r))}.\end{aligned}$$

To be a little more precise, let us now prove an equivalent for $\|u_k\|^2_{L^2(B(N,r))}$ as $k \to \infty$, which is uniform in r.

Lemma 3.2. *For all $k \in \mathbb{N}^*$ and all $r \in \left[0, \frac{\pi}{2}\right)$, we have*

$$\|u_k\|^2_{L^2(B(N,r))} = \frac{\pi}{k+1} \frac{\sin(r)^{2k+2}}{\cos(r)} (1+R), \quad \textit{with } |R| \leq \frac{\tan(r)^2}{2k+2}.$$

This furnishes an optimal lower/upper bound for this quantity which is uniform with respect to r in any compact set $[0, \alpha]$ with $\alpha < \frac{\pi}{2}$.

Proof. We write $a = -\log \sin r > 0$, use the change of variable $y = -\log \sin s$, and want to have an asymptotic expansion of

$$\tfrac{1}{2\pi} \|u_k\|^2_{L^2(B(N,r))} = \int_0^r (\sin s)^{2k+1}\, ds = \int_a^{+\infty} e^{-(2k+2)y} \frac{1}{\sqrt{1-e^{-2y}}}\, dy.$$

This integral is of the form

$$\mathcal{I}(a,k) := \int_a^{+\infty} e^{-(2k+2)y} f(y)\, dy,$$

where $f(y) = 1/\sqrt{1-e^{-2y}}$ is smooth on $[a, +\infty)$. Writing

$$|f(y) - f(a)| \leq (y-a) \sup_{[a,\infty)} |f'| \leq (y-a) \frac{e^{-2a}}{(1-e^{-2a})^{\frac{3}{2}}},$$

since $f'(y) = -e^{-2y}(1-e^{-2y})^{-3/2}$ and integrating on $(a, +\infty)$, we obtain

$$\left| \mathcal{I}(a,k) - f(a) \frac{e^{-(2k+2)a}}{2k+2} \right| \leq \frac{e^{-(2k+2)a}}{(2k+2)^2} \frac{e^{-2a}}{(1-e^{-2a})^{\frac{3}{2}}}.$$

Coming back to the original notation, this is precisely

$$\left| \frac{1}{2\pi} \|u_k\|^2_{L^2(\mathcal{O}_\epsilon)} - \frac{\sin(r)^{2k+2}}{(2k+2)\cos(r)} \right| \leq \frac{\sin(r)^{2k+4}}{(2k+2)^2 \cos(r)^3} = \frac{\sin(r)^{2k+2}}{(2k+2)^2 \cos(r)} \tan(r)^2,$$

which concludes the proof of the lemma. $\square$

Note that the eigenfunctions we have constructed are complex-valued. Yet, since $u_k = (\sin(s))^k e^{ik\theta}$, its real part $\mathrm{Re}(u_k) = (\sin(s))^k \cos(k\theta)$ is a real-valued eigenfunction to which the same estimates hold, except that $\int_{\mathbb{S}^1} |e^{ik\theta}|^2\, d\theta = 2\pi$ should be replaced by $\int_{\mathbb{S}^1} \cos(k\theta)^2\, d\theta = \pi$.

3B. ***General surfaces of revolution.*** In this section we consider a revolution surface $\mathcal{S} \subset \mathbb{R}^3$ being diffeomorphic to a sphere $\mathbb{S}^2$, generalizing the results of Section 3A. We follow [Besse 1978, Chapter 4B, p. 95] for the precise geometric description of such manifolds.

Assume that $(\mathcal{S}, g)$ is an embedded submanifold of $\mathbb{R}^3$ (endowed with the induced Euclidean structure), having $\mathbb{S}^1 = (\mathbb{R}/2\pi\mathbb{Z}) \sim \mathrm{SO}(2)$ as an effective isometry group. The action of $\mathbb{S}^1$ on $\mathcal{S}$, denoted by $\theta \mapsto \mathcal{R}_\theta$ (such that $\mathcal{R}_\theta \mathcal{S} = \mathcal{S}$), has exactly two fixed points denoted by $N, S \in \mathcal{S}$ (the so-called north and south poles).

We now describe a nice parametrization of $(\mathcal{S}, g)$. Let $L = \mathrm{dist}_g(N, S)$ and γ_0 be a geodesic from N to S (thus with length L). For any $\theta \in \mathbb{S}^1$, the isometry $\mathcal{R}_\theta$ transforms the geodesic γ_0 into $\mathcal{R}_\theta(\gamma_0)$, which is another geodesic joining N to S. Set $U = \mathcal{S} \setminus \{N, S\}$. For every $m \in U$, there exists a unique $\theta \in \mathbb{S}^1$ such that m belongs to $\mathcal{R}_\theta(\gamma_0)$. The geodesic $\mathcal{R}_\theta(\gamma_0)$ can be parametrized by arclength

$$\rho : [0, L] \to \mathcal{R}_\theta(\gamma_0), \quad \rho(0) = N, \quad \rho(L) = S, \quad s = \mathrm{dist}_g(\rho(s), N) = L - \mathrm{dist}_g(\rho(s), S),$$

and there exists a unique $s \in (0, L)$ such that $\rho(s) = m$. We use (s, θ) as a parametrization of $U \subset \mathcal{S}$:

$$\begin{aligned} \zeta : U = \mathcal{S} \setminus \{N, S\} &\to (0, L) \times \mathbb{S}^1, \\ m &\mapsto \zeta(m) = (s, \theta). \end{aligned}$$

We define two other exponential charts (U_N, ζ_N) and (U_S, ζ_S) centered at the fixed points N and S by

$$\begin{gathered} U_N = \{N\} \cup \zeta^{-1}\big((0, \tfrac{1}{2}L) \times \mathbb{S}^1\big) = B_g\big(N, \tfrac{1}{2}L\big) \subset \mathcal{S}, \\ U_S = \{S\} \cup \zeta^{-1}\big((\tfrac{1}{2}L, L) \times \mathbb{S}^1\big) = B_g\big(S, \tfrac{1}{2}L\big) \subset \mathcal{S}, \\ \zeta_N : U_N \to B_{\mathbb{R}^2}\big(0, \tfrac{1}{2}L\big), \quad \zeta_N(N) = 0, \quad \zeta_S : U_S \to B_{\mathbb{R}^2}\big(0, \tfrac{1}{2}L\big), \quad \zeta_S(S) = 0, \end{gathered}$$

with the transition maps

$$\begin{aligned} \zeta_N \circ \zeta^{-1} : \zeta(U \cap U_N) = \big(0, \tfrac{1}{2}L\big) \times \mathbb{S}^1 &\to \zeta_N(U \cap U_N) = B_{\mathbb{R}^2}\big(0, \tfrac{1}{2}L\big) \setminus \{0\}, \\ (s, \theta) &\mapsto (s\cos(\theta), s\sin(\theta)), \end{aligned}$$

and

$$\begin{aligned} \zeta_S \circ \zeta^{-1} : \zeta(U \cap U_S) = \big(\tfrac{1}{2}L, L\big) \times \mathbb{S}^1 &\to \zeta_S(U \cap U_S) = B_{\mathbb{R}^2}\big(0, \tfrac{1}{2}L\big) \setminus \{0\}, \\ (s, \theta) &\mapsto ((L-s)\cos(\theta), (L-s)\sin(\theta)). \end{aligned}$$

On the cylinder $(0, L) \times \mathbb{S}^1$, the metric g is given by

$$(\zeta^{-1})^* g = ds^2 + R(s)^2 d\theta^2$$

for some smooth function $R : (0, L) \to \mathbb{R}_*^+$ (see below Remark 3.4 for the geometric interpretation of R). Since g is a smooth metric on $\mathcal{S}$, [Besse 1978, Proposition 4.6] gives that R extends to a C^∞ function

$[0, L] \to \mathbb{R}^+$ satisfying

$$R(0)=R(L)=0, \qquad R'(0)=1, \qquad R'(L)=-1, \qquad R^{(2p)}(0)=R^{(2p)}(L)=0 \quad \text{for any } p\in\mathbb{N}. \tag{47}$$

In these coordinates, the Riemannian volume form is hence $R(s)\,ds\,d\theta$, the Riemannian gradient of a function is

$$\nabla_g f = \partial_s f \frac{\partial}{\partial s} + \frac{1}{R(s)^2}\partial_\theta f \frac{\partial}{\partial \theta}, \quad \text{with } g(\nabla_g f, \nabla_g f) = |\partial_s f|^2 + \frac{1}{R(s)^2}|\partial_\theta f|^2 \tag{48}$$

and the Laplace–Beltrami operator is given by

$$\Delta_{s,\theta} = \frac{1}{R(s)}\partial_s R(s)\partial_s + \frac{1}{R(s)^2}\partial_\theta^2. \tag{49}$$

Another important operator is the infinitesimal generator X_θ of the group $(\mathcal{R}_\theta)_{\theta\in\mathbb{S}^1}$, defined, for $f \in C^\infty(\mathcal{S})$, by

$$X_\theta f = \lim_{\theta\to 0} \theta^{-1}(f\circ\mathcal{R}_\theta - f). \tag{50}$$

In the chart (U, ζ), the action of $\mathcal{R}_\theta$ is given by $(\zeta^{-1})^*\mathcal{R}_\theta(u, \theta') = (u, \theta'+\theta)$, so that $(\zeta^{-1})^* X_\theta = \partial_\theta$. Let us now check that X_θ is a smooth vector field. Indeed, we have

$$(\zeta_N^{-1})^* X_\theta = (\zeta_N^{-1})^*\zeta^*\partial_\theta = d(\zeta_N\circ\zeta^{-1})\cdot\partial_\theta,$$

and hence

$$(\zeta_N^{-1})^* X_\theta(s\cos(\theta), s\sin(\theta)) = (-s\sin(\theta)\partial_{x_1} + s\cos(\theta)\partial_{x_2})(s\cos(\theta), s\sin(\theta)),$$

that is,

$$((\zeta_N^{-1})^* X_\theta)(x_1, x_2) = -x_2\partial_{x_1} + x_1\partial_{x_2}.$$

Since $((\zeta_N^{-1})^* X_\theta)(0) = 0$ (and since the computation is similar in U_S), we have obtained that X_θ is smooth. Note also that $X_\theta(N) = X_\theta(S) = 0$ and that its norm is given by $\sqrt{g(X_\theta, X_\theta)(s,\theta)} = R(s)$ (in the coordinates of U).

We define by $L^2(\mathcal{S}) := L^2(\mathcal{S}, d\,\mathrm{Vol}_g)$ the space of square integrable functions, which is also invariant by the action of $(\mathcal{R}_\theta)_{\theta\in\mathbb{S}^1}$.

Now, note that $(\mathcal{R}_\theta)_{\theta\in\mathbb{S}^1}$ acts as a (periodic) one-parameter unitary group on $L^2(\mathcal{S})$ by $f \mapsto f\circ\mathcal{R}_\theta$. The Stone theorem (see, e.g., [Reed and Simon 1980, Theorem VIII-8 p266]) hence implies that its infinitesimal generator is iA, where A is a self-adjoint operator on $L^2(\mathcal{S})$ with domain $D(A)\subset L^2(\mathcal{S})$. Since $iAf = X_\theta f$ for $f\in C^\infty(\mathcal{S})$ (which is dense in $D(A)$) according to (50), we have that A is the self-adjoint extension of X_θ/i. From now on, we slightly abuse the notation and still denote by X_θ/i its self-adjoint extension A.

Since g is invariant by the action of $\mathcal{R}_\theta$, we have

$$[X_\theta, \Delta_g] = 0.$$

Moreover, Δ_g has compact resolvent, so that the operators Δ_g and X_θ share a common basis of eigenfunctions: indeed, X_θ/i is self-adjoint and preserves each (finite-dimensional) eigenspace of Δ_g, and it can be diagonalized on these spaces. In U a common eigenfunction can be written as $e^{ik\theta} f(s)$ with $k \in \mathbb{Z}$, $f \in C^\infty(0, L) \cap L^2((0, L), R(s)\, ds)$ a solution of

$$-\frac{1}{R(s)} \partial_s (R(s)\, \partial_s f) + \frac{k^2}{R(s)^2} f = \lambda f \tag{51}$$

for some $\lambda \geq 0$ an eigenvalue for $-\Delta_g$. To prove this assertion, take u to be a necessarily smooth common eigenfunction of Δ_g and X_θ. In U (with the coordinates (s, θ)), we have $u = u(s, \theta)$ with (see (49) for the definition of $\Delta_{s,\theta}$)

$$-\Delta_{s,\theta} u(s, \theta) = \lambda u(s, \theta) \quad \text{and} \quad \frac{\partial_\theta}{i} u(s, \theta) = \mu u(s, \theta), \quad (s, \theta) \in (0, L) \times \mathbb{S}^1, \tag{52}$$

for some $\lambda, \mu \in \mathbb{R}$. Setting $f(s) := u(s, 0)$, the second identity in (52) implies $u(s, \theta) = e^{i\mu\theta} f(s)$. The function u being smooth on $(0, L) \times \mathbb{S}^1$, it is 2π-periodic in θ so that $\mu = k \in \mathbb{Z}$. Hence, $u(s, \theta) = e^{ik\theta} f(s)$ and the first identity in (52) directly yields (51).

We will call these normalized eigenfunctions $\varphi_{k,n} = e^{ik\theta} f_{k,n}(s)$ with eigenvalues $\lambda_{k,n}$ for $-\Delta_g$, where $n \in \mathbb{N}$. In particular, we can write $L^2(\mathcal{S}) = \bigoplus^{\perp}_{(k,n) \in \mathbb{Z} \times \mathbb{N}} \operatorname{span}(\varphi_{k,n})$.

We will define

$$L^2_k = \ker(X_\theta - ik)) = \{\varphi \in L^2(\mathcal{S}) : \varphi_{|U} = e^{ik\theta} f(s),\ f \in L^2((0, L),\ R(s)\, ds)\}$$

and $H^2_k = H^2(\mathcal{S}) \cap L^2_k$. The commutation property implies that $\Delta_g H^2_k \subset L^2_k$, so we can define the operator $\Delta_k = \Delta_{g|L^2_k}$, which is self-adjoint with domain H^2_k. This can be seen for instance directly on the simultaneous diagonalization, which implies an isometry $L^2(\mathcal{S}) \approx \ell^2(\mathbb{Z} \times \mathbb{N})$, where $L^2_k \approx \{(k, n) : n \in \mathbb{N}\}$ as a closed subspace of $\ell^2(\mathbb{Z} \times \mathbb{N})$. The fact that Δ_g has compact resolvent implies the same for Δ_k.

Remark 3.3. Note that the introduction of X_θ allows us to give a more intrinsic definition of d_A introduced in (17): given any point m_0 on the "strict global nondegenerate equator" of $\mathcal{S}$, the Agmon distance d_A is the unique continuous function such that

$$X_\theta d_A = 0, \quad d_A(m_0) = 0, \quad |\nabla_g d_A|^2_g(m) - \left(\frac{1}{g(X_\theta, X_\theta)(m)} - \frac{1}{g(X_\theta, X_\theta)(m_0)} \right) = 0.$$

All properties of Lemma 3.8 can be formulated intrinsically since s measures the geodesic distance to the north pole, and hence $s(m) = \operatorname{dist}_g(m, N)$, $L - s(m) = \operatorname{dist}_g(m, S)$, and $s(m) - s_0 = \operatorname{dist}_g(m, \text{equator})$.

Remark 3.4 (another possible parametrization). Some of the surfaces of revolution described above admit the following "cylindrical" parametrization on the set U: with $z_- < z_+$ and the two poles $N = (0, 0, z_+)$ and $S = (0, 0, z_-)$, we have

$$(z_-, z_+) \times \mathbb{S}^1 \to U = \mathcal{S} \setminus \{N, S\} \subset \mathbb{R}^3,$$
$$(z, \theta) \mapsto (\mathsf{R}(z) \cos\theta, \mathsf{R}(z) \sin\theta, z),$$

where $\mathsf{R} : [z_-, z_+] \to (0, \infty)$ is the profile of the surface, that is, a smooth positive function on (z_-, z_+) satisfying $\mathsf{R}(z_\pm) = 0$ and $\lim_{z \to z_\pm} \mathsf{R}'(z) = \mp\infty$. Note that $\mathsf{R}(z)$ represents the distance of $\mathcal{S}$ to the revolution axis $\{x = y = 0\}$ at height z. Note that (except for the shape/topology of the surface) this parametrization is the same as that of [Allibert 1998]; see Section 1C3. We have

$$\begin{cases} dx_1 = \mathsf{R}'(z)\cos\theta\, dz - \mathsf{R}(z)\sin\theta\, d\theta, \\ dx_2 = \mathsf{R}'(z)\sin\theta\, dz + \mathsf{R}(z)\cos\theta\, d\theta, \\ dx_3 = dz, \end{cases}$$

so that the metric on $\mathcal{S}$ induced by the Euclidean structure is given by

$$g = dx_1^2 + dx_2^2 + dx_3^2 = (1 + \mathsf{R}'(z)^2)\, dz^2 + \mathsf{R}(z)^2\, d\theta^2.$$

As a consequence, the Riemannian volume element is $\mathsf{V}(z)\, dz\, d\theta$ with $\mathsf{V}(z) = \mathsf{R}(z)\sqrt{1 + \mathsf{R}'(z)^2}$ and the Laplace–Beltrami operator is given in these coordinates by

$$\Delta_{z,\theta} = \frac{1}{\mathsf{V}(z)}\, \partial_z \left(\frac{\mathsf{V}(z)}{1 + \mathsf{R}'(z)^2}\, \partial_z \right) + \frac{1}{\mathsf{R}(z)^2} \partial_\theta^2, \tag{53}$$

with a suitable self-adjoint extension on $L^2((z_-, z_+) \times \mathbb{S}^1, \mathsf{V}(z)\, dz\, d\theta)$. The link between s and z is the diffeomorphism

$$s(z) = \int_{z_-}^{z} \sqrt{1 + \mathsf{R}'(t)^2}\, dt,$$

and we have $L = \int_{z_-}^{z_+} \sqrt{1 + \mathsf{R}'(t)^2}\, dt$, together with $R(s(z)) = \mathsf{R}(z) (= \sqrt{g(X_\theta, X_\theta)})$. In particular, we see that $R(s)$ indeed measures the distance to the axis of revolution.

Remark 3.5 (the sphere). Note that, in the z-parametrization, the sphere is given by $z_\pm = \pm 1$ and $r(z) = \sqrt{1 - z^2}$ and hence $r'(z) = -z/\sqrt{1 - z^2}$ and $\mathsf{V}(z) = 1$ is smooth (which is not the general case if the surface is flat near the poles).

Let us first prove existence of the particular eigenfunctions under interest in Theorem 1.9. We then study their concentration/deconcentration properties.

Lemma 3.6. *Assume that $s \mapsto R(s)$ admits a nondegenerate local maximum at $s_0 \in (0, L)$. Then, for all $k \in \mathbb{N}$, there exists $\psi_k \in C^\infty(\mathcal{S}) \cap L_k^2$ and $\mu_k \in \mathbb{R}$ such that*

$$\mu_k = \frac{1}{R(s_0)^2} + \frac{1}{k} \sqrt{\frac{|R''(s_0)|}{R^3(s_0)}} + O\left(\frac{1}{k^{3/2}} \right),$$

$\|\psi_k\|_{L^2(\mathcal{S})} = 1$, and we have $-\Delta_g \psi_k = k^2 \mu_k \psi_k$.

Note that the assumption of the lemma is $R'(s_0) = 0$ and $R''(s_0) < 0$. In the proofs below, we shall often consider $h = k^{-1}$ as a semiclassical parameter.

