UNIFORM CONTROLLABILITY OF SCALAR CONSERVATION LAWS IN THE VANISHING VISCOSITY LIMIT[*]

MATTHIEU LÉAUTAUD[†]

Abstract. We deal with viscous perturbations of scalar conservation laws on a bounded interval with a general flux function f and a small dissipation coefficient ε. Acting on this system on both endpoints of the interval, we prove global exact controllability to constant states with nonzero speed. More precisely, we construct boundary controls so that the solution is driven to the targeted constant state, and we moreover require these controls to be uniformly bounded as $\varepsilon \to 0^+$ in an appropriate space. For general (nonconvex) flux functions this can be done for sufficiently large time, and for convex fluxes f, we have a precise estimate on the minimal time needed to control.

Key words. controllability, scalar conservation laws, vanishing viscosity limit

AMS subject classifications. 93B05, 93C10, 93C20, 35Q93, 35L65, 35B25

DOI. 10.1137/100803043

1. Introduction.

1.1. Motivation and main results.
We are concerned with the controllability of the nonlinear parabolic equation

$$(1) \qquad u_t + [f(u)]_x - \varepsilon u_{xx} = 0 \quad \text{in } (0,T) \times (0,L),$$

where T is a positive time, L a positive length, and ε a positive viscosity coefficient. On both endpoints of the interval, we act on the system through the boundary conditions

$$(2) \qquad u|_{x=0} = g_0(t) \quad \text{in } (0,T), \qquad u|_{x=L} = g_L(t) \quad \text{in } (0,T).$$

The problem of exact controllability is the following: given an initial datum

$$(3) \qquad u|_{t=0} = u_0 \quad \text{in } (0,L),$$

a time $T > 0$, and a prescribed state u_T, is it possible to find control functions $g_0 = g_0^\varepsilon$ and $g_L = g_L^\varepsilon$ so that the associated solution u of the system (1), (2), (3) is steered to u_T in time T? The aim of this paper is to prove such a controllability result uniformly with respect to the viscosity coefficient ε in a sufficiently small range. That is, is it possible to find such control functions $g_0 = g_0^\varepsilon$, $g_L = g_L^\varepsilon$ whose norms in a suitable Banach space remain bounded as $\varepsilon \to 0^+$? More precisely, we shall only consider the uniform controllability of constant targets $u_T = M \in \mathbb{R}$. The relevance of this assumption is discussed in Remark 1.5.

Uniform controllability problems for singular perturbations of partial differential equations have already been considered in several works, beginning with [26, Chapter 3]. In the context of a transport/heat equation (i.e., $f(u) = Vu$ for some constant $V \in \mathbb{R}$) in a vanishing viscosity limit, this study was initiated by Coron and Guerrero

[†]Laboratoire Jacques-Louis Lions, Université Pierre et Marie Curie Paris 6, UMR 7598, Paris, F-75005 France; CNRS, UMR 7598 LJLL, Paris, F-75005 France (leautaud@ann.jussieu.fr), and Laboratoire POEMS, INRIA Paris-Rocquencourt/ENSTA, CNRS UMR 2706, Paris, France.