Proof. We first construct a family of sufficiently accurate quasimodes (i.e., approximate eigenfunctions) compactly supported in U and of the form (in the coordinates (s, θ) of U) $e^{ik\theta} u_k(s)$. The function $u_k(s)$ should thus solve (51) approximately. Setting $h = k^{-1}$ and $\mu = \lambda h^2$ in that equation, we are left with the following semiclassical eigenvalue (or approximate eigenvalue) problem in the limit $h \to 0^+$:

$$(P_h - \mu) f = -\frac{h^2}{R(s)}\, \partial_s (R(s)\, \partial_s f) + \left(\frac{1}{R(s)^2} - \mu \right) f = 0.$$

Now, according to the assumption, the potential $1/R(s)^2$ is positive, tends to plus infinity near 0 and L, and admits $1/R(s_0)^2$ as a nondegenerate local minimum. Namely, this is $R'(s_0) = 0$ and $R''(s_0) < 0$. The construction is classical (harmonic approximation) and follows, e.g., that of [Dimassi and Sjöstrand 1999, Theorem 4.23] in a simpler setting. The idea is to approximate the operator P_h by its harmonic approximation at s_0, namely

$$\begin{aligned} \widetilde{P}_h := & -\frac{h^2}{R(s_0)}\, \partial_s R(s_0)\, \partial_s + \frac{1}{R(s_0)^2} + \left(\frac{1}{R^2} \right)''(s_0) \frac{(s-s_0)^2}{2} \\ = & -h^2 \partial_s^2 + \frac{1}{R(s_0)^2} - \frac{2R''(s_0)}{R^3(s_0)} \frac{(s-s_0)^2}{2}. \end{aligned} \tag{54}$$

Recall that the spectrum of the operator $-h^2 \partial_y^2 + c_0 y^2$ on $\mathbb{R}$ ($c_0 > 0$) is given by

$$E_n(h) = h E_n(1) = h(2n+1)\sqrt{c_0},$$

associated with the eigenfunctions

$$u_n^h(y) = h^{-\frac{1}{4}} u_n^1 \left(\frac{y}{\sqrt{h}} \right), \quad \text{where } u_n^1(y) = p_n(y) e^{-\sqrt{c_0}\frac{y^2}{2}}$$

(p_n being a Hermite polynomial). Here, this applies with

$$c_0 = \frac{|R''(s_0)|}{R^3(s_0)}.$$

We consider a cutoff function $\chi \in C_c^\infty(0, L)$ such that $\chi = 1$ in a neighborhood of s_0. We set

$$u^h(s) = \chi(s) u_0^h(s), \quad \text{with } u_0^h(s) = C h^{-\frac{1}{4}} e^{-\sqrt{c_0}\frac{(s-s_0)^2}{2h}}, \tag{55}$$

where C is a normalizing constant, and prove this is an approximate eigenfunction (quasimode). First notice that we have, with $\widetilde{P}_h$ defined in (54), that

$$\widetilde{P}_h u^h = \chi \widetilde{P}_h u_0^h + [\widetilde{P}_h, \chi] u_0^h = \left(\frac{1}{R(s_0)^2} + h\sqrt{c_0} \right) \chi u_0^h + [-h^2 \partial_s^2, \chi] u_0^h.$$

In this expression, $[-h^2 \partial_s^2, \chi]$ is a first-order differential operator supported away from zero, where u_0^h and its derivatives are exponentially small. This yields

$$\left\| \widetilde{P}_h u^h - \left(\frac{1}{R(s_0)^2} + h\sqrt{c_0} \right) u^h \right\|_{L^2((0,L), R(s)\, ds)} = O(e^{-\frac{c}{h}}).$$

Now we consider, with norms $L^2((0,L),R(s)ds)$,

$$\begin{aligned}\left\|\left(P_h-\left(\frac{1}{R(s_0)^2}+h\sqrt{c_0}\right)\right)u^h\right\|L^2 \\ &\leq \|(P_h-\widetilde{P}_h)u^h\|L^2+\left\|\left(\widetilde{P}_hu^h-\left(\frac{1}{R(s_0)^2}+h\sqrt{c_0}\right)\right)u^h\right\|L^2 \\ &\leq \left\|\left(\frac{h^2}{R(s)}\partial_sR(s)\partial_s-h^2\partial_s^2\right)u^h\right\|L^2+\left\|\left(\frac{1}{R(s)^2}-\frac{1}{R(s_0)^2}-c_0(s-s_0)^2\right)u^h\right\|L^2+Ce^{-\frac{c}{h}}.\end{aligned}$$

According to the Taylor formula and the definition of c_0, we have

$$\frac{1}{R(s)^2}-\frac{1}{R(s_0)^2}-c_0(s-s_0)^2=O((s-s_0)^3)$$

on the support of χ, so that

$$\left\|\left(\frac{1}{R(s)^2}-\frac{1}{R(s_0)^2}-c_0(s-s_0)^2\right)u^h\right\|L^{2\,2}\leq C\int_{\mathbb{R}}|(s-s_0)^3h^{-\frac14}e^{-\sqrt{c_0}\frac{(s-s_0)^2}{2h}}|^2\,dz\leq Ch^3.$$

We now estimate the term

$$\left\|\left(\frac{h^2}{R(s)}\partial_sR(s)\partial_s-h^2\partial_s^2\right)u^h\right\|L^2=\left\|\frac{hR'(s)}{R(s)}h\partial_su^h\right\|L^2.$$

Notice that $h\partial_su^h=h\chi'u_0^h+h\chi\partial_su_0^h=O_{L^2}(e^{-c/h})-\sqrt{c_0}(s-s_0)\chi u_0^h$, where we have used the expression of u_0^h in (55). Moreover, since $R'(s_0)=0$, the Taylor formula yields

$$\left\|\frac{hR'(s)}{R(s)}h\partial_su^h\right\|L^2\leq Ce^{-\frac{c}{h}}+C\|h(s-s_0)^2\chi u_0^h\|_{L^2}\leq Ch^2.$$

Now, combining the above estimates finally yields the existence of constants $D,h_0>0$ such that for all $h<h_0$ we have, with $\nu_h=1/R(s_0)^2+h\sqrt{c_0}$,

$$\|(P_h-\nu_h)u^h\|L^2((0,L),R(s)\,ds)\leq Dh^{\frac32}\approx Dh^{\frac32}\|u^h\|L^2((0,L),R(s)\,ds).$$

Now, we define in coordinates in $U\subset\mathcal{S}$, $f_k(s,\theta)=e^{ik\theta}u^h(s)$, $h=k^{-1}$. This function is smooth and compactly supported in U thanks to the cutoff χ, and can therefore be extended as a function in $C^\infty(\mathcal{S})\cap L_k^2$, still denoted by f_k, which satisfies

$$\|(h^2\Delta_k-\nu_h)f_k\|_{L_k^2}\leq Dh^{\frac32}\approx Dh^{\frac32}\|f_k\|_{L_k^2}.$$

Hence, if $\nu_h\notin\operatorname{Sp}(-h^2\Delta_k)$, this implies

$$\|(-h^2\Delta_k-\nu_h)^{-1}\|_{L_k^2\to L_k^2}\geq\frac{1}{Dh^{3/2}}.$$

Finally, the operator $h^2\Delta_k$ being self-adjoint on L_k^2, we have for $z\in\mathbb{C}\setminus\operatorname{Sp}(-h^2\Delta_k)$

$$\|(-h^2\Delta_k-z)^{-1}\|=\frac{1}{d(z,\operatorname{Sp}(-h^2\Delta_k))},$$

so that, if $\nu_h \notin \mathrm{Sp}(-h^2\Delta_k)$,

$$\frac{1}{d(\nu_h, \mathrm{Sp}(-h^2\Delta_k))} \geq \frac{1}{Dh^{3/2}}.$$

In any case, this implies $d(\nu_h, \mathrm{Sp}(-h^2\Delta_k)) \leq Dh^{3/2}$, and using that the spectrum of $h^2\Delta_k$ is purely pointwise, this proves the sought result. □

The next step is to study the behavior of the eigenfunction ψ_k constructed in the previous lemma (and under a stronger assumption on the point s_0). This is the goal of the so-called Agmon estimates. We first need the following integration-by-parts lemma.

Lemma 3.7. *For all $\Psi \in W^{1,\infty}(\mathcal{S})$ real-valued and all $w \in H^2(\mathcal{S})$, we have*

$$\int_{\mathcal{S}} |\nabla_g(\Psi w)|_g^2 \, d\,\mathrm{Vol}_g - \int_{\mathcal{S}} |\nabla_g \Psi|_g^2 |w|^2 \, d\,\mathrm{Vol}_g = \mathrm{Re}\Big(\int_{\mathcal{S}} |\Psi|^2(-\Delta_g w)\bar{w} \, d\,\mathrm{Vol}_g\Big).$$

Proof. For $\Psi \in C^2(\mathcal{S})$, this is a direct consequence of the integration by parts formula (also valid when $\mathcal{S}$ has a boundary $\partial\mathcal{S}$ and $w|_{\partial\mathcal{S}} = 0$)

$$\begin{aligned}
\int_{\mathcal{S}} |\nabla_g(\Psi w)|_g^2 \, d\,\mathrm{Vol}_g &= -\int_{\mathcal{S}} \Delta_g(\Psi w)\Psi\bar{w} \, d\,\mathrm{Vol}_g \\
&= \mathrm{Re}\Big(\int_{\mathcal{S}} \big(-\Psi(\Delta_g w) - (\Delta_g\Psi)w - 2\nabla_g\Psi\cdot\nabla_g w\big)\Psi\bar{w} \, d\,\mathrm{Vol}_g\Big) \\
&= \mathrm{Re}\Big(\int_{\mathcal{S}} |\Psi|^2(-\Delta_g w)\bar{w} \, d\,\mathrm{Vol}_g\Big) + A,
\end{aligned}$$

with

$$\begin{aligned}
A &= \mathrm{Re}\Big(\int_{\mathcal{S}} \big(-(\Delta_g\Psi)\Psi|w|^2 - 2\nabla_g\Psi\cdot\nabla_g w\Psi\bar{w}\big) \, d\,\mathrm{Vol}_g\Big) \\
&= \mathrm{Re}\Big(\int_{\mathcal{S}} \big(|\nabla_g\Psi|^2|w|^2 + \nabla_g\Psi\cdot\nabla_g(|w|^2)\Psi - 2\nabla_g\Psi\cdot\nabla_g w\Psi\bar{w}\big) \, d\,\mathrm{Vol}_g\Big) \\
&= \int_{\mathcal{S}} |\nabla_g\Psi|^2|w|^2 \, d\,\mathrm{Vol}_g,
\end{aligned}$$

where we integrated by parts in the second line. This is the sought estimate in the case $\Psi \in C^2(\mathcal{S})$. The result of the lemma follows by a classical approximation argument; see, e.g., [Dimassi and Sjöstrand 1999, Proof of Proposition 6.1]. □

We shall now assume that R reaches at s_0 a *strict global nondegenerate* maximum, and introduce the relevant Agmon distance to the "equator" $s = s_0$. The latter is defined in the coordinates of U by the eikonal equation (17), or, more explicitly, for $s \in (0, L)$, by (18).

Lemma 3.8 (properties of d_A). *Assume that R reaches at s_0 a **strict global nondegenerate** maximum. Then, $d_A \in C^2(0, L)$, and we have*

$$d_A(s) = -\log(s) + O(1) \quad \text{as } s \to 0^+, \qquad d_A(s) = -\log(L-s) + O(1) \quad \text{as } s \to L^-, \tag{56}$$

$$d_A(s) = \frac{1}{2}\sqrt{\frac{-R''(s_0)}{R^3(s_0)}}(s-s_0)^2 + O((s-s_0)^3) \quad \text{as } s \to s_0. \tag{57}$$

Proof. Note that according to (47), we have $1/R(y) \to +\infty$ as $y \to 0^+$ or $y \to L^-$, with

$$R(s) = s + O(s^3) \quad \text{when } s \to 0^+ \qquad \text{and} \qquad R(s) = L - s + O((L-s)^3) \quad \text{when } s \to L^-.$$

As a consequence, with (18), we obtain

$$d_A(s) = \left| \int_{s_0}^{s} \frac{1}{y}(1 + O(y^2))\, dy \right| = -\log(s) + O(1)$$

as $s \to 0^+$ (and similarly when $s \to L^-$), that is, (56).

Let us also study the behavior of d_A near s_0. Setting

$$V(s) = \frac{1}{R(s)^2} - \frac{1}{R(s_0)^2},$$

we have $V(s_0) = V'(s_0) = 0$ and

$$V''(s_0) = \frac{-2R''(s_0)}{R^3(s_0)} > 0.$$

This implies (57) and that d_A is of class C^2 near s_0, by Taylor expansion of d_A and its derivatives. □

We can now state the following relatively precise result. All results concerning surfaces of revolution are corollaries of this one.

Theorem 3.9 (Agmon estimate). *Assume that R reaches at s_0 a* strict global nondegenerate *maximum, and consider the associated numbers μ_k and functions ψ_k given by Lemma 3.6. There exist $C, C_0, k_0 > 0$ such that for all $k \in \mathbb{N}$, $k \geq k_0$, we have $\psi_k \in L^2(\mathcal{S}, e^{kd_A}\, d\operatorname{Vol}_g)$ with the estimate*

$$\int_{\mathcal{S}} e^{2kd_A(m)} |\psi_k|^2(m)\, d\operatorname{Vol}_g(m) \leq C k^{2C_0}.$$

Here, we have used $d_A(m)$ to denote $d_A(s(m))$ with a slight abuse of notation. Note that $d_A(N) = d_A(S) = +\infty$. We first draw corollaries of this result, concluding the proof of Theorem 1.9, and then prove Theorem 3.9 at the end of the section. Using that d_A is decreasing on $(0, s_0]$, we obtain the following direct corollary.

Corollary 3.10. *Under the assumptions of Theorem 3.9, there exist $C, C_0, k_0 > 0$ such that for all $k \in \mathbb{N}$, $k \geq k_0$ and all $s_1 \leq s_0$ we have*

$$\int_{B(N,s_1)} |\psi_k|^2\, d\operatorname{Vol}_g \leq C k^{2C_0} e^{-2d_A(s_1)k}.$$

From this result, we may now derive a proof of Theorem 1.9 and Corollary 1.11.

Proof of Theorem 1.9 and Corollary 1.11. The eigenfunctions constructed in Lemma 3.6 satisfy

$$\lambda_k = k^2 \left(\frac{1}{R(s_0)^2} + \frac{1}{k} \sqrt{\frac{|R''(s_0)|}{R^3(s_0)}} + O\left(\frac{1}{k^{3/2}} \right) \right).$$

In particular, for any $C_* > \frac{1}{2}\sqrt{|R''(s_0)| R(s_0)}$, there is $k_0 \in \mathbb{N}$ such that $k \geq \sqrt{\lambda_k} R(s_0) - C_*$ for $k \geq k_0$. This gives $e^{-2kd_A(s_1)} \leq e^{2C_* d_A(s_1)} e^{-2d_A(s_1) R(s_0) \sqrt{\lambda_k}}$. Then, Theorem 1.9 follows directly from

Corollary 3.10 up to changing the constants involved. The second part of Theorem 1.9 follows directly from Proposition 1.6.

Corollary 1.11 follows from the asymptotics (56) of d_A and the fact than Theorem 1.9 is uniform for r small. Indeed, for an appropriate constant C, we have $d_A(s) \geq -\log(s) - C$ for all $0 < s_1 \leq s_0$.

Finally, for fixed λ_k and using the uniformity for r small, we obtain the order of vanishing using the general Lemma B.1 of Appendix B. □

We will need a very simple lemma.

Lemma 3.11. *Let $\varphi \in W^{1,\infty}(\mathcal{S}) \cap L^2_k$. Then, we have the pointwise estimate on U*

$$|\nabla_g(\varphi)|_g^2 \geq \frac{k^2}{g(X_\theta, X_\theta)}|\varphi|^2.$$

Proof. We have, in the coordinates of U, that φ can be written as $\varphi(s,\theta) = e^{ik\theta} f(s)$, with, according to (48),

$$\begin{aligned}|\nabla_g(\varphi)|_g^2 &= |\partial_s f|^2 + \frac{1}{R(s)^2}|\partial_\theta(e^{ik\theta} f(s))|^2 = |\partial_s f|^2 + \frac{k^2}{R(s)^2}|e^{ik\theta} f(s)|^2 \\ &\geq \frac{k^2}{R(s)^2}|e^{ik\theta} f(s)|^2 = \frac{k^2}{g(X_\theta, X_\theta)}|\varphi|^2,\end{aligned}$$

which is the sought result. □

Let us now give a proof of Theorem 3.9, following that of [Helffer 1988, Proposition 3.3.5].

Proof of Theorem 3.9. As in the above proof, we use the notation $h = k^{-1}$, considered as a semiclassical parameter. We define for some constant $C_0 > 1$, $h_0 > 0$ and $h \in (0, h_0)$ the sets

$$\Omega_- = \{s \in (0, L) : d_A(s) \leq C_0 h\}, \quad \Omega_+ = \{s \in (0, L) : d_A(s) > C_0 h\},$$

We set

$$\begin{aligned}\phi(s) &= d_A(s) - C_0 h \log(C_0) \quad \text{for } s \in \Omega_-, \\ &= d_A(s) - C_0 h \log(d_A(s)/h) \quad \text{for } s \in \Omega_+.\end{aligned}$$

For $M > 1$, set $\phi_M = \min(\phi, M)$ and $\Omega_M = \phi_M^{-1}(\{M\})$. Moreover, on Ω_-, we have

$$\phi = d_A - C_0 h \log(C_0) \leq d_A \leq C_0 h < C_0 h_0,$$

so for $M \geq C_0 h_0$, we have $\Omega_- \cap \Omega_M = \varnothing$. Hence, we have a partition $\Omega_- \sqcup (\Omega_+ \setminus \Omega_M) \sqcup (\Omega_+ \cap \Omega_M)$.

Note that it will be very important in what follows that all the estimates are independent of M, while C_0 will be defined later on. The function ϕ_M is Lipschitz on $(0, L)$, and can be pulled back to an (R_θ)-invariant Lipschitz function defined on U, and extended to $\mathcal{S}$ by $\phi_M(N) = \phi_M(S) = M$. We call $\mathcal{S}_+, \mathcal{S}_-, \mathcal{S}_M \subset \mathcal{S}$, the (R_θ)-invariant regions on $\mathcal{S}$ associated to $\Omega_-, \Omega_+, \Omega_M$, respectively, so that

$$\mathcal{S} = \mathcal{S}_- \sqcup (\mathcal{S}_+ \setminus \mathcal{S}_M) \sqcup (\mathcal{S}_+ \cap \mathcal{S}_M).$$

We now apply the formula of Lemma 3.7 with $\Psi = e^{\phi_M/h}$ with ϕ_M given above and M large, and $w = \psi_h$ (note that $\psi_h \in C^\infty(\mathcal{S})$ since it is an eigenfunction of Δ_g, so the lemma applies):

$$\int_{\mathcal{S}} |\nabla_g(\Psi\psi_h)|_g^2 \, d\,\mathrm{Vol}_g - \int_{\mathcal{S}} |\nabla_g \Psi|_g^2 |\psi_h|^2 \, d\,\mathrm{Vol}_g = k^2 \mu_h \int_{\mathcal{S}} |\Psi|^2 |\psi_h|^2 \, d\,\mathrm{Vol}_g \,.$$

Applying now Lemma 3.11 since $\Psi\psi_h \in W^{1,\infty}(\mathcal{S}) \cap L_k^2$ and using $|\nabla_g \Psi|_g^2 = k^2 |\phi_M'(s)|^2 e^{2\phi_M/h}$ in U and so almost everywhere in $\mathcal{S}$, we get

$$\int_{\mathcal{S}} \left(\frac{1}{R(s)^2} - |\phi_M'(s)|^2 - \mu_h \right) e^{2\phi_M/h} |\psi_h|^2 \, d\,\mathrm{Vol}_g \le 0.$$

Using the expression of ϕ_M on Ω_- and of $\mu_h = 1/R(s_0)^2 + O(h)$, this yields, for some $C > 0$ (independent of h and M),

$$\begin{aligned} \int_{\mathcal{S}_+} \left(\frac{1}{R(s)^2} - |\phi_M'(s)|^2 - \mu_h \right) e^{2\phi/h} |\psi_h|^2 \, d\,\mathrm{Vol}_g &\le Ch \int_{\mathcal{S}_-} e^{2d_A(s)/h} |\psi_h|^2 \, d\,\mathrm{Vol}_g \\ &\le Che^{2C_0} \int_{\mathcal{S}_-} |\psi_h|^2 \, d\,\mathrm{Vol}_g \le Che^{2C_0}, \end{aligned}$$

since ψ_h is normalized.

Note also that on $\Omega_M \cap \Omega_+$, we have $d_A \ge C_0 h$ and so $d_A \ge d_A - C_0 h \log(C_0) \ge \phi \ge M \ge 1$. Hence, since d_A is continuous, there is a constant $\epsilon > 0$ so that $s \in \Omega_M \cap \Omega_+$ implies $|s - s_0| \ge \epsilon$. In particular, since s_0 is a nondegenerate maximum for R, there is $\eta > 0$ so that it also implies

$$\frac{1}{R(s)^2} - \frac{1}{R(s_0)^2} \ge \eta.$$

On $\mathcal{S}_M \cap \mathcal{S}_+$, we thus have

$$\frac{1}{R(s)^2} - |\phi_M'(s)|^2 - \mu_h = \frac{1}{R(s)^2} - \frac{1}{R(s_0)^2} + O(h) \ge 0$$

for $h < h_0$ for h_0 only depending on the geometry, and not on M. Therefore, we have obtained

$$\int_{\mathcal{S}_+ \setminus \mathcal{S}_M} \left(\frac{1}{R(s)^2} - |\phi'(s)|^2 - \mu_h \right) e^{2\phi/h} |\psi_h|^2 \, d\,\mathrm{Vol}_g \le Che^{2C_0}. \tag{58}$$

Next, on $\Omega_+ \setminus \Omega_M$, we have $\phi' = d_A' - C_0 h (d_A'/d_A)$ and hence

$$\begin{aligned} \frac{1}{R(s)^2} - |\phi'|^2 - \mu_h &= -h \sqrt{\frac{|R''(s_0)|}{R^3(s_0)}} + O(h^{\frac{3}{2}}) + 2C_0 h \frac{(d_A')^2}{d_A} - C_0^2 h^2 \frac{(d_A')^2}{d_A^2} \\ &\ge -h \sqrt{\frac{|R''(s_0)|}{R^3(s_0)}} + O(h^{\frac{3}{2}}) + C_0 h \frac{(d_A')^2}{d_A}, \end{aligned}$$

where we used that $d_A \ge C_0 h$. According to (57),

$$\frac{(d_A')^2}{d_A} \to 2\sqrt{\frac{-R''(s_0)}{R(s_0)^3}} > 0$$

and $(d_A')^2/d_A$ can thus be extended by continuity at s_0. Since $d_A'(s) = 0$ if and only if $s = s_0$ (R reaches at s_0 its *unique* global maximum), the extended function is uniformly bounded from below on any compact subset of $(0, L)$. Moreover, according to (56), we have

$$\frac{(d_A')^2}{d_A}(s) \sim_{s\to 0^+} \frac{1}{s^2 \log(s^{-1})} \quad \text{and} \quad \frac{(d_A')^2}{d_A}(s) \sim_{s\to L^-} \frac{1}{(L-s)^2 \log((L-s)^{-1})}.$$

Hence, there is a constant $C_1 > 0$ such that $(d_A')^2/d_A \geq C_1$ on $(0, L)$, and we have

$$\frac{1}{R(s)^2} - |\phi'|^2 - \mu_h \geq h\left(C_0 \frac{(d_A')^2}{d_A} - \sqrt{\frac{|R''(s_0)|}{R^3(s_0)}} + O(h^{\frac{1}{2}})\right) \geq \frac{C_0}{2} h \frac{(d_A')^2}{2d_A},$$

when taking C_0 large with respect to C_1^{-1} and $h \leq h_0$ with h_0 depending on C_0, C_1. We can now fix C_0, h_0. From (58), we have thus obtained

$$Ch \int_{\mathcal{S}_+\setminus\mathcal{S}_M} \frac{(d_A')^2}{d_A} e^{\frac{2\phi}{h}} |\psi_h|^2 \, d\operatorname{Vol}_g \leq Che^{2C_0}.$$

Our next task is to replace ϕ by d_A in this expression. Note that

$$e^{2\phi(s)/h} = e^{2d_A(s)/h} \left(\frac{h}{d_A(s)}\right)^{2C_0}.$$

In particular, this yields

$$Ch \int_{\mathcal{S}_+\setminus\mathcal{S}_M} \frac{(d_A')^2}{d_A} e^{2d_A(z)/h} \left(\frac{h}{d_A(s)}\right)^{2C_0} |\psi_h|^2 \, d\operatorname{Vol}_g \leq Ch.$$

Now, the function $(d_A')^2/d_A^{1+2C_0}$ is positive on $(0, s_0) \cup (s_0, L)$, tends to $+\infty$ at s_0, and satisfies, as above,

$$\frac{(d_A')^2}{d_A^{1+2C_0}} \sim \frac{1}{s^2(\log(s^{-1}))^{1+2C_0}} \to +\infty \quad \text{as } s \to 0^+,$$

and similarly

$$\frac{(d_A')^2}{d_A^{1+2C_0}} \sim \frac{1}{(L-s)^2(\log((L-s)^{-1}))^{1+2C_0}} \to +\infty \quad \text{as } s \to L^-.$$

Hence, it is bounded from below on $(0, L)$ by a constant, and we obtain

$$\int_{\mathcal{S}_+\setminus\mathcal{S}_M} e^{2d_A(z)/h} |\psi_h|^2 \, d\operatorname{Vol}_g \leq Ch^{-2C_0},$$

which, combined with the already-remarked fact that $\int_{\mathcal{S}_-} e^{2d_A(z)/h}|\psi_h|^2 \operatorname{Vol}_g \leq Cte$, gives

$$\int_{\mathcal{S}\setminus\mathcal{S}_M} e^{2d_A(z)/h} |\psi_h|^2 \, d\operatorname{Vol}_g \leq Ch^{-2C_0}.$$

Since all the constants are independent of M, it gives the sought result by dominated convergence (for fixed h) making M tend to infinity. $\square$

3C. ***The disk.*** Denote by $\mathbb{D} = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \le 1\} \subset \mathbb{R}^2$ the unit disk. We denote by Δ the (negative) flat Laplace operator in $\mathbb{R}^2$. In polar coordinates, $x = r\cos\theta$, $y = r\sin\theta$, we have

$$\Delta = \partial_x^2 + \partial_y^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}.$$

Then, it can be seen that

$$\psi_{n,k}(r,\theta) = J_n(z_{n,k} r) e^{in\theta} \tag{59}$$

is an orthogonal basis of $L^2(\mathbb{D})$, where

- J_n is the Bessel function of order n, namely
$$J_n(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{iz\sin\theta} e^{-in\theta}\, d\theta, \quad n \in \mathbb{Z},\ z \in \mathbb{C} \setminus \mathbb{R}_-, \tag{60}$$
- $0 < z_{n,1} < z_{n,2} < z_{n,3} < \cdots$ is the sequence of the positive zeros of J_n.

We refer for instance to [Vasy 2015, Chapters 14.4 and 15] for an elementary introduction. In particular, the functions defined in (59) satisfy

$$-\Delta\psi_{n,k} = \lambda_{n,k}\psi_{n,k} \quad \text{in } \operatorname{Int}(\mathbb{D}), \qquad \text{with } \lambda_{n,k} = z_{n,k}^2 \text{ and } \psi_{n,k}|_{\partial\mathbb{D}} = 0.$$

Roughly speaking, the index n encodes the oscillation in the θ-variable, while the index k will contain an oscillation in the radial variable. We refer to [Anantharaman et al. 2016] for a description of concentration/delocalization properties of general eigenfunctions (or, more generally, quasimodes) on the disk. Here, we want to analyze some eigenfunctions corresponding to the so-called whispering gallery modes that are concentrated close to the boundary of $\mathbb{D}$. They "rotate" very fast and concentrate towards one of the two trajectories of the billiard contained in $S^*\partial\mathbb{D}$. This phenomenon corresponds to $n \to +\infty$ and k small, typically $k = 1$. In the following, we thus focus on

$$\psi_{n,1}(r,\theta) = J_n(z_{n,1} r) e^{in\theta},$$

and hence on the function $J_n(z_{n,1} r)$. This requires information on $z_{n,1}$.

A huge amount of information is known on the Bessel functions and its zeros. But we will need very few of them. First, we need to normalize them. For instance, [Burq et al. 2003, Lemma 5.1] taken for $k = 1$ (which is that of interest for us) yields

$$\|\psi_{n,1}\|_{L^2(\mathbb{D})} \approx n^{-\frac{2}{3}}.$$

We also need a rough estimate on the asymptotic of the $z_{n,1}$, see [Burq et al. 2003, Lemma 4.3] for instance, namely,

$$z_{n,1} = n + \mathcal{O}(n^{\frac{1}{3}}), \quad z_{n,1} > n.$$

To estimate the norm of $\psi_{n,1}$ on $B(0,\epsilon)$, $\epsilon < 1$, we first prove the following lemma.

Lemma 3.12. *For all $\alpha \ge 0$ and $n \in \mathbb{N}$, we have*

$$\left| J_n\left(\frac{n}{\cosh(\alpha)}\right)\right| \le e^{n(\tanh(\alpha) - \alpha)}.$$

Note that in [Copson 1965, Section 32, p. 79], for fixed α, a full asymptotics in terms of n is proved, with principal term

$$J_n\left(\frac{n}{\cosh(\alpha)}\right) \approx \frac{e^{n(\tanh(\alpha)-\alpha)}}{\sqrt{2\pi n \tanh(\alpha)}}. \tag{61}$$

Here, we need only the principal term but also a uniform bound in terms of α. Note that the short proof below is not very informative, and the reader is referred to [Copson 1965, Section 32] for a complete steepest descent approach to this asymptotic expansion.

Proof of Lemma 3.12. We start from formula (60), in which we write $\nu = n/\cosh(\alpha)$, and use the holomorphy of the integrand, together with the fact that $e^{i\nu(\sin z - z\cosh\alpha)}$ is a periodic function of $\mathrm{Re}(z)$ to change the contour. This yields

$$\begin{aligned} J_n(\nu) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i\left(\frac{n}{\cosh(\alpha)}\right)\sin\theta} e^{-in\theta}\,d\theta = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i\nu(\sin\theta-\theta\cosh\alpha)}\,d\theta \\ &= \frac{1}{2\pi}\int_{-\pi-i\alpha}^{\pi-i\alpha} e^{i\nu(\sin z - z\cosh\alpha)}\,dz = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i\nu(\sin x\cosh\alpha - i\cos x\sinh\alpha - x\cosh\alpha + i\alpha\cosh\alpha)}\,dx. \end{aligned}$$

This implies

$$|J_n(\nu)| \le \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{\nu(\cos x\sinh\alpha-\alpha\cosh\alpha)}\,dx \le e^{\nu(\sinh\alpha-\alpha\cosh\alpha)} = e^{n(\tanh\alpha-\alpha)},$$

and concludes the lemma. □

Lemma 3.13. *There exist $C, \beta, n_0 > 0$ such that for all $n \ge n_0$ and $0 < r \le 1 - \beta n^{-2/3}$ we have*

$$\|\psi_{n,1}\|_{L^\infty(B(0,r))} \le \exp(-n d_A(r) + C n^{\frac{1}{3}}).$$

Recall the definition of d_A in (20). See also Remark 1.14. Note that for $r \in (0, 1)$ fixed, the asymptotic formula (61) implies that such eigenfunctions have indeed the decay given by Lemma 3.13.

Proof. We have $z_{n,1}/n = 1 + O(n^{-2/3})$ and $z_{n,1}/n > 1$. Hence recalling that $|d_A'|$ is decreasing on $(0, 1]$, we have, as long as $r z_{n,1}/n \le 1$,

$$\left| d_A\left(\frac{r z_{n,1}}{n}\right) - d_A(r)\right| \le C n^{-\frac{2}{3}} r |d_A'(r)| = C n^{-\frac{2}{3}} r \sqrt{\frac{1}{r^2}-1} = C n^{-\frac{2}{3}}\sqrt{1-r^2}.$$

Thus we obtain from Lemma 3.12

$$|J_n(z_{n,1} r)| = \left| J_n\left(n\frac{z_{n,1}}{n} r\right)\right| \le \exp\left(-n d_A\left(\frac{z_{n,1}}{n} r\right)\right) \le \exp(-n d_A(r) + C n^{\frac{1}{3}})$$

for all $n \in \mathbb{N}$ and $0 < r \le n/z_{n,1}$. □

The combination of the previous estimates gives Theorem 1.12.

4. Maximal vanishing rate of sums of eigenfunctions, and observability from small balls

In this section, we prove Theorem 1.15, i.e., the Lebeau–Robbiano spectral inequality with observation in balls of (small) radius r and constants uniform in r.

We follow the proof proposed in [Jerison and Lebeau 1999, middle of p. 231]. There are three main steps, that we summarize in three lemmas. We then prove Theorem 1.15 from these lemmas, and prove the lemmas afterwards.

In the following, for $\beta > 0$, we set $X_\beta = (-\beta, \beta) \times \mathcal{M}$, and define $P = -\partial_s^2 - \Delta_g$. In the set $X_{2S} = (-2S, 2S) \times \mathcal{M}$, we denote by (s, x) the running point and by B_r a geodesic ball (for the metric $\mathrm{Id} \otimes g$) of radius r (its center being implicit in the notation). We also use the rescaled H^1-norm on an open set U, denoted by $H_r^1(U)$ and defined by

$$\|F\|^2_{H_r^1(U)} = \|F\|^2_{L^2(U)} + r^2\|\nabla_g F\|^2_{L^2(U)}. \tag{62}$$

This will only be used on small geodesic balls or annuli, namely $U = B_{\alpha r}$ or $U = B_{\alpha r} \setminus B_{\beta r}$.

4A. *The three key lemmas.* In this section, we state the three key lemmas needed for the proof of Theorem 1.15.

The first lemma is a classical global Lebeau–Robbiano interpolation inequality [1995, Section 3, estimate (1)].

Lemma 4.1 (global interpolation inequality from unit balls to the whole space). *Let $S > 0$ and let $U \subset X_{2S}$ be any nonempty open set. Then there is $C > 0$ and $\alpha_0 \in (0, 1)$ such that we have*

$$\|F\|_{H^1(X_S)} \le C(\|PF\|_{L^2(X_{2S})} + \|F\|_{H^1(U)})^{\alpha_0}\|F\|^{1-\alpha_0}_{H^1(X_{2S})}$$

for all $F \in H^2(X_{2S})$ such that $F|_{(-2S,2S)\times\partial M} = 0$.

The next lemma states a local interpolation inequality. Its specificity is that the observation term is on a small ball B_r and the constants are uniform in r small. For this, the exponent has to depend on r as $|\log(r)|^{-1}$.

Lemma 4.2 (local interpolation inequality from small balls to unit balls). *Let $P = -\partial_s^2 - \Delta_g$ and let B_r denote balls centered at $(s_0, x_0) \in X_T$, away from the boundary. Then, there exists $r_1 > 0$ such that for all $0 < r_0 \le r_1$ there is $C > 0$ such that for all $r \in \big(0, \frac{r_0}{10}\big)$, and $F \in H^2(B_{r_0})$, we have*

$$\|F\|_{H^1(B_{r_0/4})} \le C(\|PF\|_{L^2(B_{r_0})} + \|F\|_{H_r^1(B_r)})^{\alpha_r}\|F\|^{1-\alpha_r}_{H^1(B_{r_0})}, \quad \alpha_r = \frac{\log 2}{\log\big(\frac{2r_0}{r}\big) + \log 2}.$$

A proof of this lemma is given in Section 4C, starting from a Carleman estimate (with singular weight) due to Aronszajn [1957]; see also [Aronszajn et al. 1962; Donnelly and Fefferman 1988; 1990].

The last lemma is an interpolation inequality with boundary observation term. All terms are taken on sets of size r, and the important feature of this estimate is that the constants are uniform in r.

Lemma 4.3 (uniform local interpolation at the boundary on small balls). *Let $(0, x_0) \in \{0\} \times \mathcal{M}$, $\mathrm{dist}_g(x_0, \partial\mathcal{M}) > 0$ and consider balls centered at $(0, x_0)$. Then, there exist $C > 0$, $r_0 > 0$ and $\alpha_0 \in (0, 1)$*

such that we have for all $0 < r < r_0$

$$\|F\|_{H^1_r(B_r)} \le C(r^2\|PF\|_{L^2(B_{2r})} + r^{\frac{3}{2}}\|\partial_s F|_{s=0}\|_{L^2(B_{2r}\cap\{0\}\times\mathcal{M})})^{\alpha_0}\|F\|^{1-\alpha_0}_{H^1_r(B_{2r})}$$

for all $F \in H^2(X_{2S})$ *such that* $F|_{(-2S,2S)\times\partial M} = 0$.

This lemma is proved in Section 4D, and is a consequence of a uniform Carleman estimate proved in Appendix A.

4B. ***Concluding the proof of Theorem 1.15 from the three lemmas.*** From these three lemmas, we may now give a proof of Theorem 1.15. We first formulate a straightforward corollary of the three lemmas to prepare the proof.

Corollary 4.4. *Let* $P = -\partial_s^2 - \Delta_g$ *and* $(0, x_0) \in \{0\} \times \mathrm{Int}(\mathcal{M})$ *and consider balls centered at* $(0, x_0)$. *Then, there exist* $r_0 > 0$, $C > 0$ *and* $\alpha_0 \in (0, 1)$ *such that, for all* $r \in \left(0, \frac{r_0}{10}\right)$ *and* $F \in H^2(X_{2S})$ *with* $PF = 0$ *and* $F|_{(-2S,2S)\times\partial M} = 0$, *we have*

$$\|F\|_{H^1(X_S)} \le C\|F\|^{\alpha_0}_{H^1(B_{r_0/4})}\|F\|^{1-\alpha_0}_{H^1(X_{2S})},$$

$$\|F\|_{H^1(B_{r_0/4})} \le C\|F\|^{\alpha_r}_{H^1(B_r)}\|F\|^{1-\alpha_r}_{H^1(X_{2S})}, \qquad \alpha_r = \frac{\log 2}{\log\left(\frac{2r_0}{r}\right) + \log 2},$$

$$\|F\|_{H^1(B_r)} \le C\|\partial_s F|_{s=0}\|^{\alpha_0}_{L^2(B_{2r}\cap\{0\}\times\mathcal{M})}\|F\|^{1-\alpha_0}_{H^1(X_{2S})}.$$

Proof of Theorem 1.15. Let us first treat the case where $\partial\mathcal{M} = \varnothing$, or $\partial\mathcal{M} \neq \varnothing$ but the center of the balls, x_0 is in $\mathrm{Int}(\mathcal{M})$. The case x_0 near $\partial\mathcal{M}$ will be treated afterwards.

We reformulate (again) these three results as (in a form close to that of [Donnelly and Fefferman 1988])

$$\frac{\|F\|_{H^1(X_{2S})}}{\|F\|_{H^1(B_{r_0/4})}} \le \left(C\frac{\|F\|_{H^1(X_{2S})}}{\|F\|_{H^1(X_S)}}\right)^{\frac{1}{\alpha_0}},$$

$$\frac{\|F\|_{H^1(X_{2S})}}{\|F\|_{H^1(B_r)}} \le \left(C\frac{\|F\|_{H^1(X_{2S})}}{\|F\|_{H^1(B_{r_0/4})}}\right)^{\frac{1}{\alpha_r}},$$

$$\frac{\|F\|_{H^1(X_{2S})}}{\|\partial_s F|_{s=0}\|_{L^2(B_{2r}\cap\{0\}\times\mathcal{M})}} \le \left(C\frac{\|F\|_{H^1(X_{2S})}}{\|F\|_{H^1(B_r)}}\right)^{\frac{1}{\alpha_0}},$$

and combine them to obtain

$$\frac{\|F\|_{H^1(X_{2S})}}{\|\partial_s F|_{s=0}\|_{L^2(B_{2r}\cap\{0\}\times\mathcal{M})}} \le C^{\frac{1}{\alpha_0}}C^{\frac{1}{\alpha_0\alpha_r}}C^{\frac{1}{\alpha_0^2\alpha_r}}\left(\frac{\|F\|_{H^1(X_{2S})}}{\|F\|_{H^1(X_S)}}\right)^{\frac{1}{\alpha_0^2\alpha_r}}. \tag{63}$$

We then follow [Lebeau and Robbiano 1995; Jerison and Lebeau 1999; Lebeau and Zuazua 1998; Le Rousseau and Lebeau 2012], and, given $\psi \in E_{\le\lambda}$ take the function

$$F(s) = \frac{\sinh(s\sqrt{-\Delta_g})}{\sqrt{-\Delta_g}}\Pi_+\psi + s\Pi_0\psi,$$

where Δ_g is the Dirichlet Laplacian, Π_0 the orthogonal projector on $\ker(\Delta_g)$ and $\Pi_+ = \mathrm{Id} - \Pi_0$; that is, F is the unique solution to

$$(-\partial_s^2 - \Delta_g)F = 0, \quad F|_{(-2S,2S)\times\partial M} = 0, \quad (F, \partial_s F)|_{s=0} = (0, \psi).$$

Classical computations (see, e.g., [Le Rousseau and Lebeau 2012, Proof of Theorem 5.4]) show that there is $C > 1$ such that for all $\lambda \geq 0$ and $\psi \in E_{\leq\lambda}$, we have

$$\frac{1}{C}\|\psi\|_{L^2(\mathcal{M})} \leq \|F\|_{H^1(X_S)} \leq \|F\|_{H^1(X_{2S})} \leq Ce^{3S\sqrt{\lambda}}\|\psi\|_{L^2(\mathcal{M})}.$$

As a consequence, (63) yields for some $C, \kappa > 0$, for all $\lambda \geq 0$, $\psi \in E_{\leq\lambda}$, and $r \in \left(0, \frac{r_0}{4}\right)$

$$\frac{\|\psi\|_{L^2(\mathcal{M})}}{\|\psi\|_{L^2(B_{\mathcal{M}}(x_0,2r))}} \leq C^{\kappa+\frac{1}{\alpha_r}} e^{(\kappa+\frac{1}{\alpha_r})\sqrt{\lambda}}. \tag{64}$$

Recalling the definition of α_r, this is the sought result of Theorem 1.15 (up to changing $2r$ into r, and the names of the constants accordingly) with the restriction $r \in \left(0, \frac{r_0}{4}\right)$. To conclude for all $r > 0$, it suffices to notice that (64) remains true with $\alpha_{r_0/16}$ on the right-hand side uniformly for observation terms $\|\psi\|_{L^2(B_{\mathcal{M}}(x_0,2r))}$ with $r \geq \frac{r_0}{8}$ (the constants are nonincreasing functions of the observation set).

To conclude the proof in the general case, we need to consider the situation $\partial\mathcal{M} \neq \varnothing$ in full generality. We again follow [Donnelly and Fefferman 1988; Jerison and Lebeau 1999]. In this case, we define the double manifold $\widetilde{\mathcal{M}} = \mathcal{M} \sqcup \mathcal{M}$, consisting in gluing two copies of $\mathcal{M}$, endowed with a smooth structure of compact manifold, as in [Lee 2013, Theorem 9.29–Example 9.32]. Then, the procedure is very well explained in [Anton 2008, Section 3] and we only sketch the proof. We extend the metric g on $\mathcal{M}$ by symmetry/parity with respect to the boundary $\partial\mathcal{M}$ as a metric $\tilde{g}$ on $\widetilde{\mathcal{M}}$. Note that even if g is smooth, the extended metric $\tilde{g}$ is only Lipschitz on $\widetilde{\mathcal{M}}$. This is not an issue since Lemmas 4.1, 4.2 and 4.3 remain valid for Lipschitz metrics (as a consequence of Appendix A, [Aronszajn et al. 1962; Donnelly and Fefferman 1990], and Appendix A, respectively). In the case of Dirichlet boundary condition on $\partial\mathcal{M}$, and given $\psi \in E_{\leq\lambda}$ we take its antisymmetric/odd extension on $\widetilde{\mathcal{M}}$, yielding a function $\tilde{\psi} \in \widetilde{E}_{\leq\lambda}$. Here, $\widetilde{E}_{\leq\lambda}$ is the counterpart of $E_{\leq\lambda}$ defined for the Laplace–Beltrami operator $\Delta_{\tilde{g}}$ on $\widetilde{\mathcal{M}}$. The above computations are then made for $\Delta_{\tilde{g}}$ on $\widetilde{\mathcal{M}}$ and the estimate (64) is proved for $\tilde{\psi}$. The same estimate for ψ follows. Similarly, in the case of Neumann boundary condition, we take the symmetric/even extension of functions, yielding the sought result. □

4C. *A proof of Lemma 4.2 from Aronszajn estimates.* In this section, we give a proof of Lemma 4.2 starting from Carleman–Aronszajn estimates as stated in [1988, Proposition 2.10; 1990, Proposition 2.10] (and slightly modified according to the remarks in [Jerison and Lebeau 1999, beginning of Section 14.3]), which we now state. An alternative proof of a closely related estimate is given in [Hörmander 1985a, inequality (17.2.11), Chapter XVII.2].

Proposition 4.5. *Let $P = -\partial_s^2 - \Delta_g$ and let $(\rho, t) \in (0, r_1) \times \mathbb{S}^n$ be geodesic polar coordinates around a point $(s_0, x_0) \in X_S$ away from the boundary. Then, there exists a function $\bar{\rho}(\rho)$ with*

$$\bar{\rho} = \rho + O(\rho^2) \quad \text{as } \rho \to 0^+, \tag{65}$$

and constants $\tau_0, C, r_0 > 0$, *such that we have*

$$C \int |\bar\rho^{-\tau} P u|^2 \rho^{-1}\, d\rho\, dt \geq \int (|\bar\rho^{-\tau}\nabla u|^2 + |\bar\rho^{-\tau} u|^2)\rho^{-1}\, d\rho\, dt \quad \textit{for all } \tau \geq \tau_0,\ u \in C_0^\infty(B_{r_0}\setminus\{0\}).$$

With this Carleman–Aronszajn estimate in hand, we now give a proof of Lemma 4.2.

Proof of Lemma 4.2. We use the estimate of Proposition 4.5 as in [Lebeau and Robbiano 1995] (see also [Le Rousseau and Lebeau 2012, Section 5]) to deduce an interpolation inequality. We introduce for this (as in [Donnelly and Fefferman 1988, beginning of Section 3]) a cutoff function $\chi_r = \chi_r(\rho)$ such that, with $0 < r < \frac{r_0}{2}$ a small parameter (appearing in the statement of the lemma),

$$\operatorname{supp}(\chi_r) \subset \left\{\frac{r}{2} < \bar\rho < r_0\right\}, \quad \chi_r = 1 \quad \text{on } \left\{r < \bar\rho < \frac{r_0}{2}\right\},$$
$$|\partial^\alpha \chi_r| \leq C_\alpha r^{-|\alpha|} \quad \text{on } \left\{\frac{r}{2} < \bar\rho < r\right\}, \qquad |\partial^\alpha \chi_r| \leq C_\alpha \quad \text{on } \left\{\frac{r_0}{2} < \bar\rho < r_0\right\}.$$

We apply Proposition 4.5 to $u = \chi_r F$. The operator $[P, \chi_r]$ is a first-order differential operator with $\operatorname{supp}[P,\chi_r] \subset \left\{\frac{r}{2} < \bar\rho < r\right\} \cup \left\{\frac{r_0}{2} < \bar\rho < r_0\right\}$, being moreover of the form $O(r^{-1})D + O(r^{-2})$ on the set $\left\{\frac{r}{2} < \bar\rho < r\right\}$. Therefore, we obtain using (65), for all $\tau \geq \tau_0$,

$$\begin{aligned}
&\int (|\bar\rho^{-\tau}\nabla(\chi_r F)|^2 + |\bar\rho^{-\tau}\chi_r F|^2)\rho^{-1}\, d\rho\, dt \\
&\qquad \leq C \int |\bar\rho^{-\tau}\chi_r P F|^2 \rho^{-1}\, d\rho\, dt + C\int |\bar\rho^{-\tau}[P,\chi_r]F|^2\rho^{-1}\, d\rho\, dt \\
&\qquad \leq C\left(\frac{r}{2}\right)^{-2\tau-1} \|PF\|^2_{L^2(\bar B_{r_0})} + C\left(\frac{r}{2}\right)^{-2\tau-2}\|F\|^2_{H^1_r(\frac{r}{2}\leq\bar\rho\leq r)} + C\left(\frac{r_0}{2}\right)^{-2\tau}\|F\|^2_{H^1(\frac{r_0}{2}\leq\bar\rho\leq r_0)},
\end{aligned}$$

where $\bar B_{r_0}$ denotes the set $\{\bar\rho \leq r_0\}$. Recall that the norm H^1_r is defined in (62). Concerning the left-hand side, we bound it from below by

$$\begin{aligned}
\int (|\bar\rho^{-\tau}\nabla(\chi_r F)|^2 + |\bar\rho^{-\tau}\chi_r F|^2)\rho^{-1}\, d\rho\, dt &\geq \int_{2r\leq\bar\rho\leq\frac{r_0}{4}} (|\bar\rho^{-\tau}\nabla(\chi_r F)|^2 + |\bar\rho^{-\tau}\chi_r F|^2)\rho^{-1}\, d\rho\, dt \\
&\geq \left(\frac{r_0}{4}\right)^{-2\tau}\|F\|^2_{H^1(r\leq\bar\rho\leq\frac{r_0}{4})}.
\end{aligned}$$

Combining the last two estimates together with the fact that $\left(\frac{r_0}{4}\right)^{-\tau}\|F\|_{H^1(\bar B_r)} \leq \left(\frac{r}{2}\right)^{-\tau}\|F\|_{H^1(\bar B_r)}$ yields, for some $\tau_0 > 0$ and all $\tau \geq \tau_0$ and $r \in \left(0, \frac{r_0}{10}\right)$,

$$\left(\frac{r_0}{4}\right)^{-\tau}\|F\|_{H^1(\bar B_{r_0/4})} \leq C\left(\frac{r}{2}\right)^{-\tau}(\|PF\|_{L^2(\bar B_{r_0})} + \|F\|_{H^1_r(\bar B_r)}) + C\left(\frac{r_0}{2}\right)^{-\tau}\|F\|_{H^1(\bar B_{r_0})}.$$

Multiplying by r_0^τ and recalling (65) to replace balls in $\bar\rho$ by real balls, we obtain, up to changing the names of the parameters r, r_0, that

$$\|F\|_{H^1(B_{r_0/4})} \leq C\left(\frac{2r_0}{r}\right)^{\tau}(\|PF\|_{L^2(B_{r_0})} + \|F\|_{H^1_r(B_r)}) + \frac{C}{2^\tau}\|F\|_{H^1(B_{r_0})}.$$

An optimization in $\tau \geq \tau_0$ [Robbiano 1995] (see also [Le Rousseau and Lebeau 2012, Lemma 5.2]) then implies the following interpolation inequality:

$$\|F\|_{H^1(B_{r_0/4})} \leq C(\|PF\|_{L^2(B_{r_0})} + \|F\|_{H^1_r(B_r)})^{\alpha_r} \|F\|_{H^1(B_{r_0})}^{1-\alpha_r}, \quad \alpha_r = \frac{\log 2}{\log\left(\frac{2r_0}{r}\right) + \log 2},$$

and concludes the proof of the lemma. □

4D. ***A proof of Lemma 4.3 from Proposition A.14.*** In this section, we give a proof of Lemma 4.3. The latter consists in performing a scaling argument to reduce the problem to fixed-size balls. However, the scaling argument yields in these fixed balls a family of metrics (converging to a fixed metric as $r \to 0$), and we need to use uniform interpolation/Carleman estimates for such families of metrics. These uniform estimates are proved in Appendix A (Proposition A.14).

Proof of Lemma 4.3. We first choose r_0 small enough so that $\bar{B}_{2r,0} \subset X_S$ and there exists a local coordinate patch on $\mathcal{M}$: $\Phi : \{x \in \mathcal{M} : \operatorname{dist}(x, x_0) < 2r_0\} \to U$ where U is a neighborhood of 0 in $\mathbb{R}^n$, with $\Phi(x_0) = 0$. Up to a multiplication by an invertible constant matrix, we may assume that $((\Phi^{-1})^* g)(0) = \mathrm{Id}$. As a consequence, $ds^2 \otimes ((\Phi^{-1})^* g)(ry)$, defined on the ball of radius 2, converges uniformly in this ball towards the flat metric on the flat ball of $\mathbb{R}^{n+1}$ in the limit $r \to 0^+$. We will thus only use the flat metric in the present proof, which behaves well with respect to scaling. The distance (hence the balls, still denoted by B_r or B_1 below, all centered at 0) will be defined with respect to the flat metric, as well as the Sobolev norms (still denoted by $H^1_r(B_r)$, $H^1(B_1)$ below). The final result we obtain will be formulated in terms of the flat metric, and associated balls and Sobolev spaces. Coming back to a formulation on the manifold $\mathbb{R} \times \mathcal{M}$ with the metric $ds^2 \otimes g$ only uses the uniform equivalence of norms in $T^*(\mathbb{R} \times \mathcal{M})$ and in $L^2(\mathbb{R} \times \mathcal{M})$ for r sufficiently small.

With this in mind, let us now proceed with the scaling argument in the coordinate chart. Denote by $F_r(x) = F(rx)$ and P_r the Laplace–Beltrami operator with respect to the metric $ds^2 \otimes ((\Phi^{-1})^* g)(ry)$ defined on the ball of radius 2, we have

$$\begin{aligned}
\|F\|_{H^1_r(B_r)} &= r^{\frac{n+1}{2}} \|F_r\|_{H^1(B_1)}, \\
r^2 \|PF\|_{L^2(B_{2r})} &= r^{\frac{n+1}{2}} \|P_r F_r\|_{L^2(B_2)}, \\
r^{\frac{3}{2}} \|\partial_s F|_{s=0}\|_{L^2(B_{2r} \cap \{0\} \times \mathcal{M})} &= r^{\frac{1}{2}} r^{\frac{n}{2}} \|\partial_s F_r|_{s=0}\|_{L^2(B_2 \cap \{0\} \times \mathcal{M})}.
\end{aligned}$$

Note that the metric $ds^2 \otimes g(r\,\cdot)$ defined on B_2 converges uniformly, when r tends to zero, to the flat metric $ds^2 \otimes g(0) = ds^2 \otimes dy_1^2 \otimes \cdots \otimes dy_n^2$ for the Lipschitz topology on metrics. So, the result follows if we are able to prove the following estimate: there exist ϵ, α_0, C such that for all Lipschitz metrics $\mathfrak{g}$ with $\|\mathfrak{g} - \mathrm{Id}\|_{W^{1,\infty}} < \epsilon$ and all $u \in H^2(B_2)$ such that $u|_{s=0} = 0$ we have

$$\|u\|_{H^1(B_1)} \leq C\left(\|(-\partial_s^2 - \Delta_{\mathfrak{g}})u\|_{L^2(B_2)} + \|\partial_s u|_{s=0}\|_{L^2(B_2 \cap \{0\} \times \mathbb{R}^n)}\right)^{\alpha_0} \|F\|_{H^1(B_2)}^{1-\alpha_0}.$$

This is the object of Proposition A.14 proved in Appendix A. Note that the result of Proposition A.14 is stated with half-balls B_k^+ but is also true with real balls B_k instead by a symmetry argument. □

5. The observability constant for positive solutions

The aim of this section is to prove the result of Theorem 1.4 concerning observability of positive solutions to the heat equation. The main tool will be the following Li–Yau estimates.

Theorem 5.1 [Li and Yau 1986, Theorem 2.3]. *Let $\mathcal{M}$ be a compact manifold. Let*

$$-K = \min(0, \min_{x\in\mathcal{M}} \mathrm{Ricci}(x)) \le 0,$$

where $\mathrm{Ricci}(x)$ *is the Ricci curvature at x. We assume that the boundary of $\mathcal{M}$ is convex, i.e., $II > 0$. Let $u(t,x)$ be a positive solution on $(0,+\infty)$ of the heat equation with Neumann boundary condition. Then for any $\alpha > 1,\ x, y \in \mathcal{M}$, and $0 < t_1 < t_2$, we have*

$$u(t_1,x) \le \left(\frac{t_2}{t_1}\right)^{\frac{n\alpha}{2}} e^{\frac{n\alpha K(t_2-t_1)}{\sqrt{2}(\alpha-1)}} e^{\alpha\frac{d(x,y)^2}{4(t_2-t_1)}} u(t_2,y).$$

Here, we have denoted by $II(x)$ the second fundamental form of ∂M with respect to outward-pointing normal at the point x.

Remark 5.2. The convexity assumption is not necessary to obtain a Li–Yau-type estimate (if the boundary is smooth), up to a loss in the exponent. Indeed, setting $-H = \min(0, \min_{x\in\partial\mathcal{M}} II(x)) \le 0$, Wang [1997, Theorem 3.1] proved the estimate

$$u(t_1,x) \le \left(\frac{t_2}{t_1}\right)^{C_\alpha} e^{C'_\alpha(t_2-t_1)} e^{\alpha\frac{d(x,y)^2}{4(t_2-t_1)}} u(t_2,y) \quad \text{for all } \alpha > (1+H)^2.$$

The proof of Theorem 1.4 below shows that the result still holds without the convexity argument, but yields

$$\|u(T)\|^2_{L^2(\mathcal{M})} \le \frac{C_\epsilon}{T} e^{(1+H+\epsilon)^2\frac{(\mathcal{L}(M,\omega)+\epsilon)^2}{2T}} \int_0^T \|u(t,\cdot)\|^2_{L^2(\omega)}\,dt,$$
$$\|u(T)\|^2_{L^2(\mathcal{M})} \le \frac{C_\epsilon}{T} e^{(1+H+\epsilon)^2\frac{(\mathcal{L}(M,z_0)+\epsilon)^2}{2T}} \int_0^T u(t,z_0)^2\,dt,$$

instead of (6)–(7) (hence with a loss of $(1+H)^2$ in the exponent). We do not know whether this is optimal. Finally, we did not find any analogous estimate in the case of Dirichlet boundary conditions.

Proof of Theorem 1.4. Along the proof, we will need the following asymptotic constants, all depending on the chosen $\epsilon > 0$. Namely, we shall use $\eta_0 > 0$ arbitrarily small, $r > 1$ arbitrarily large, $\lambda \in (0,1)$ arbitrarily close to 1, and $\alpha > 1$ arbitrary close to 1. Given $\epsilon > 0$, they will all be fixed at the end so that

$$\frac{r\alpha}{(r-1)\lambda}(\mathcal{L}(\mathcal{M},\omega)+3\eta_0)^2 \le (1+\epsilon)(\mathcal{L}(\mathcal{M},\omega)+\epsilon)^2.$$

For any $x_0 \in \mathcal{M}$ and for any $\eta_0 > 0$, there exist $\eta = \eta(x_0,\eta_0) \in (0,\eta_0)$ and $y_0 \in \omega$ such that

$$d(x_0,y_0) \le \mathcal{L}(\mathcal{M},\omega) + \eta \quad \text{and} \quad B(y_0,\eta) \subset \omega.$$

In particular, we have $\mathcal{M} \subset \bigcup_{x_0\in\mathcal{M}} B(x_0,\eta)$ so that the compactness of $\mathcal{M}$ yields the following statement: given $\eta_0 > 0$, there exist a finite set J and families $(x_j)_{j\in J} \in \mathcal{M}^J,\ (y_j)_{j\in J} \in \omega^J$ and $(\eta_j)_{j\in J} \in (0,\eta_0)^J$

such that

$$\mathcal{M} \subset \bigcup_{j \in J} B(x_j, \eta_j), \quad d(x_j, y_j) \le \mathcal{L}(\mathcal{M}, \omega) + \eta_j \quad \text{and} \quad B(y_j, \eta_j) \subset \omega \quad \text{for all } j \in J.$$

Now, fix $j \in J$, and take $x \in B(x_j, \eta_j)$ and $y \in B(y_j, \eta_j) \subset \omega$, and we have

$$d(x, y) \le \eta_j + \mathcal{L}(\mathcal{M}, \omega) + \eta_j + \eta_j \le \mathcal{L}(\mathcal{M}, \omega) + 3\eta_0 =: d_m.$$

For $t \in \left[0, \frac{T}{r}\right]$, Theorem 5.1 with $t_1 = t$ and $t_2 = rt_1 = rt$ then yields

$$u(t, x)^2 \le r^{n\alpha} e^{\frac{2n\alpha K t(r-1)}{\sqrt{2}(\alpha-1)}} e^{\frac{\alpha d_m^2}{2(r-1)t}} u(rt, y)^2.$$

Defining

$$\gamma := \frac{2n\alpha K(r-1)}{\sqrt{2}(\alpha-1)},$$

this may be rewritten as

$$u(t, x)^2 e^{-\frac{\alpha d_m^2}{2(r-1)t}} \le r^{n\alpha} e^{\gamma t} u(rt, y)^2. \tag{66}$$

We may now integrate this estimate for $x \in B(x_j, \eta_j)$ and $y \in B(y_j, \eta_j) \subset \omega$:

$$\begin{aligned} e^{-\frac{\alpha d_m^2}{2(r-1)t}} \|u(t)\|^2_{L^2(B(x_j,\eta_j))} &\le \frac{|B(x_j, \eta_j)|}{|B(y_j, \eta_j)|} r^{n\alpha} e^{\gamma t} \|u(rt)\|^2_{L^2(B(x_j,\eta_j))} \\ &\le \frac{|B(x_j, \eta_j)|}{|B(y_j, \eta_j)|} r^{n\alpha} e^{\gamma t} \|u(rt)\|^2_{L^2(\omega)}. \end{aligned}$$

Summing all these estimates for $j \in J$ yields, for a constant $C(\eta_0)$ depending only on the geometry of $(\mathcal{M}, g)$, on ω, and on the constant η_0,

$$e^{-\frac{\alpha d_m^2}{2(r-1)t}} \|u(t)\|^2_{L^2(\mathcal{M})} \le C(\eta_0) r^{n\alpha} e^{\gamma t} \|u(rt)\|^2_{L^2(\omega)}.$$

Given $\lambda \in (0, 1)$, integrating this on the interval $t \in \left[\lambda \frac{T}{r}, \frac{T}{r}\right]$ yields

$$\begin{aligned} \int_{\lambda \frac{T}{r}}^{\frac{T}{r}} e^{-\frac{\alpha d_m^2}{2(r-1)t}} \|u(t)\|^2_{L^2} \, dt &\le C(\eta_0) r^{n\alpha} \int_{\lambda \frac{T}{r}}^{\frac{T}{r}} e^{\gamma t} \|u(rt)\|^2_{L^2(\omega)} \, dt \\ &\le C(\eta_0) r^{n\alpha} e^{\gamma \frac{T}{r}} \int_{\lambda \frac{T}{r}}^{\frac{T}{r}} \|u(rt)\|^2_{L^2(\omega)} \, dt = C(\eta_0) r^{n\alpha} e^{\gamma \frac{T}{r}} \int_{\lambda T}^{T} \|u(s)\|^2_{L^2(\omega)} \, ds, \end{aligned}$$

after the change of variables $s = rt$. Concerning the left-hand side, we use the decay of the L^2-norm of solutions to the heat equation to write

$$\|u(t)\|_{L^2(\mathcal{M})} \ge \left\| u\left(\frac{T}{r}\right) \right\| L^2(\mathcal{M}) \ge \|u(T)\|_{L^2(\mathcal{M})}, \tag{67}$$

for all $t \in \left[\lambda \frac{T}{r}, \frac{T}{r}\right]$ since $r > 1$. Noting also that $t \mapsto e^{-\alpha d_m^2/(2(r-1)t)}$ is increasing in $t > 0$, we have

$$\int_{\lambda \frac{T}{r}}^{\frac{T}{r}} e^{-\frac{\alpha d_m^2}{2(r-1)t}} \, dt \ge \frac{T(1-\lambda)}{r} e^{-\frac{r\alpha d_m^2}{2(r-1)\lambda T}}.$$

Combining the above three estimates yields

$$\frac{T(1-\lambda)}{r}e^{-\frac{r\alpha d_m^2}{2(r-1)\lambda T}}\|u(T)\|^2_{L^2(\mathcal{M})}\leq C(\eta_0)r^{n\alpha}e^{\gamma\frac{T}{r}}\int_{\lambda T}^{T}\|u(s)\|^2_{L^2(\omega)}\,ds;$$

that is, for all $\eta>0$, $r>1$, $\lambda\in(0,1)$, and $\alpha>1$,

$$\|u(T)\|^2_{L^2(\mathcal{M})}\leq\frac{C(\eta)r^{n\alpha+1}}{T(1-\lambda)}e^{\frac{2n\alpha K(r-1)}{\sqrt{2}(\alpha-1)}\frac{T}{r}}e^{\frac{r\alpha(\mathcal{L}(\mathcal{M},\omega)+\eta)^2}{2(r-1)\lambda T}}\int_{\lambda T}^{T}\|u(s)\|^2_{L^2(\omega)}\,ds.$$

But $\frac{r}{r-1}=1+\frac{1}{r-1}$ can be made arbitrary close to 1^+ for large r, λ close to 1^-, α close to 1^+, and η to 0^+, so that

$$\frac{r\alpha(\mathcal{L}(\mathcal{M},\omega)+\eta)^2}{2(r-1)\lambda T}\leq\frac{\mathcal{L}(\mathcal{M},\omega)^2+\epsilon}{2T}.$$

We have thus proved the first statement.

To be a little more precise, we can choose α,r such that $\frac{1}{r}+\frac{1}{\alpha}=1$. This yields

$$\|u(T)\|^2_{L^2(\mathcal{M})}\leq\frac{C(\eta)\big(\frac{\alpha}{\alpha-1}\big)^{n\alpha+1}}{T(1-\lambda)}e^{\frac{2nK}{\sqrt{2}(\alpha-1)}T}e^{\frac{\alpha^2(\mathcal{L}(\mathcal{M},\omega)+\eta)^2}{2\lambda T}}\int_{\lambda T}^{T}\|u(s)\|^2_{L^2(\omega)}\,ds,$$

or, with $\alpha=1+\epsilon$ and $\lambda=1-\epsilon$, we obtain for all $\epsilon\in(0,1)$

$$\begin{aligned}\|u(T)\|^2_{L^2(\mathcal{M})}&\leq\frac{C(\eta)\big(\frac{1+\epsilon}{\epsilon}\big)^{(1+\epsilon)n+1}}{T\epsilon}e^{\frac{2nK}{\sqrt{2\epsilon}}T}e^{\frac{(1+\epsilon)^2}{1-\epsilon}\frac{(\mathcal{L}(\mathcal{M},\omega)+\eta)^2}{2T}}\int_{(1-\epsilon)T}^{T}\|u(s)\|^2_{L^2(\omega)}\,ds\\&\leq\frac{C(\eta)}{T\epsilon^{2n+2}}e^{\frac{2nK}{\sqrt{2\epsilon}}T}e^{\frac{(1+\epsilon)^2}{1-\epsilon}\frac{(\mathcal{L}(\mathcal{M},\omega)+\eta)^2}{2T}}\int_{(1-\epsilon)T}^{T}\|u(s)\|^2_{L^2(\omega)}\,ds.\end{aligned}$$

So we have proved the first estimate of the theorem. The second can be obtained similarly by integrating (66) in the x-variable only, and not in the y-variable. □

Remark 5.3. In fact, note that from (67) on we could also put $\big\|u\big(\frac{T}{r}\big)\big\|L^2(\mathcal{M})^2$ on the left-hand side of all estimates of the proof, which amounts to $\big\|u\big(T\frac{\epsilon}{1+\epsilon}\big)\big\|L^2(\mathcal{M})^2$, and, in particular, we have the stronger statement

$$\|u((1-\epsilon)T)\|^2_{L^2(\mathcal{M})}\leq\frac{C(\eta)r^{n\alpha+1}}{T\epsilon}e^{\frac{2nK}{\sqrt{2\epsilon}}T}e^{\frac{(1+\epsilon)^2}{1-\epsilon}\frac{(\mathcal{L}(\mathcal{M},\omega)+\eta)^2}{2T}}\int_{(1-\epsilon)T}^{T}\|u(s)\|^2_{L^2(\omega)}\,ds.$$

Remark 5.4. All constants can be made explicit. We denote by $K:=\min\{0,-\min_{x\in\mathcal{M}}\mathrm{Ricci}(x)\}$. For instance, we have for all $\eta>0$

$$\|u(T)\|^2_{L^2(\mathcal{M})}\leq\frac{C(\eta)r^{n\alpha+1}}{T(1-\lambda)}e^{\frac{2nK}{\sqrt{2}(\alpha-1)}T}e^{\frac{\alpha^2(\mathcal{L}(\mathcal{M},\omega)+\eta)^2}{2\lambda T}}\int_{\lambda T}^{T}\|u(s)\|^2_{L^2(\omega)}\,ds,$$

Choosing the constants, we have, for all $\epsilon\in(0,1)$, for all $\eta>0$

$$\left\|u\left(T\frac{\epsilon}{1+\epsilon}\right)\right\|L^2(\mathcal{M})^2\leq\frac{C(\eta)}{T\epsilon^{2n+2}}e^{\frac{2nK}{\sqrt{2\epsilon}}T}e^{(1+\epsilon)^3\frac{(\mathcal{L}(\mathcal{M},\omega)+\eta)^2}{2T}}\int_{(1-\epsilon)T}^{T}\|u(s)\|^2_{L^2(\omega)}\,ds.$$

Note that for nonnegatively (Ricci) curved manifolds (this is the case of a convex domain in $\mathbb{R}^n$) $K=0$ and the constant is
$$\frac{C(\eta)}{T\epsilon^{2n+2}}e^{(1+\epsilon)^3\frac{(\mathcal{L}(\mathcal{M},\omega)+\eta)^2}{2T}}$$
and hence decays like $\frac{1}{T}$ for T large.

Appendix A: Uniform Lipschitz Carleman estimates

In this appendix, we produce Carleman estimates for a Laplace–Beltrami operator on a Riemannian manifold M with boundary ∂M. Our proof presents several advantages with respect to the existing proofs of similar results:

- It is relatively short.
- It is completely geometric and, we hope, is relatively readable.
- As we already said, it requires the minimum of regularity for the metric (in dimension ≥ 3), namely only Lipschitz regularity. Indeed, it is known that in dimension ≥ 3, local uniqueness does not hold for general elliptic operators (even in divergence form) with $C^{0,\alpha}$ coefficients for all $\alpha<1$; see [Pliś 1963; Miller 1974].

The proof, using formulae from Riemannian geometry, is inspired by Carleman estimates for the Schrödinger equation proved by the first author [Laurent 2010].

There have been several works about such Carleman estimates for Lipschitz metrics (but without boundary). The oldest result seems to be [Aronszajn et al. 1962] for elliptic operators. Another one, which actually falls short of the Lipschitz regularity is the very general result of [Hörmander 1963, Section 8.3], which requires C^1 regularity, but applies to many more operators than elliptic ones. A proof for general elliptic operators with order $2m$ and Lipschitz coefficients is written in [Hörmander 1985a, Proposition 17.2.3]. For Lipschitz regularity of the coefficients, we can also mention for instance [Nakić et al. 2019], with explicit dependence. One can also mention doubling estimates directly for the parabolic equation; see [Canuto et al. 2002; Escauriaza and Vessella 2003] for instance.

A1. ***Toolbox of Riemannian geometry.*** The definitions given in this section have a deep geometric meaning; see [Gallot et al. 1987]. We will however only use the associated calculus rules, which we recall below. Note that they are usually written for smooth metrics, but they still make sense for Lipschitz metric, as we shall see below. We follow the notation of [Gallot et al. 1987].

Here and in all estimates below, M is a (not necessarily compact) smooth d-dimensional manifold with boundary ∂M, so that $M=\partial M\sqcup \mathrm{Int}(M)$.

Given an open set $U\subset M$ such that $\overline{U}$ is compact in M (note that this definition holds not only for open sets of $\mathrm{Int}(M)$), we denote by $L^p(U)$, $H^k(U)$, $W^{k,\infty}(U)$ the usual Sobolev spaces. These are defined intrinsically once U is fixed, even if the associated norms may depend on the metric or the charts chosen. The notation $L^p_{\mathrm{loc}}(M)$, $H^k_{\mathrm{loc}}(M)$, $W^{k,\infty}_{\mathrm{loc}}(M)$ will be used for functions belonging to $L^p(U)$, etc. for any open set U such that $\overline{U}$ is compact in M (and not $\mathrm{Int}(M)$).

We denote by g a locally Lipschitz metric on M, (that is, $x \mapsto g_x(\cdot,\cdot)$ is a locally Lipschitz section of the bundle of symmetric bilinear forms on TM that is uniformly bounded from below by a positive constant on any compact set).

Given a local regularity space B as above, and $U \subset M$ such that $\overline{U}$ is compact in M, we define

$$\mathcal{T}_B^2(U) = \Gamma_B(T^2T^*M)|_U$$

to be the space of sections of 2-tensors on T^*M having regularity B on a neighborhood of U. In local charts, such a tensor $t \in \mathcal{T}_B^2(M)$ can be written as $t = (t_{ij})$, with t_{ij} having the regularity of B. Typically, a locally Lipschitz metric g satisfies $g \in \mathcal{T}^2_{W^{1,\infty}_{\mathrm{loc}}}(M)$.

We denote by $\langle\cdot,\cdot\rangle_g = g(\cdot,\cdot)$ the inner product in TM. Note that this notation omits mention of the point $x \in M$ at which the inner products takes place: this allows us to write $\langle X, Y\rangle_g$ as a function on M (the dependence on x is omitted here as well) when X and Y are two vector fields on M. For a vector field X, we also define $|X|_g^2 = \langle X, X\rangle_g$.

We recall that the Riemannian gradient ∇_g of a function f is defined by

$$\langle \nabla_g f, X\rangle_g = df(X) \quad \text{for any vector field } X.$$

For a function f on M, we denote by $\int f = \int_M f(x)\, d\,\mathrm{Vol}_g(x)$ its integral on M, where $d\,\mathrm{Vol}_g(x)$ is the Riemannian density. We denote by div_g the associated divergence, defined on a vector field X by

$$\int u \,\mathrm{div}_g X = -\int \langle \nabla_g u, X\rangle_g \quad \text{for all } u \in C_c^\infty(\mathrm{Int}(M)).$$

We denote by $\Delta_g = \mathrm{div}_g \nabla_g$ the associated (nonpositive) Laplace–Beltrami operator. We also denote by D the Levi-Civita connection associated to the metric g; see [Gallot et al. 1987, Chapter II, Section B].

Let us now recall how these objects can be written in local coordinates.

Formula 1. In coordinates, for f a smooth function and $X = \sum_i X^i \frac{\partial}{\partial x_i}$, $Y = \sum_i Y^i \frac{\partial}{\partial x_i}$ smooth vector fields on M, we have

$$\langle X, Y\rangle_g = \sum_{i=1}^n g_{ij} X^i Y^j, \qquad \int f = \int f\, d\,\mathrm{Vol}_g = \int f(x)\sqrt{\det g(x)}\, dx,$$

$$\nabla_g f = \sum_{i,j=1}^n g^{ij}(\partial_j f)\frac{\partial}{\partial x_i}, \qquad \mathrm{div}_g(X) = \sum_{i=1}^n \frac{1}{\sqrt{\det g}}\,\partial_i(\sqrt{\det g}\, X_i),$$

$$\Delta_g f = \sum_{i,j=1}^n \frac{1}{\sqrt{\det g}}\,\partial_i(\sqrt{\det g}\, g^{ij}\partial_j f), \qquad D_X Y = \sum_{i=1}^n \left(\sum_{j=1}^n X^j \frac{\partial Y^i}{\partial x_j} + \sum_{j,k=1}^n \Gamma^i_{j,k} X^j Y^k\right)\frac{\partial}{\partial x_i},$$

where $(g^{-1})_{ij} = g^{ij}$ and the Christoffel symbols are defined by

$$\Gamma^i_{j,k} = \frac{1}{2}\sum_{l=1}^n g^{il}(\partial_j g_{kl} + \partial_k g_{lj} - \partial_l g_{jk});$$

see for instance [Gallot et al. 1987, p. 71].

Note in particular that the Lipschitz regularity of g can be written as, on any local chart U with $\overline{U}$ compact, $g_{ij} \in W^{1,\infty}(U)$, and implies $g^{ij} \in W^{1,\infty}(U)$. This gives, if f, X, Y are smooth, that $\langle X, Y\rangle_g \in W^{1,\infty}_{\mathrm{loc}}(M)$, $\nabla_g f$ is a locally Lipschitz vector field, $\Delta_g f \in L^\infty_{\mathrm{loc}}(M)$ and $D_X Y$ is an L^∞_{loc} vector field on M, since the definitions of Δ_g and D_X involve one derivative of the coefficients of g.

In view of the properties of D_X, it is natural to set $D_X f = Xf = df(X)$ for a function f on M. Let us now collect some properties of these objects, that we shall use below.

Formula 2. For f, h smooth functions and $X = \sum_i X^i \frac{\partial}{\partial x_i}$, $Y = \sum_i Y^i \frac{\partial}{\partial x_i}$ smooth vector fields on M, we have

$$\begin{aligned} \nabla_g(fh) &= (\nabla_g f)h + f(\nabla_g h), \\ \operatorname{div}_g(fX) &= \langle \nabla_g f, X\rangle_g + f \operatorname{div}_g(X), \\ D_X(fY) &= (Xf)Y + f D_X Y, \quad \text{where } Xf := df(X), \\ D_X(\langle Y, Z\rangle_g) &= \langle D_X Y, Z\rangle_g + \langle Y, D_X Z\rangle_g. \end{aligned}$$

That D_X acts on functions as well as on vector fields suggests extending the definition of D_X to more general vector bundles; see [Gallot et al. 1987, Proposition 2.58]. In particular, for a one-form ω, $D_X\omega$ is defined (by duality) to be the one-form acting as

$$(D_X\omega)(Y) = X(\omega(Y)) - \omega(D_X Y) \quad \text{for all vector fields } Y.$$

This allows us to define the Hessian of a function, see [Gallot et al. 1987, Exercise 2.65],

$$\operatorname{Hess}(f)(X, Y) = (D_X df)(Y) \quad \text{for vector fields } X, Y$$

(which only involves the values of X, Y and not their derivatives). In local charts, note that we have

$$\operatorname{Hess}(f)(X, Y) = \sum_{i,j} X^i Y^j [\partial^2_{ij} f - \Gamma^k_{ij} \partial_k f],$$

which again is in $L^\infty_{\mathrm{loc}}(M)$ for a locally Lipschitz metric g and L^∞_{loc} vector fields X, Y. Note also that the Hessian of f is symmetric; that is, $\operatorname{Hess}(f)(X, Y) = \operatorname{Hess}(f)(Y, X)$.

Lemma A.1. *For any function f and any vector field X and Y, we have*

$$\operatorname{Hess}(f)(X, Y) = \langle D_X \nabla_g f, Y\rangle_g.$$

Proof. According to the above calculus rules, we compute in two different ways the quantity

$$D_X(\langle \nabla_g f, Y\rangle_g) = D_X\big(df(Y)\big) = (D_X df)(Y) + df(D_X Y) = \operatorname{Hess}(f)(X, Y) + df(D_X Y).$$

We also have

$$D_X(\langle \nabla_g f, Y\rangle_g) = \langle D_X \nabla_g f, Y\rangle_g + \langle \nabla_g f, D_X Y\rangle_g = \langle D_X \nabla_g f, Y\rangle_g + df(D_X Y),$$

which, combined with the previous computation yields the result. ☐

Finally, we recall an integration-by-parts formula in the present context.

Formula 3 (Riemannian Stokes formula). Assume ∂M is piecewise C^1 and graph-Lipschitz. Then, for all $f \in H^2_{\mathrm{loc}}(M)$ and $h \in H^1_{\mathrm{loc}}(M)$, one of which is compactly supported, we have

$$\int (\Delta_g f)h = \int_{\partial M} \langle \nabla_g f, \nu \rangle_g h - \int \langle \nabla_g f, \nabla_g h \rangle_g .$$

Here, the boundary ∂M is endowed with the Riemannian metric induced by g, and $\int_{\partial M}$ is the integral with respect to the associated surface measure (defined as in Formula 1). The vector field ν is the unit normal vector to ∂M which is outgoing. It is defined almost everywhere if ∂M is piecewise C^1. In a local coordinate chart $(x_1, \dots, x_n)$ centered at 0, and in which $\partial M \subset \{x_n = 0\}$ and $M \subset \{x_n \leq 0\}$, we have

$$\nu = \sum_{j=1}^{n} \frac{g^{jn}}{\sqrt{g^{nn}}} \frac{\partial}{\partial x_j}.$$

With the prescribed regularity of the boundary, the space $L^\infty_{\mathrm{loc}}(\partial M)$ is defined intrinsically. We denote by $\partial_\nu f = \langle \nabla_g f, \nu \rangle_g$ the normal derivative at the boundary, which is only $L^\infty_{\mathrm{loc}}(\partial M)$ since ∂M is piecewise C^1.

Note that in the above coordinate system, we have

$$\partial_\nu f = \sum_{j=1}^{n} \frac{g^{jn}}{\sqrt{g^{nn}}} \partial_{x_j} f.$$

In particular, if f satisfies Dirichlet boundary conditions, this is $\partial_\nu f = \sqrt{g^{nn}}\, \partial_{x_n} f$.

Note finally the vector field $X - \langle X, \nu \rangle_g \nu$ is tangential to $\partial \mathcal{M}$, so that we may decompose a vector field as its normal and tangential parts. In particular, we shall decompose the gradient $\nabla_g f = \partial_\nu f \nu + \nabla_T f$, where $\nabla_T f|_{\partial \mathcal{M}} \in T\partial \mathcal{M}$.

A2. *The Carleman estimate.* We stress the fact that functions $u \in C^\infty(M)$ are smooth up to the boundary of M (as opposed to functions $u \in C^\infty(\mathrm{Int}(M))$). We will first estimate the Carleman conjugate operator in Theorem A.2 and then give the desired estimate under appropriate assumptions in Theorem A.5.

Theorem A.2. *Assume g is a Lipschitz metric on M and ∂M is piecewise C^1 and graph-Lipschitz. Let U be an open subset of M such that $\overline{U}$ is compact (in the topology of $M \supset \partial M$) and define $\Sigma = \partial M \cap U$. Then, for any $f \in W^{1,\infty}(U)$, $\varphi \in W^{2,\infty}(U)$, $u \in H^2_{\mathrm{comp}}(U)$ and $\tau \geq 0$, we have*

$$\int |e^{\tau\varphi} \Delta_g (e^{-\tau\varphi} u)|^2 + R(u) \geq \tau^3 \int \big[2\,\mathrm{Hess}(\varphi)(\nabla_g \varphi, \nabla_g \varphi) + (\Delta_g \varphi)|\nabla_g \varphi|_g^2 - f|\nabla_g \varphi|_g^2\big]|u|^2$$
$$+ \tau \int 2\,\mathrm{Hess}(\varphi)(\nabla_g u, \nabla_g u) - (\Delta_g \varphi)|\nabla_g u|_g^2 + f|\nabla_g u|_g^2 + BT(u),$$

with boundary terms

$$BT(u) = -2\tau \int_\Sigma \langle \nabla_g u, \nu \rangle_g \langle \nabla_g \varphi, \nabla_g u \rangle_g + \tau \int_\Sigma \langle \nabla_g \varphi, \nu \rangle_g |\nabla_g u|_g^2$$
$$- \tau^3 \int_\Sigma \langle \nabla_g \varphi, \nu \rangle_g |u|^2 |\nabla_g \varphi|_g^2 + \tau \int_\Sigma \langle \nabla_g u, \nu \rangle_g f u, \tag{68}$$

and remainder $R(u)$ satisfying

$$|R(u)| \leq \big(\|f - \Delta_g \varphi\|^2_{L^\infty(U)} + \tfrac{1}{2}\|\nabla_g f\|_{L^\infty(U)}\big)\tau^2 \|u\|^2_{L^2} + \tfrac{1}{2}\|\nabla_g f\|_{L^\infty(U)} \|\nabla_g u\|^2_{L^2}. \tag{69}$$

Note that the last term in (68) is actually of lower order. We keep it here since it vanishes in the case of Dirichlet boundary conditions.

Remark A.3. It is very important for our purpose to notice that all terms in this identity only involve derivatives of order 0 or 1 of the metric. This will be important when we will consider stability issues with respect to Lipschitz perturbations of the metric.

This identity suggests introducing and studying the following two important quantities, given X a smooth vector field on M:

$$\mathcal{B}_{g,\varphi,f}(X) = 2\,\mathrm{Hess}(\varphi)(X,X) - (\Delta_g\varphi)|X|_g^2 + f|X|_g^2,$$
$$\mathcal{E}_{g,\varphi,f} = 2\,\mathrm{Hess}(\varphi)(\nabla_g\varphi,\nabla_g\varphi) + (\Delta_g\varphi)|\nabla_g\varphi|_g^2 - f|\nabla_g\varphi|_g^2.$$

Note that for a Lipschitz metric g, we have $\mathcal{E}_{g,\varphi,f} \in L^\infty_{\mathrm{loc}}(M)$ and $\mathcal{B}_{g,\varphi,f}(X) \in L^\infty_{\mathrm{loc}}(M)$ for any locally bounded vector field X.

Remark A.4. At this level, it would be very tempting to set $F = -\Delta_g\varphi + f$ and work with the associated simplified expressions of $\mathcal{B}_{g,\varphi,f}(X)$ and $\mathcal{E}_{g,\varphi,f}$. From a conceptual point of view, this is completely fine; see Remark A.8 below. However, since we consider the limiting Lipschitz regularity of the metric, this change of additional function is not admissible. Indeed, the remainder term $R(u)$ in Theorem A.2 requires the regularity $\nabla_g f \in L^\infty$ and $f = F + \Delta_g\varphi$ is already in L^∞ and consumes one derivative of the metric g. Having $\nabla_g F \in L^\infty$ would then require g to be $W^{2,\infty}$.

We define $\|w\|_{L^2}^2 = \int |w|^2$ (see Formula 1 for the notation $\int$) for a function w and $\|X\|_{L^2}^2 = \int |X|_g^2$ for a vector field X.

We can now state the Carleman estimate.

Theorem A.5. *Let U be an open subset of M such that $\overline{U}$ is compact* (*in the topology of $M \supset \partial M$*) *and define $\Sigma = \partial M \cap U$. Assume that the functions (φ, f) satisfy $f \in W^{1,\infty}(U)$, $\varphi \in W^{2,\infty}(U)$, $|\nabla_g\varphi|_g^2 > 0$ on $\overline{U}$, and there exists $C_0 > 0$ such that for any vector field X we have almost everywhere on U*

$$\mathcal{B}_{g,\varphi,f}(X) \geq 2C_0|X|_g^2, \tag{70}$$
$$\mathcal{E}_{g,\varphi,f} \geq 2C_0|\nabla_g\varphi|_g^2. \tag{71}$$

Then, setting $c(\varphi) = \max\{1, (\min_{\overline{U}} |\nabla_g\varphi|_g^2)^{-1}\}$, we have the following statements:

(1) *For all $\tau \geq \frac{c(\varphi)}{C_0}\big(\|f - \Delta_g\varphi\|_{L^\infty(U)}^2 + \frac{1}{2}\|\nabla_g f\|_{L^\infty(U)}\big)$ and all $v \in C_c^\infty(U)$ we have the estimate*

$$\tfrac{1}{3}C_0(\tau^3\|e^{\tau\varphi}v\nabla_g\varphi\|_{L^2(U)}^2 + \tau\|e^{\tau\varphi}\nabla_g v\|_{L^2(U)}^2) \leq \|e^{\tau\varphi}\Delta_g v\|_{L^2(U)}^2 + \tau(\|e^{\tau\varphi}\nabla_g v\|_{L^2(\Sigma)}^2 + \tau^2\|e^{\tau\varphi}v\nabla_g\varphi\|_{L^2(\Sigma)}^2)K_{f,\varphi}, \tag{72}$$

with $K_{f,\varphi} = 3\big(\frac{c(\varphi)}{\tau}\|f\|_{L^\infty(\Sigma)} + 3\|\nabla_g\varphi\|_{L^\infty(\Sigma)}\big)$.

(2) *For all $\tau \geq \frac{c(\varphi)}{C_0}\big(\|f - \Delta_g\varphi\|_{L^\infty(U)}^2 + \frac{1}{2}\|\nabla_g f\|_{L^\infty(U)}\big)$ and all $v \in C_c^\infty(U)$ such that $v = 0$ on Σ we have*

$$\tfrac{1}{3}C_0(\tau^3\|e^{\tau\varphi}v\nabla_g\varphi\|_{L^2(U)}^2 + \tau\|e^{\tau\varphi}\nabla_g v\|_{L^2(U)}^2) \leq \|e^{\tau\varphi}\Delta_g v\|_{L^2(U)}^2 + \tau\int_\Sigma e^{2\tau\varphi}\partial_\nu\varphi|\partial_\nu v|^2. \tag{73}$$

(3) *If $\varphi|_\Sigma$ is constant and* $-m(\varphi) := \max_\Sigma \partial_\nu \varphi < 0$, *then setting* $M(\varphi) := \max_\Sigma(-\partial_\nu\varphi) > 0$, *we have for all*

$$\tau \geq \max\left\{\frac{c(\varphi)}{C_0}\left(\|f-\Delta_g\varphi\|^2_{L^\infty(U)} + \tfrac{1}{2}\|\nabla_g f\|_{L^\infty(U)}\right), \frac{\sqrt{\|f\|_{L^\infty(\Sigma)}}}{m(\varphi)}\right\}$$

and all $v \in C^\infty_c(U)$

$$\|e^{\tau\varphi}\Delta_g v\|^2_{L^2} + M(\varphi)\tau\int_\Sigma e^{2\tau\varphi}|\nabla_T v|^2_g \geq \frac{C_0}{3}(\tau^3\|e^{\tau\varphi} v\nabla_g\varphi\|^2_{L^2(U)} + \tau\|e^{\tau\varphi}\nabla_g v\|^2_{L^2(U)}) + \frac{\tau}{8}\frac{m(\varphi)^3}{M(\varphi)^2}\int_\Sigma e^{2\tau\varphi}|\partial_\nu v|^2 + \tau^3\frac{m(\varphi)^3}{4}\int_\Sigma |v|^2. \quad (74)$$

Remark A.6. In the last two statements of this result, we assume boundary conditions (either for v or for φ) on the whole boundary Σ. Since the integrals involved are local, we could also assume different conditions on parts of the boundary, obtaining the associated terms in the estimates.

For simplicity, in the proof, we shall denote by

$$\|u\|^2_{H^1_\tau} = \tau^2\|u\nabla_g\varphi\|^2_{L^2} + \|\nabla_g u\|^2_{L^2}$$

the semiclassical norm (recall that $|\nabla_g\varphi|^2_g > 0$ here).

Proof of Theorem A.5. We first let $v = e^{-\tau\varphi}u$, and apply the estimate of Theorem A.2. The latter, together with our assumption (70)–(71) (applied almost everywhere in M to $X = \nabla_g u$) implies for all $\tau \geq 0$ and $u \in C^\infty_c(U)$

$$\begin{aligned}\|e^{\tau\varphi}\Delta_g(e^{-\tau\varphi}u)\|^2_{L^2} + R(u) &\geq 2C_0\tau^3\|u\nabla_g\varphi\|^2_{L^2} + 2C_0\tau\|\nabla_g u\|^2_{L^2} + BT(u)\\ &= 2C_0\tau\|u\|^2_{H^1_\tau} + BT(u),\end{aligned}$$

where $BT(u)$ is defined in (68) and $R(u)$ estimated in (69). Now, we have

$$|R(u)| \leq c(\varphi)\left(\|f-\Delta_g\varphi\|^2_{L^\infty} + \tfrac{1}{2}\|\nabla_g f\|_{L^\infty}\right)\|u\|^2_{H^1_\tau},$$

which implies that if $\tau C_0 \geq c(\varphi)\left(\|f-\Delta_g\varphi\|^2_{L^\infty} + \tfrac{1}{2}\|\nabla_g f\|_{L^\infty}\right)$, we obtain

$$\|e^{\tau\varphi}\Delta_g(e^{-\tau\varphi}u)\|^2_{L^2} \geq C_0\tau\|u\|^2_{H^1_\tau} + BT(u). \quad (75)$$

We now consider the boundary terms. Without any assumption on the boundary, we have

$$|BT(u)| \leq 3\tau\|\nabla_g\varphi\|_{L^\infty(\Sigma)}(\|\nabla_g u\|^2_{L^2(\Sigma)} + \tau^2\|u\nabla_g\varphi\|^2_{L^2(\Sigma)}) + \tfrac{1}{2}\|f\|_{L^\infty(\Sigma)}(\|\partial_\nu u\|^2_{L^2(\Sigma)} + \tau^2\|u\|^2_{L^2(\Sigma)}),$$

and hence obtain in this case

$$C_0\tau\|u\|^2_{H^1_\tau} \leq \|e^{\tau\varphi}\Delta_g(e^{-\tau\varphi}u)\|^2_{L^2} + (c(\varphi)\|f\|_{L^\infty(\Sigma)} + 3\tau\|\nabla_g\varphi\|_{L^\infty(\Sigma)})(\|\nabla_g u\|^2_{L^2(\Sigma)} + \tau^2\|u\nabla_g\varphi\|^2_{L^2(\Sigma)}).$$

Recalling that $u = e^{\tau\varphi}v$, this implies $\nabla_g u = e^{\tau\varphi}\nabla_g v + \tau u \nabla_g \varphi$, and hence

$$\begin{aligned} \|e^{\tau\varphi}\nabla_g v\|_{L^2}^2 &\leq 2\|\nabla_g u\|_{L^2}^2 + 2\tau^2\|u\nabla_g\varphi\|_{L^2}^2 = 2\|u\|_{H^1_\tau}^2, \\ \|\nabla_g u\|_{L^2}^2 &\leq 2\|e^{\tau\varphi}\nabla_g v\|_{L^2}^2 + 2\tau^2\|e^{\tau\varphi}v\nabla_g\varphi\|_{L^2}^2. \end{aligned} \tag{76}$$

The last four estimates imply

$$\begin{aligned} &\tfrac{1}{3}C_0(\tau^3\|e^{\tau\varphi}v\|_{L^2}^2 + \tau\|e^{\tau\varphi}\nabla_g v\|_{L^2}^2) \\ &\quad\leq C_0\tau\|u\|_{H^1_\tau}^2 \\ &\quad\leq \|e^{\tau\varphi}\Delta_g v\|_{L^2}^2 + 3\big(c(\varphi)\|f\|_{L^\infty(\Sigma)} + 3\tau\|\nabla_g\varphi\|_{L^\infty(\Sigma)}\big)\big(\|e^{\tau\varphi}\nabla_g v\|_{L^2(\Sigma)}^2 + \tau^2\|e^{\tau\varphi}v\nabla_g\varphi\|_{L^2(\Sigma)}^2\big), \end{aligned}$$

and hence (72).

Second, we assume the Dirichlet boundary condition $v|_\Sigma = 0$. This implies $u|_\Sigma = 0$ and $\nabla_g u|_\Sigma = \partial_\nu u|_\Sigma \partial_\nu$, so that we obtain

$$BT(u) = -\tau\int_\Sigma \partial_\nu\varphi|\partial_\nu u|^2 = -\tau\int_\Sigma \partial_\nu\varphi e^{2\tau\varphi}|\partial_\nu v|^2.$$

Estimate (75) then reads

$$\|e^{\tau\varphi}\Delta_g(e^{-\tau\varphi}u)\|_{L^2}^2 + \int_\Sigma \partial_\nu\varphi e^{2\tau\varphi}|\partial_\nu v|^2 \geq C_0\tau\|u\|_{H^1_\tau}^2.$$

Using again (76) to come back to the variable v yields (73).

Finally, we consider the case where $\varphi|_\Sigma$ is constant and $\partial_\nu\varphi \leq -m(\varphi) < 0$, in which case we obtain from (68)

$$\begin{aligned} BT(u) &= -2\tau\int_\Sigma \partial_\nu\varphi|\partial_\nu u|^2 + \tau\int_\Sigma \partial_\nu\varphi|\nabla_g u|_g^2 - \tau^3\int_\Sigma(\partial_\nu\varphi)^3|u|^2 + \tau\int_\Sigma \partial_\nu u f u \\ &= -\tau\int_\Sigma \partial_\nu\varphi|\partial_\nu u|^2 + \tau\int_\Sigma \partial_\nu\varphi|\nabla_T u|_g^2 - \tau^3\int_\Sigma(\partial_\nu\varphi)^3|u|^2 + \tau\int_\Sigma \partial_\nu u f u. \end{aligned}$$

Estimate (75) then reads

$$\|e^{\tau\varphi}\Delta_g(e^{-\tau\varphi}u)\|_{L^2}^2 + \tau\int_\Sigma \partial_\nu\varphi|\partial_\nu u|^2 - \tau\int_\Sigma \partial_\nu\varphi|\nabla_T u|_g^2 + \tau^3\int_\Sigma(\partial_\nu\varphi)^3|u|^2 - \tau\int_\Sigma \partial_\nu u f u \geq C_0\tau\|u\|_{H^1_\tau}^2,$$

and hence, using $-M(\varphi) \leq \partial_\nu\varphi \leq -m(\varphi) < 0$,

$$\begin{aligned} &\|e^{\tau\varphi}\Delta_g(e^{-\tau\varphi}u)\|_{L^2}^2 + M(\varphi)\tau\int_\Sigma |\nabla_T u|_g^2 - \tau\int_\Sigma \partial_\nu u f u \\ &\qquad\qquad \geq C_0\tau\|u\|_{H^1_\tau}^2 + m(\varphi)\tau\int_\Sigma |\partial_\nu u|^2 + \tau^3 m(\varphi)^3\int_\Sigma |u|^2. \end{aligned}$$

Now, we estimate

$$\begin{aligned} \left|\int_\Sigma \partial_\nu u f u\right| &\leq \|f\|_{L^\infty(\Sigma)}\|u\|_{L^2(\Sigma)}\|\partial_\nu u\|_{L^2(\Sigma)} \\ &\leq \frac{\|f\|_{L^\infty(\Sigma)}}{2m(\varphi)\tau}\|\partial_\nu u\|_{L^2(\Sigma)}^2 + \frac{\|f\|_{L^\infty(\Sigma)}m(\varphi)\tau}{2}\|u\|_{L^2(\Sigma)}^2, \end{aligned}$$

so that for $\tau \geq \sqrt{\|f\|_{L^\infty(\Sigma)}}/m(\varphi)$ this term is absorbed in the right-hand side, and we obtain

$$\|e^{\tau\varphi}\Delta_g(e^{-\tau\varphi}u)\|_{L^2}^2 + M(\varphi)\tau\int_\Sigma |\nabla_T u|_g^2 \geq C_0\tau\|u\|_{H_\tau^1}^2 + \frac{m(\varphi)}{2}\tau\int_\Sigma |\partial_\nu u|^2 + \tau^3\frac{m(\varphi)^3}{2}\int_\Sigma |u|^2.$$

Recalling that $u = e^{\tau\varphi}v$ and $\nabla_T\varphi|_\Sigma = 0$, this implies $\nabla_T u = e^{\tau\varphi}\nabla_T v$ and

$$e^{2\tau\varphi}|\partial_\nu v|^2 \leq 2|\partial_\nu u|^2 + 2\tau^2|\partial_\nu\varphi|^2|u|^2;$$

hence

$$\frac{1}{4}\frac{m(\varphi)^2}{M(\varphi)^2}\int_\Sigma e^{2\tau\varphi}|\partial_\nu v|^2 \leq \frac{1}{2}\int_\Sigma |\partial_\nu u|^2 + \tau^2\frac{m(\varphi)^2}{2}\int_\Sigma |u|^2.$$

Finally using again (76) with the last two estimates implies (74), concluding the proof of Theorem A.5. ☐

Proof of Theorem A.2. The statement of the theorem is a lower bound for the L^2-norm of the quantity $e^{\tau\varphi}\Delta_g(e^{-\tau\varphi}u)$, which we may compute as

$$P_\varphi u := e^{\tau\varphi}\Delta_g(e^{-\tau\varphi}u) = \Delta_g u - 2\tau\langle\nabla_g\varphi, \nabla_g u\rangle_g - \tau(\Delta_g\varphi)u + \tau^2|\nabla_g\varphi|_g^2 u.$$

We then decompose the conjugated operator P_φ as

$$P_\varphi = Q_2 + Q_1,$$

with

$$\begin{aligned} Q_1 u &:= -2\tau\langle\nabla_g\varphi, \nabla_g u\rangle_g - \tau f u, \\ Q_2 u &:= \Delta_g u + \tau^2|\nabla_g\varphi|_g^2 u - \tau(\Delta_g\varphi)u + \tau f u = \tilde{Q}_2 u + R_2 u, \end{aligned}$$

where $\tilde{Q}_2$ is the principal part of Q_2, that is,

$$\tilde{Q}_2 u = \Delta_g u + \tau^2|\nabla_g\varphi|_g^2 u \quad \text{and} \quad R_2 u = \tau(-\Delta_g\varphi + f)u.$$

Now, we write ($\|\cdot\|$ denotes the L^2-norm for short)

$$2\|P_\varphi u\|^2 + 2\|R_2 u\|^2 \geq \|P_\varphi u - R_2 u\|^2 = \|Q_1 u + \tilde{Q}_2 u\|^2,$$

where we estimate the remainder as

$$\|R_2 u\|^2 \leq \tau^2\|f - \Delta_g\varphi\|_{L^\infty}^2\|u\|_{L^2}^2. \tag{77}$$

Hence, we are left to produce a lower bound for

$$\|Q_1 u + \tilde{Q}_2 u\|^2 = \|Q_1 u\|^2 + \|\tilde{Q}_2 u\|^2 + 2\,\mathrm{Re}(Q_1 u, \tilde{Q}_2 u).$$

Now, note that the all differential operators $P_\varphi, Q_1, \tilde{Q}_2$ involved have real coefficients. Hence, if we consider complex-valued functions $u = u_R + iu_I$, we have $\|P_\varphi u\|^2 = \|P_\varphi u_R\|^2 + \|P_\varphi u_I\|^2$, $\|u\|^2 = \|u_R\|^2 + \|u_I\|^2$ so that proving $\|P_\varphi u\|^2 \geq c_0\|u\|^2$ for real-valued functions u implies the same inequality for complex-valued ones. As a consequence, we only prove the result for a real-valued function u, and associated real inner product. We now provide an explicit computation for $(Q_1 u, \tilde{Q}_2 u)$, which is the key step in the proof.

Lemma A.7. *For all functions $\varphi \in W^{2,\infty}_{\mathrm{loc}}(M)$, $f \in W^{1,\infty}_{\mathrm{loc}}(M)$ and $u \in H^2_{\mathrm{comp}}(M)$, we have*

$$\begin{aligned}(Q_1 u, \widetilde{Q}_2 u) = \tau^3 \int \big(2\,\mathrm{Hess}(\varphi)(\nabla_g \varphi, \nabla_g \varphi) + (\Delta_g \varphi)|\nabla_g \varphi|_g^2 - f|\nabla_g \varphi|_g^2\big)|u|^2 \\ + \tau \int 2\,\mathrm{Hess}(\varphi)(\nabla_g u, \nabla_g u) - (\Delta_g \varphi)|\nabla_g u|_g^2 + f|\nabla_g u|_g^2 \\ + \tau \int u \langle \nabla_g u, \nabla_g f\rangle_g + BT(u),\end{aligned}$$

with

$$\begin{aligned}BT(u) = -2\tau \int_\Sigma \langle \nabla_g u, \nu\rangle_g \langle \nabla_g \varphi, \nabla_g u\rangle_g + \tau \int_\Sigma \langle \nabla_g \varphi, \nu\rangle_g |\nabla_g u|_g^2 \\ - \tau^3 \int_\Sigma \langle \nabla_g \varphi, \nu\rangle_g |u|^2 |\nabla_g \varphi|_g^2 - \tau \int_\Sigma \langle \nabla_g u, \nu\rangle_g f u.\end{aligned}$$

To conclude the proof of the theorem, we now simply write

$$2\|P_\varphi u\|^2 + 2\|R_2 u\|^2 \geq \|Q_1 u + \widetilde{Q}_2 u\|^2 \geq 2(Q_1 u, \widetilde{Q}_2 u). \tag{78}$$

In the estimates of Lemma A.7, the remainder term is

$$R_3(u) = -\tau \int u\langle \nabla_g u, \nabla_g f\rangle_g, \quad |R_3(u)| \leq \tfrac{1}{2}\|\nabla_g f\|_{L^\infty}(\|\nabla_g u\|_{L^2}^2 + \tau^2\|u\|_{L^2}^2),$$

which, combined with (78), (77) and Lemma A.7, concludes the proof of Theorem A.2 with

$$R(u) = \|R_2 u\|^2 + |R_3(u)|. \qquad \square$$

For the proof of Theorem A.2 to be complete, it only remains to prove Lemma A.7.

Proof of Lemma A.7. We have

$$(Q_1 u, \widetilde{Q}_2 u) = \int (-2\tau\langle \nabla_g \varphi, \nabla_g u\rangle_g - \tau f u)(\Delta_g u + \tau^2 |\nabla_g \varphi|_g^2 u) = -\tau(2J + 2\tau^2 K + L), \tag{79}$$

with

$$J = \int \langle \nabla_g \varphi, \nabla_g u\rangle_g \Delta_g u, \quad K = \int \langle \nabla_g \varphi, \nabla_g u\rangle_g |\nabla_g \varphi|_g^2 u, \quad L = \int f u(\Delta_g u + \tau^2 |\nabla_g \varphi|_g^2 u).$$

We now perform one (and only one, which is the most we can do with the Lipschitz regularity of g) integration by parts in each of these integrals. Firstly, we compute J as

$$J = \int_\Sigma \langle \nabla_g u, \nu\rangle_g \langle \nabla_g \varphi, \nabla_g u\rangle_g - \int \langle \nabla_g u, \nabla_g(\langle \nabla_g \varphi, \nabla_g u\rangle_g)\rangle_g.$$

But, we also have

$$\begin{aligned}\langle \nabla_g u, \nabla_g(\langle \nabla_g \varphi, \nabla_g u\rangle_g)\rangle_g &= D_{\nabla_g u}(\langle \nabla_g \varphi, \nabla_g u\rangle_g) \\ &= \langle D_{\nabla_g u} \nabla_g \varphi, \nabla_g u\rangle_g + \langle \nabla_g \varphi, D_{\nabla_g u} \nabla_g u\rangle_g \\ &= \mathrm{Hess}(\varphi)(\nabla_g u, \nabla_g u) + \mathrm{Hess}(u)(\nabla_g u, \nabla_g \varphi),\end{aligned}$$

so that

$$J = \int_\Sigma \langle \nabla_g u, \nu\rangle_g \langle \nabla_g \varphi, \nabla_g u\rangle_g - \int \mathrm{Hess}(\varphi)(\nabla_g u, \nabla_g u) - \int \mathrm{Hess}(u)(\nabla_g u, \nabla_g \varphi).$$

We then notice that

$$\begin{aligned}\langle \nabla_g \varphi, \nabla_g |\nabla_g u|_g^2\rangle_g &= d(|\nabla_g u|_g^2)(\nabla_g \varphi) = D_{\nabla_g \varphi}(\langle \nabla_g u, \nabla_g u\rangle_g) \\ &= \langle D_{\nabla_g\varphi}\nabla_g u, \nabla_g u\rangle_g + \langle \nabla_g u, D_{\nabla_g\varphi}\nabla_g u\rangle_g = 2\langle D_{\nabla_g\varphi}\nabla_g u, \nabla_g u\rangle_g \\ &= 2\,\mathrm{Hess}(u)(\nabla_g\varphi, \nabla_g u), \end{aligned} \tag{80}$$

so that we have in particular

$$2\int \mathrm{Hess}(u)(\nabla_g\varphi, \nabla_g u) = \int \langle \nabla_g\varphi, \nabla_g|\nabla_g u|_g^2\rangle_g = -\int (\Delta_g\varphi)|\nabla_g u|_g^2 + \int_\Sigma \langle \nabla_g\varphi, \nu\rangle_g |\nabla_g u|_g^2.$$

Coming back to J, this finally implies the expression

$$\begin{aligned}2J = 2\int_\Sigma \langle \nabla_g u, \nu\rangle_g \langle \nabla_g\varphi, \nabla_g u\rangle_g - 2\int \mathrm{Hess}(\varphi)(\nabla_g u, \nabla_g u) \\ + \int (\Delta_g\varphi)|\nabla_g u|_g^2 - \int_\Sigma \langle \nabla_g\varphi, \nu\rangle_g |\nabla_g u|_g^2. \end{aligned}\tag{81}$$

Secondly, remarking that $\nabla_g |u|^2 = 2u\nabla_g u$, we write K as

$$K = \int \langle \nabla_g\varphi, \nabla_g u\rangle_g |\nabla_g\varphi|_g^2 u = \frac{1}{2}\int |\nabla_g\varphi|_g^2 \langle \nabla_g\varphi, \nabla_g|u|^2\rangle_g.$$

An integration by parts yields

$$\begin{aligned}\int (\Delta_g\varphi)|u|^2|\nabla_g\varphi|_g^2 &= \int_\Sigma \langle \nabla_g\varphi, \nu\rangle_g |u|^2|\nabla_g\varphi|_g^2 - \int \langle \nabla_g\varphi, \nabla_g(|u|^2|\nabla_g\varphi|_g^2)\rangle_g \\ &= \int_\Sigma \langle \nabla_g\varphi, \nu\rangle_g |u|^2|\nabla_g\varphi|_g^2 - \int |\nabla_g\varphi|_g^2\langle \nabla_g\varphi, \nabla_g|u|^2\rangle_g - \int |u|^2\langle \nabla_g\varphi, \nabla_g|\nabla_g\varphi|_g^2\rangle_g.\end{aligned}$$

Combining these two formulas, we obtain

$$\begin{aligned}2K &= -\int (\Delta_g\varphi)|u|^2|\nabla_g\varphi|_g^2 + \int_\Sigma \langle \nabla_g\varphi, \nu\rangle_g |u|^2|\nabla_g\varphi|_g^2 - \int \langle \nabla_g\varphi, \nabla_g|\nabla_g\varphi|_g^2\rangle_g |u|^2 \\ &= -\int (\Delta_g\varphi)|u|^2|\nabla_g\varphi|_g^2 + \int_\Sigma \langle \nabla_g\varphi, \nu\rangle_g |u|^2|\nabla_g\varphi|_g^2 - 2\int \mathrm{Hess}(\varphi)(\nabla_g\varphi, \nabla_g\varphi)|u|^2, \end{aligned}\tag{82}$$

where we have used as in (80) that

$$\langle \nabla_g\varphi, \nabla_g|\nabla_g\varphi|_g^2\rangle_g = D_{\nabla_g\varphi}\langle \nabla_g\varphi, \nabla_g\varphi\rangle_g = 2\langle D_{\nabla_g\varphi}\nabla_g\varphi, \nabla_g\varphi\rangle_g = 2\,\mathrm{Hess}(\varphi)(\nabla_g\varphi, \nabla_g\varphi).$$

Thirdly, let us compute L with one integration by parts as

$$\begin{aligned}L &= \int fu(\Delta_g u + \tau^2|\nabla_g\varphi|_g^2 u) \\ &= \int_\Sigma \langle \nabla_g u, \nu\rangle_g fu - \int \langle \nabla_g u, \nabla_g(fu)\rangle_g + \tau^2\int |\nabla_g\varphi|_g^2 f|u|^2 \\ &= \int_\Sigma \langle \nabla_g u, \nu\rangle_g fu - \int f|\nabla_g u|_g^2 + \tau^2\int |\nabla_g\varphi|_g^2 f|u|^2 - \int u\langle \nabla_g u, \nabla_g f\rangle_g. \end{aligned}\tag{83}$$

Coming back to (79) and combining the computations of J, K, L in (81)–(83), we have obtained the statement of Lemma A.7. □

Remark A.8. We wish to compare the above proof with the more usual proofs of Carleman estimates [Hörmander 1985b, Chapter 23; Lebeau and Robbiano 1995; Le Rousseau and Lebeau 2012]. Note first that the fact that operators and functions are real-valued implies, for $u \in C_c^\infty(\mathrm{Int}(M))$, that $(Q_1 u, Q_2 u) = (Q_2 Q_1 u, u) = -(u, Q_1 Q_2 u) = \frac{1}{2}([Q_2, Q_1]u, u)$. Note also that the principal symbol of the conjugated operator P_φ is given by

$$p_\varphi(x,\xi) = \sigma(P_\varphi)(x,\xi) = |\xi^\sharp|_g^2 - \tau^2 |\nabla_g \varphi|_g^2 + 2i\tau \langle \nabla_g \varphi, \xi^\sharp \rangle_g = |\xi|_{g^*}^2 - \tau^2 |d\varphi|_{g^*}^2 + 2i\tau \langle d\varphi, \xi \rangle_{g^*},$$

where g^* is the dual metric on T^*M, i.e., $g^* = (g^{ij})$, and $\xi^\sharp$ is defined by $\langle \xi^\sharp, X \rangle_g = \xi(X)$.

Here, a computation shows that we have

$$\{\mathrm{Re}\, p_\varphi, \mathrm{Im}\, p_\varphi\}(x,\xi) = 4\tau\, \mathrm{Hess}(\varphi)(\xi^\sharp, \xi^\sharp) + 4\tau^3\, \mathrm{Hess}(\varphi)(\nabla_g \varphi, \nabla_g \varphi).$$

As a consequence, the important quantity in the Carleman estimate of Theorem A.5 is

$$\begin{aligned} \mathcal{B}_{g,\varphi,f}(\xi^\sharp) + \tau^2 \mathcal{E}_{g,\varphi,f} &= (f - \Delta_g \varphi)(|\xi^\sharp|_g^2 - \tau^2 |\nabla_g \varphi|_g^2) + 2\,\mathrm{Hess}(\varphi)(\xi^\sharp, \xi^\sharp) + 2\tau^2\, \mathrm{Hess}(\varphi)(\nabla_g \varphi, \nabla_g \varphi) \\ &= (f - \Delta_g \varphi)\, \mathrm{Re}\, p_\varphi + \frac{1}{2\tau} \{\mathrm{Re}\, p_\varphi, \mathrm{Im}\, p_\varphi\}. \end{aligned}$$

The main assumption under which the Carleman estimate of Theorem A.5 holds is hence the existence of a function $F = F(x)$ (of the position variable only) so that

$$F\, \mathrm{Re}\, p_\varphi + \frac{1}{2\tau} \{\mathrm{Re}\, p_\varphi, \mathrm{Im}\, p_\varphi\} \geq C(|\xi|^2 + \tau^2). \tag{84}$$

The choice of F under the form $F = f - \Delta\varphi$ is only made in order not to consume regularity of the metric g; see above Remark A.4.

Of course, assumption (84) is stronger than the usual subellipticity of the Hörmander theorem [1985b, Chapter 23]:

$$\{\mathrm{Re}\, p_\varphi, \mathrm{Im}\, p_\varphi\} > 0 \quad \text{on the set } \{\mathrm{Re}\, p_\varphi = 0, \mathrm{Im}\, p_\varphi = 0\}.$$

The proof of the Hörmander theorem [1985b, Section 23.3] then uses a symbol $F(x,\xi)$ instead of just a function $F(x)$, for instance having the form

$$F(x,\xi) = \frac{\mathrm{Re}\, p_\varphi}{\xi^2 + \tau^2}.$$

We refer to [Le Rousseau and Lebeau 2012, Section 3.1] for a related discussion regarding the Fursikov–Imanuvilov approach to Carleman estimates.

A3. *Constructing weight functions via convexification.* In this section, we explain how to construct weight functions (φ, f) that satisfy the assumption of Theorem A.5, via the usual convexification procedure. In the present context (as opposed to the usual situation), this also requires a very specific choice of the function f.

Lemma A.9 (explicit convexification). *Let $\Psi \in W^{2,\infty}_{\mathrm{loc}}(M;\mathbb{R})$ and $G \in W^{2,\infty}_{\mathrm{loc}}(\mathbb{R})$, and choose*

$$\varphi = G(\Psi) \quad \textit{and} \quad f = 2G''(\Psi)|\nabla_g \Psi|_g^2. \tag{85}$$

Then we have

$$\mathcal{B}_{g,\varphi,f}(X) = 2G'(\Psi)\operatorname{Hess}(\Psi)(X,X) + 2G''(\Psi)|\langle\nabla_g\Psi, X\rangle_g|^2 + \big(G''(\Psi)|\nabla_g\Psi|_g^2 - G'(\Psi)\Delta_g\Psi\big)|X|_g^2,$$
$$\mathcal{E}_{g,\varphi,f} = G'(\Psi)^2\big[2G'(\Psi)\operatorname{Hess}(\Psi)(\nabla_g\Psi,\nabla_g\Psi) + G''(\Psi)|\nabla_g\Psi|_g^4 + G'(\Psi)\Delta_g\Psi|\nabla_g\Psi|_g^2\big].$$

To state the next corollary, for $B \in \mathcal{T}^2_{L^\infty_{\mathrm{loc}}}(M)$ an L^∞_{loc} section of bilinear forms on TM, we define

$$|B|_g(x) = \sup_{X\in T_xM\setminus 0} \frac{|B(x,X,X)|}{|X|_g^2},$$

which yields an L^∞_{loc} function on M.

Corollary A.10. *Let $\Psi \in W^{2,\infty}_{\mathrm{loc}}(M;\mathbb{R})$, $\lambda > 0$, and define φ, f as in (85) with $G(t) = e^{\lambda t}$. Then, for any $\lambda > 0$ and any vector field X, we have almost everywhere on M*

$$\mathcal{B}_{g,\varphi,f}(X) \geq \lambda e^{\lambda\Psi}|X|_g^2\big(\lambda|\nabla_g\Psi|_g^2 - 2|\operatorname{Hess}(\Psi)|_g - \Delta_g\Psi\big),$$
$$\mathcal{E}_{g,\varphi,f} \geq \lambda e^{\lambda\Psi}|\nabla_g\varphi|_g^2\big(\lambda|\nabla_g\Psi|_g^2 - 2|\operatorname{Hess}(\Psi)|_g + \Delta_g\Psi\big).$$

Proof of Corollary A.10. With this choice of G, Lemma A.9 gives

$$\mathcal{B}_{g,\varphi,f}(X) = \lambda e^{\lambda\Psi}[2\operatorname{Hess}(\Psi)(X,X) + 2\lambda|\langle\nabla_g\Psi, X\rangle_g|^2 - \Delta_g\Psi|X|_g^2 + \lambda|\nabla_g\Psi|_g^2|X|_g^2],$$

together with

$$\mathcal{E}_{g,\varphi,f} = \lambda^3 e^{3\lambda\Psi}[2\operatorname{Hess}(\Psi)(\nabla_g\Psi,\nabla_g\Psi) + \lambda|\nabla_g\Psi|_g^4 + \Delta_g\Psi|\nabla_g\Psi|_g^2],$$

which yields the sought result. $\square$

Proof of Lemma A.9. We first have $d\varphi = G'(\Psi)d\Psi$ and $\nabla_g\varphi = G'(\Psi)\nabla_g\Psi$. We then compute the Hessian and the Laplacian as

$$\begin{aligned}\operatorname{Hess}(\varphi)(X,Y) &= \langle D_X\nabla_g\varphi, Y\rangle_g = \langle D_X(G'(\Psi)\nabla_g\Psi), Y\rangle_g\\ &= G'(\Psi)\langle D_X\nabla_g\Psi, Y\rangle_g + G''(\Psi)d\Psi(X)\langle\nabla_g\Psi, Y\rangle_g\\ &= G'(\Psi)\operatorname{Hess}(\Psi)(X,Y) + G''(\Psi)\langle\nabla_g\Psi, X\rangle_g\langle\nabla_g\Psi, Y\rangle_g,\end{aligned}$$

and

$$\Delta_g\varphi = \operatorname{div}_g(G'(\Psi)\nabla_g\Psi) = G'(\Psi)\Delta_g\Psi + G''(\Psi)|\nabla_g\Psi|_g^2.$$

In particular, we have

$$\begin{aligned}\operatorname{Hess}(\varphi)(\nabla_g\varphi,\nabla_g\varphi) &= G'(\Psi)\operatorname{Hess}(\Psi)(\nabla_g\varphi,\nabla_g\varphi) + G''(\Psi)|\langle\nabla_g\Psi,\nabla_g\varphi\rangle_g|^2\\ &= G'(\Psi)^2\big[G'(\Psi)\operatorname{Hess}(\Psi)(\nabla_g\Psi,\nabla_g\Psi) + G''(\Psi)||\nabla_g\Psi|_g^2|^2\big],\end{aligned}$$

together with

$$\Delta_g\varphi|\nabla_g\varphi|_g^2 = G'(\Psi)^2|\nabla_g\Psi|_g^2\big(G'(\Psi)\Delta_g\Psi + G''(\Psi)|\nabla_g\Psi|_g^2\big).$$

As a consequence, we obtain

$$\begin{aligned}\mathcal{B}_{g,\varphi,f}(X) = 2G'(\Psi)\operatorname{Hess}(\Psi)(X,X) + 2G''(\Psi)|\langle\nabla_g\Psi, X\rangle_g|^2&\\ + \big(-G'(\Psi)\Delta_g\Psi - G''(\Psi)|\nabla_g\Psi|_g^2 + f\big)|X|_g^2,&\end{aligned}$$

as well as

$$\mathcal{E}_{g,\varphi,f}(x) = G'(\Psi)^2\big[2G'(\Psi)\operatorname{Hess}(\Psi)(\nabla_g\Psi,\nabla_g\Psi) + 2G''(\Psi)|\nabla_g\Psi|_g^4 + \big(G'(\Psi)\Delta_g\Psi + G''(\Psi)|\nabla_g\Psi|_g^2 - f\big)|\nabla_g\Psi|_g^2\big].$$

Now, recalling the choice $f = 2G''(\Psi)|\nabla_g\Psi|_g^2$ concludes the proof of the lemma. □

Remark A.11. Note that in this proof, the choice $f = \alpha G''(\Psi)|\nabla_g\Psi|_g^2$ yields a useful lower bound only if $\alpha \in (1,3)$. See also [Le Rousseau and Lebeau 2012, Section 3.1] for a related discussion.

A4. *Uniformity with respect to the metric.* Until this point, all calculations are exact for a fixed metric. In the present section, we prove uniform estimates in a class of metrics. For this, even though the manifold with boundary M is not assumed compact, we will consider only open subsets U of M such that $\overline{U}$ is compact in M (not in $\operatorname{Int}(M)$). On the compact set $\mathcal{K}$, the spaces $W^{k,\infty}(\mathcal{K})$ are defined intrinsically, even if the associated norms may depend on the metric or the charts chosen. We fix one of these norms $\|\cdot\|_{W^{1,\infty}(\mathcal{K})}$ for functions on M, as well as for forms on M (still denoted by $\|\cdot\|_{W^{1,\infty}(\mathcal{K})}$).

Now, given a reference metric g_0 and two constants $D \geq \epsilon > 0$, we consider the class

$$\Gamma_{\epsilon,D}(\mathcal{K},g_0) = \{g \text{ metric in } \mathcal{T}^2_{W^{1,\infty}_{\text{loc}}}(M) : \|g\|_{W^{1,\infty}(\mathcal{K})} \leq D,\ \epsilon g_0 \leq g \leq Dg_0\}.$$

Lemma A.12. *Let U be an open subset of M such that $\overline{U}$ is compact (in the topology of $M \supset \partial M$) and define $\Sigma = \partial M \cap U$. Given a metric $g_0 \in \mathcal{T}^2_{W^{1,\infty}_{\text{loc}}}(M)$, $D \geq \epsilon > 0$, and a function $\Psi \in W^{2,\infty}(U)$ such that $|\nabla_{g_0}\Psi|^2_{g_0} > 0$ on $\overline{U}$, there exists $C_0 > 0$ and $\lambda > 0$ such that for any $g \in \Gamma_{\epsilon,D}(\overline{U},g_0)$, the functions $\varphi = e^{\lambda\Psi}$, $f = 2\lambda^2|\nabla_g\Psi|_g^2$ satisfy*

$$\mathcal{B}_{g,\varphi,f}(X) \geq 2C_0|X|_g^2 \quad \textit{for all vector fields } X, \tag{86}$$

$$\mathcal{E}_{g,\varphi,f} \geq 2C_0|\nabla_g\varphi|_g^2 \tag{87}$$

almost everywhere in U.

Note that the constant C_0 involved is explicitly computable in terms of D and ϵ, which we do not write for the sake of readability. Yet, if one is interested in obtaining explicit constants, the choice $G(t) = e^{\lambda t}$ of convexifying function is probably not the best one.

Proof. Denote by $g^* = (g^{ij})$ the metric on T^*M induced by g. For $g \in \Gamma_{\epsilon,D}(\overline{U},g_0)$, we have $\frac{1}{D}g_0^* \leq g^* \leq \frac{1}{\epsilon}g_0^*$. With this notation, we have

$$\frac{1}{D}|\nabla_{g_0}\Psi|^2_{g_0} = \frac{1}{D}|d\Psi|^2_{g_0^*} \leq |\nabla_g\Psi|_g^2 = |d\Psi|^2_{g^*} \leq \frac{1}{\epsilon}|d\Psi|^2_{g_0^*} = \frac{1}{\epsilon}|\nabla_{g_0}\Psi|^2_{g_0}, \tag{88}$$

where $|\omega|^2_{g^*} = \langle\omega,\omega\rangle_{g^*}$ is the cotangent squared norm. Next, using the uniform $W^{1,\infty}(U)$ bound in $\Gamma_{\epsilon,D}(\overline{U},g_0)$, we have

$$|\Delta_g\Psi| \leq C(\epsilon,D)\|\Psi\|_{W^{2,\infty}(U)}, \qquad |\operatorname{Hess}(\Psi)|_g \leq C(\epsilon,D)\|\Psi\|_{W^{2,\infty}(U)}.$$

Now, the compactness of $\overline{U}$ with the assumption yields $c_0 > 0$ such that $|\nabla_{g_0}\Psi|^2_{g_0} \geq c_0$ everywhere on $\overline{U}$. According to Corollary A.10 and the above two estimates, we obtain for any $\lambda > 0$ and any vector field X

$$\begin{aligned}\mathcal{B}_{g,\varphi,f}(X) &\geq \lambda e^{\lambda\Psi}|X|^2_g(\lambda|\nabla_g\Psi|^2_g - 2|\operatorname{Hess}(\Psi)|_g - \Delta_g\Psi),\\ &\geq \lambda e^{\lambda \min_{\mathcal{K}}\Psi}|X|^2_g\Big(\lambda\frac{c_0}{D} - 3C(\epsilon, D)\|\Psi\|_{W^{2,\infty}(U)}\Big),\end{aligned}$$

which yields (86) when taking λ large enough. Similarly, (87) follows from taking λ large enough in

$$\mathcal{E}_{g,\varphi,f} \geq \lambda e^{\lambda \min_{\mathcal{K}}\Psi}|\nabla_g\varphi|^2_g\Big(\lambda\frac{c_0}{D} - 3C(\epsilon, D)\|\Psi\|_{W^{2,\infty}(U)}\Big). \qquad \square$$

We directly deduce the following uniform Carleman estimate in the class $\Gamma_{\epsilon,D}(\overline{U}, g_0)$. We only state it with the Dirichlet boundary condition here for conciseness (the case without boundary condition can be written similarly).

Theorem A.13 (uniform Lipschitz Carleman estimate). *Let U be an open subset of M such that $\overline{U}$ is compact (in the topology of $M \supset \partial M$) and define $\Sigma = \partial M \cap U$. Given a metric $g_0 \in \mathcal{T}^2_{W^{1,\infty}_{\mathrm{loc}}}(M)$, $D \geq \epsilon > 0$, and a function $\Psi \in W^{2,\infty}(U)$ such that $|\nabla_{g_0}\Psi|^2_{g_0} > 0$ on $\overline{U}$, there exist $\lambda > 0$, $C_1 > 0$, $\tau_0 > 0$ such that for $\varphi = e^{\lambda\Psi}$ and for any $g \in \Gamma_{\epsilon,D}(\overline{U}, g_0)$, for all $\tau \geq \tau_0$ and all $v \in C^\infty_c(U)$ such that $v = 0$ on Σ we have*

$$C_1(\tau^3\|e^{\tau\varphi}v\|^2_{L^2(U)} + \tau\|e^{\tau\varphi}\nabla_g v\|^2_{L^2(U)}) \leq \|e^{\tau\varphi}\Delta_g v\|^2_{L^2(U)} + \tau\int_\Sigma e^{2\tau\varphi}\partial_\nu\varphi|\partial_\nu v|^2, \tag{89}$$

$$C_1(\tau^3\|e^{\tau\varphi}v\|^2_{L^2(U)} + \tau\|e^{\tau\varphi}\nabla_{g_0} v\|^2_{L^2(U)}) \leq \|e^{\tau\varphi}\Delta_g v\|^2_{L^2(U)} + \tau\int_\Sigma e^{2\tau\varphi}\partial_\nu\varphi|\partial_\nu v|^2. \tag{90}$$

Note that in the second inequality (90), we implicitly wrote

$$\|e^{\tau\varphi}\nabla_{g_0}v\|^2_{L^2(U)} = \int_U e^{2\tau\varphi}|\nabla_{g_0}v|^2_{g_0}\, d\operatorname{Vol}_{g_0}$$

in the left-hand side, which no longer depends on the metric g. Hence, the sole dependence on the metric g in (90) is through Δ_g and ∂_ν.

Proof. We choose $f = 2\lambda^2|\nabla_g\Psi|^2_g$ and according to Lemma A.12, the bounds (70)–(71) with constant C_0 are satisfied for λ large enough uniformly in the class $g \in \Gamma_{\epsilon,D}(\overline{U}, g_0)$. According to Theorem A.5, this implies (89) with $C_1 = \frac{1}{3}C_0c(\varphi)$ for all $\tau \geq \tau_0(g)$, with

$$\tau_0(g) = \frac{c(\varphi)}{C_0}\big(\|f - \Delta_g\varphi\|^2_{L^\infty(U)} + \tfrac{1}{2}\|\nabla_g f\|_{L^\infty(U)}\big),$$

with $c(\varphi) = \max\{1, (\min_{\overline{U}}|\nabla_g\varphi|^2_g)^{-1}\}$. Now, (88) implies that

$$\max\Big\{1, \epsilon\big(\min_{\overline{U}}|e^{\lambda\Psi}\nabla_{g_0}\Psi|^2_{g_0}\big)^{-1}\Big\} \leq c(\varphi) \leq \max\Big\{1, D\big(\min_{\overline{U}}|e^{\lambda\Psi}\nabla_{g_0}\Psi|^2_{g_0}\big)^{-1}\Big\}$$

uniformly for $g \in \Gamma_{\epsilon,D}(\overline{U}, g_0)$, and, similarly

$$\tau_0(g) \leq C(\epsilon, D, \Psi, g_0)$$

uniformly for $g \in \Gamma_{\epsilon,D}(\overline{U}, g_0)$. This concludes the proof of (89). The proof of (90) follows again from (88) (applied to v) and the fact that $d\,\mathrm{Vol}_{g_0} \le \epsilon^{-d/2} d\,\mathrm{Vol}_g$ (recall that $d = \dim M$). □

Note that for the application that we have in Proposition A.14 below, it is sufficient to have some stability results in the following sense: if an interpolation inequality or a Carleman inequality is true for some metric g_0, it is still true for any metric in a suitable neighborhood. This is of course a byproduct of our results.

A5. *Uniform interpolation estimate at the boundary.* In this section, we consider a very particular case of the above Carleman estimate to prove a local interpolation estimate in a neighborhood of a boundary point for metrics g in the neighborhood of the constant flat metric. The manifold M considered is $\mathbb{R}^{n+1}_+ = \mathbb{R}^n \times \mathbb{R}_+$ (that is, $d = n+1$) and the reference metric is $g_0 = \mathrm{Id}$. The proof follows [Lebeau and Robbiano 1995; Lebeau and Zuazua 1998; Le Rousseau and Lebeau 2012]. Note that the above sections prove much more than what is actually needed for this argument.

Below, we set $B_r = B(0,r) \subset \mathbb{R}^{n+1}$ and $B_r^+ = B(0,r) \cap \mathbb{R}^{n+1}_+$.

Proposition A.14. *There exists $\epsilon > 0$, $C > 0$ and $\alpha_0 \in (0,1)$ so that for any metric $\mathfrak{g} \in \Gamma_{\epsilon,D}(\overline{B}_{\mathbb{R}^n}(0,2), \mathrm{Id})$ we have*

$$\|v\|_{H^1(B_1^+)} \le C\big(\|(-\partial_s^2 - \Delta_{\mathfrak{g}})v\|_{L^2(B_2^+)} + \|\partial_s v|_{s=0}\|_{L^2(B_2\cap\{0\}\times\mathbb{R}^n)}\big)^{\alpha_0} \|v\|^{1-\alpha_0}_{H^1(B_2^+)}$$

for any $v \in H^2(B_2^+)$ such that $v|_{s=0} = 0$.

Proof. In the proof, we shall denote (with a slight abuse of notation) by $x = (s,x) \in \mathbb{R}_+ \times \mathbb{R}^n$ the overall variable and recall that all balls are centered at zero. We choose a point $x^a = (-a, 0, \dots, 0) \notin \overline{\mathbb{R}}^{n+1}_+$. We define the weight function $\Psi(x) = -|x - x^a|$, which is smooth and satisfies $\Psi < 0$ and $d\Psi \neq 0$ in $\overline{B}_2^+$.

For a sufficiently small, there exist $0 < \rho_1 < \rho_2$ such that we have

$$\overline{B}_1^+ \subset W_1 \subset \overline{W}_1 \subset W_2 \subset \overline{W}_2 \subset B_2^+, \quad \text{with } W_j = \{\Psi > -\rho_j\} \cap \overline{\mathbb{R}}^{n+1}_+,\ j = 1,2. \tag{91}$$

As a consequence of Theorem A.13, there exist $\lambda > 0$, $C_1 > 0$, $\tau_0 > 0$ such that for $\varphi = e^{\lambda\Psi}$ and for any $g = \mathrm{Id} \otimes \mathfrak{g} \in \Gamma_{\epsilon,D}(\overline{B}_2^+, \mathrm{Id})$, for all $\tau \ge \tau_0$ and all $u \in C_c^\infty(B_2^+)$ such that $u = 0$ on $\{s = 0\}$, we have

$$C_1(\tau^3\|e^{\tau\varphi}u\|^2_{L^2(B_2^+)} + \tau\|e^{\tau\varphi}\nabla u\|^2_{L^2(B_2^+)}) \le \|e^{\tau\varphi}\Delta_g u\|^2_{L^2(B_2^+)} + \tau\int_{\{s=0\}} e^{2\tau\varphi}\partial_\nu\varphi|\partial_\nu u|^2. \tag{92}$$

Here, the ball, the gradient and the volume element are taken with respect to the Euclidean metric. Moreover, the normal vector field ∂_ν is that associated to the metric $g = \mathrm{Id} \otimes \mathfrak{g}$, and hence $\partial_\nu = -\partial_s$ (and does not depend on $\mathfrak{g}$). The sole dependence on the metric in (92) is thus in $\Delta_g = \partial_s^2 + \Delta_{\mathfrak{g}}$.

Note that level sets of φ are those of Ψ, i.e., pieces of spheres. Note also that we have $\varphi \le \varphi(0)$ on B_2^+ and define

$$\varphi(0) > \varphi_1 := \min_{\overline{B}_1^+}\varphi > \varphi_1' := \min_{\overline{W}_1}\varphi = e^{-\lambda\rho_1} = \max_{\overline{W}_2\setminus W_1}\varphi,$$

which only depend the geometric setting (not on the metric).

We let $\chi \in C_c^\infty(\mathbb{R}^{n+1})$ such that, with W_j as in (91), $\chi = 1$ on W_1 and $\chi = 0$ on $B_2^+ \setminus W_2$, and apply (92) to $u = \chi v \in C_c^\infty(B_2^+)$ with $v \in C^\infty(B_2^+)$ satisfying $v|_{s=0} = 0$. We have $\partial_\nu u|_{s=0} = -\chi|_{s=0}\partial_s v|_{s=0}$ since $v|_{s=0} = 0$ and hence

$$\int_{\{s=0\}} e^{2\tau\varphi}\partial_\nu\varphi|\partial_\nu u|^2 \le Ce^{2\tau\varphi(0)}\|\chi|_{s=0}\partial_s v|_{s=0}\|^2_{L^2(W_2\cap\{s=0\})}.$$

Using that $\chi = 1$ on $W_1 \supset B_1^+$, we have

$$\begin{aligned}\tau^3\|e^{\tau\varphi}u\|^2_{L^2(B_2^+)} + \tau\|e^{\tau\varphi}\nabla u\|^2_{L^2(B_2^+)} &\ge \tau^3\|e^{\tau\varphi}u\|^2_{L^2(B_1^+)} + \tau\|e^{\tau\varphi}\nabla u\|^2_{L^2(B_1^+)}\\ &\ge \tau e^{2\tau\varphi_1}\|v\|^2_{H^1(B_1^+)}.\end{aligned}$$

Finally, we have $\Delta_g \chi v = \chi\Delta_g v + [\Delta_g, \chi]v$, where $[\Delta_g, \chi]$ (recall $\Delta_g = \partial_s^2 + \Delta_{\mathfrak{g}}$) is a first-order differential operator with L^∞ coefficients supported in $\overline{W}_2 \setminus W_1$, and such that $\|[\Delta_g, \chi]\|_{H^1\to L^2} \le CD$ on that set uniformly for $\mathfrak{g} \in \Gamma_{\epsilon,D}(\bar{B}_{\mathbb{R}^n}(0,2), \mathrm{Id})$. Moreover, we have $\varphi \le \varphi_1'$ on $\overline{W}_2 \setminus W_1$. Thus, we have

$$\begin{aligned}\|e^{\tau\varphi}\Delta_g u\|^2_{L^2(B_2^+)} &\le \|e^{\tau\varphi}\chi\Delta_g v\|^2_{L^2(B_2^+)} + \|e^{\tau\varphi}[\Delta_g,\chi]v\|^2_{L^2(B_2^+)}\\ &\le e^{2\tau\varphi(0)}\|\Delta_g v\|^2_{L^2(B_2^+)} + CDe^{2\tau\varphi_1'}\|v\|^2_{H^1(B_2^+)}.\end{aligned}$$

Combining the last three estimates with (92), we find that there are $C, \tau_0 > 0$ such that for all $g = \mathrm{Id} \otimes \mathfrak{g} \in \Gamma_{\epsilon,D}(\bar{B}_2^+, \mathrm{Id})$, for all $\tau \ge \tau_0$ and all $v \in C^\infty(B_2^+)$ such that $v = 0$ on $\{s = 0\}$, we have

$$e^{2\tau\varphi_1}\|v\|^2_{H^1(B_1^+)} \le Ce^{2\tau\varphi(0)}(\|\partial_s v|_{s=0}\|^2_{L^2(B_2^+\cap\{s=0\})} + \|\Delta_g v\|^2_{L^2(B_2^+)}) + Ce^{2\tau\varphi_1'}\|v\|^2_{H^1(B_2^+)}.$$

Recalling that $\varphi(0) > \varphi_1 > \varphi_1'$ and after an optimization in the parameter τ, see [Robbiano 1995], this yields the result of the lemma. ☐

A6. *A uniform Lebeau–Robbiano spectral inequality.* In this section, we give a proof of Theorem 1.16. For this, we follow the strategy of proof of [Boyer et al. 2010, Section 2] with our uniform Carleman estimates (Theorem A.13). The original proof of [Lebeau and Robbiano 1995] also works (see the above Section 4) but is less straightforward in the present setting. We recall that $\mathcal{M}$ is the ambient compact manifold with boundary $\partial\mathcal{M}$, and set $M = [0, S_0] \times \mathcal{M}$, having piecewise C^1 and graph-Lipschitz boundary $\partial M = \{0\} \times \mathcal{M} \cup \{S_0\} \times \mathcal{M} \cup [0, S_0] \times \partial\mathcal{M}$. We denote by (s, x) the variable in M. The metric is $g = \mathrm{Id} \otimes \mathfrak{g}$. Note finally that $\partial_\nu = \partial_{\nu_x}$ on $[0, S_0] \times \partial\mathcal{M}$, where ν_x denotes here the outward unit normal to $\mathcal{M}$ at $\partial\mathcal{M}$, that $\partial_\nu = \partial_s$ on $\{S_0\} \times \mathcal{M}$, and that $\partial_\nu = -\partial_s$ on $\{0\} \times \mathcal{M}$.

Lemma A.15. *Let $\mathfrak{g}_0 \in \mathcal{T}^2_{W^{1,\infty}(\mathcal{M})}$ be a metric on $\mathcal{M}$ and write $g_0 = \mathrm{Id} \otimes \mathfrak{g}_0$. Then, there exists a function $\psi \in C^2(M; \mathbb{R})$ and $c > 0$ such that*

$$\begin{aligned} |\nabla_{g_0}\psi|_{g_0} &\ge c \quad \text{in } M, & \partial_{\nu_x}\psi &< 0 \quad \text{on } [0, S_0] \times \partial\mathcal{M},\\ \partial_s\psi &\ge c \quad \text{on } \{0\} \times (\mathcal{M} \setminus \omega), & \nabla_{\mathfrak{g}_0}\psi = 0 \text{ and } \partial_s\psi &\le -c \quad \text{on } \{S_0\} \times \mathcal{M}.\end{aligned}$$

We refer to [Boyer et al. 2010, Appendix C] for the proof of this result in the case $\mathcal{M} \subset \mathbb{R}^n$, and [Le Rousseau and Robbiano 2011, Section 5] for the adaptation to the case of a manifold. With this weight function in hand, we obtain the following global uniform Carleman estimate.

Theorem A.16 (global uniform Lipschitz Carleman estimate). *Given a metric $\mathfrak{g}_0 \in \mathcal{T}^2_{W^{1,\infty}(\mathcal{M})}$, and Ψ as in Lemma A.15, for any $D \geq \epsilon > 0$, there exist $\lambda > 0, C_1 > 0, \tau_0 > 0$ such that for $\varphi = e^{\lambda\Psi}$ and for any $\mathfrak{g} \in \Gamma_{\epsilon,D}(\mathcal{M}, \mathfrak{g}_0)$, for all $\tau \geq \tau_0$ and all $v \in H^2([0, S_0] \times \mathcal{M})$ such that $v = 0$ on $\{0\} \times \mathcal{M} \cup [0, S_0] \times \partial\mathcal{M}$, we have, with $M = [0, S_0] \times \mathcal{M}$ and $g = \mathrm{Id} \otimes \mathfrak{g}$,*

$$\begin{aligned}\tau^3\|e^{\tau\varphi}v\|^2_{L^2(M)} &+ \tau\|e^{\tau\varphi}\nabla_g v\|^2_{L^2(M)} \\ &+\tau e^{2\tau\varphi(S_0)}\left(\int_{\mathcal{M}}|\partial_s v(S_0,\cdot)|^2 + \tau^2\int_{\mathcal{M}}|v(S_0,\cdot)|^2\right) + \tau\int_{\mathcal{M}\setminus\omega} e^{2\tau\varphi(0,\cdot)}|\partial_s v(0,\cdot)|^2 \\ &\leq C\left(\|e^{\tau\varphi}(-\partial_s^2 - \Delta_{\mathfrak{g}})v\|^2_{L^2(M)} + \tau\int_\omega e^{2\tau\varphi(0,\cdot)}|\partial_s v(0,\cdot)|^2 + \tau e^{2\tau\varphi(S_0)}\int_{\mathcal{M}}|\nabla_{\mathfrak{g}} v(S_0,\cdot)|^2_{\mathfrak{g}}\right). \quad (93)\end{aligned}$$

Proof. We use the Carleman estimates (73)–(74) together with Remark A.6 and Lemma A.12 for the uniformity in the metric. More precisely, on the boundary $\{0\} \times \mathcal{M} \cup [0, S_0] \times \partial\mathcal{M}$, the Dirichlet boundary condition is prescribed and the only boundary term is $+\tau\int_\Sigma e^{2\tau\varphi}\partial_\nu\varphi|\partial_\nu v|^2$, according to (73). That $\partial_\nu\varphi \leq -c < 0$ on $\{0\} \times (\mathcal{M} \setminus \omega) \cup [0, S_0] \times \partial\mathcal{M}$ implies that the associated integral is dominated on that set, whereas the only observation term on that part of the boundary is $-\tau\int_\omega e^{2\tau\varphi(0,\cdot)}\partial_s\varphi(0,\cdot)|\partial_s v(0,\cdot)|^2$.

Now, on the part $\{S_0\} \times \mathcal{M}$ of the boundary, we have the observation term

$$\tau\int_\Sigma e^{2\tau\varphi}|\nabla_T v|^2_g = \tau e^{2\tau\varphi(S_0)}\int_{\mathcal{M}}|\nabla_{\mathfrak{g}} v(S_0,\cdot)|^2_{\mathfrak{g}}.$$

On the other side of the inequality, we have the two observed terms

$$\frac{\tau}{8}\frac{m(\varphi)^3}{M(\varphi)^2}\int_\Sigma e^{2\tau\varphi}|\partial_\nu v|^2 + \tau^3\frac{m(\varphi)^3}{4}\int_\Sigma |v|^2 \geq C\tau e^{2\tau\varphi(S_0)}\left(\int_{\mathcal{M}}|\partial_s v(S_0,\cdot)|^2 + \tau^2\int_{\mathcal{M}}|v(S_0,\cdot)|^2\right).$$

Finally, we are left with the existence of $C, \tau_0 > 0$ such that for all $v \in H^2([0, S_0]\times\mathcal{M})$, $\mathfrak{g} \in \Gamma_{\epsilon,D}(\mathcal{M}, \mathfrak{g}_0)$, and $\tau \geq \tau_0$, we have (93). □

From Theorem A.16, we now deduce a proof of Theorem 1.16, following closely (and carefully) [Boyer et al. 2010, Proof of Theorem 1.1].

Proof of Theorem 1.16. Given $w \in E^{\mathfrak{g}}_{\leq\lambda}$ take the function

$$v(s) = \frac{\sinh(s\sqrt{-\Delta_{\mathfrak{g}}})}{\sqrt{-\Delta_{\mathfrak{g}}}}\Pi^{\mathfrak{g}}_+ w + s\Pi^{\mathfrak{g}}_0 w,$$

where $\Delta_{\mathfrak{g}}$ is the Dirichlet Laplacian, $\Pi^{\mathfrak{g}}_0$ the orthogonal projector on $\ker(\Delta_{\mathfrak{g}})$ (in the case $\partial\mathcal{M} = \varnothing$, otherwise $\Pi^{\mathfrak{g}}_0 = 0$) and $\Pi^{\mathfrak{g}}_+ = \mathrm{Id} - \Pi^{\mathfrak{g}}_0$, that is, v is the unique solution to

$$(-\partial_s^2 - \Delta_{\mathfrak{g}})v = 0, \quad v|_{(0,S_0)\times\partial M} = 0, \quad (v, \partial_s v)|_{s=0} = (0, w).$$

We may now apply (93), keeping only the penultimate term in the left-hand side:

$$e^{2\tau\varphi(S_0)}\tau^3\int_{\mathcal{M}}|v(S_0,\cdot)|^2 \le C\left(\tau\int_\omega e^{2\tau\varphi(0,\cdot)}|\partial_s v(0,\cdot)|^2 + \tau e^{2\tau\varphi(S_0)}\int_{\mathcal{M}}|\nabla_{\mathfrak{g}} v(S_0,\cdot)|_{\mathfrak{g}}^2\right).$$

Now, we have

$$\int_\omega e^{2\tau\varphi(0,\cdot)}|\partial_s v(0,\cdot)|^2 \le e^{2\tau\sup_{\mathcal{M}}\varphi(0,\cdot)}\|w\|^2_{L^2(\omega)},$$

together with (using an integration by parts, together with $w \in E^{\mathfrak{g}}_{\le\lambda}$),

$$\int_{\mathcal{M}}|\nabla_{\mathfrak{g}} v(S_0,\cdot)|_{\mathfrak{g}}^2 = (-\Delta_{\mathfrak{g}} v(S_0,\cdot), v(S_0,\cdot))_{L^2(\mathcal{M},d\,\mathrm{Vol}_{\mathfrak{g}})} \le \lambda(v(S_0,\cdot), v(S_0,\cdot))_{L^2(\mathcal{M},d\,\mathrm{Vol}_{\mathfrak{g}})}.$$

The last three inequalities imply for all $\tau \ge \tau_0$

$$\tau^2\|v(S_0,\cdot)\|^2_{L^2} \le C(e^{2\tau(\sup_{\mathcal{M}}\varphi(0,\cdot)-\varphi(S_0))}\|w\|^2_{L^2(\omega)} + \lambda\|v(S_0,\cdot)\|^2_{L^2}),$$

and hence, when choosing $\tau = \max\{2\sqrt{\lambda}, \tau_0\}$, we obtain

$$\|v(S_0,\cdot)\|^2_{L^2} \le Ce^{4\sqrt{\lambda}(\sup_{\mathcal{M}}\varphi(0,\cdot)-\varphi(S_0))}\|w\|^2_{L^2(\omega)}.$$

Finally, using $\sinh(S_0\ell)/\ell \ge S_0$ and the orthogonality of the eigenfunctions, we also have

$$\int_{\mathcal{M}}|v(S_0,\cdot)|^2 = \left(\frac{\sinh^2(S_0\sqrt{-\Delta_{\mathfrak{g}}})}{-\Delta_{\mathfrak{g}}}\Pi^{\mathfrak{g}}_+ w, \Pi^{\mathfrak{g}}_+ w\right)_{L^2(\mathcal{M},d\,\mathrm{Vol}_{\mathfrak{g}})} + S_0^2\|\Pi^{\mathfrak{g}}_0 w\|^2_{L^2(\mathcal{M},d\,\mathrm{Vol}_{\mathfrak{g}})} \ge S_0^2\|w\|^2_{L^2}.$$

The last two inequalities conclude the proof of the theorem. □

Appendix B: Local behavior of vanishing functions

In this appendix, we give an explicit link between the different definitions of the vanishing rate of a function.

Lemma B.1. *Let $f \in C^\infty(B_{\mathbb{R}^n}(0,1))$ and assume that there are $C, D > 0$ such that we have uniformly for $0 < r < 1$ the estimate*

$$\|f\|_{L^2(B_{\mathbb{R}^n}(0,r))} \le Cr^D. \tag{94}$$

Then, we have $\partial^\alpha f(0) = 0$ for all $|\alpha| < D - \frac{n}{2}$.

Conversely, assume $f \in C^\infty(B_{\mathbb{R}^n}(0,1))$ satisfies $\partial^\alpha f(0) = 0$ for all $|\alpha| \le k$, $k \in \mathbb{N}$. Then we have (94) with $D = k + 1 + \frac{n}{2}$.

Proof. Define $k = \inf\{|\alpha| : \partial^\alpha f(0) \ne 0\} \in \mathbb{N} \cup \{\infty\}$ and, in the case $k < \infty$, write the Taylor expansion of f at zero as $f = P_k + R_k$ with P_k homogeneous of degree k and $|R_k| \le C|x|^{k+1}$. We obtain

$$\|P_k\|_{L^2(B(0,r))} = r^{\frac{n}{2}+k}\|P_k\|_{L^2(B(0,1))} \quad \text{and} \quad \|R_k\|_{L^2(B(0,r))} \le Cr^{\frac{n}{2}+k+1}.$$

Using (94) for r small implies $\frac{n}{2} + k \ge D$ and thus $\partial^\alpha f(0) = 0$ for all $|\alpha| < D - \frac{n}{2}$.

Conversely, if $\partial^\alpha f(0) = 0$ for all $|\alpha| \le k$, then we have $|f(x)| \le C|x|^{k+1}$ and thus

$$\|f\|_{L^2(B_{\mathbb{R}^n}(0,r))} \le C\||x|^{k+1}\|_{L^2(B_{\mathbb{R}^n}(0,r))} \le Cr^{k+1+\frac{n}{2}}.$$ □

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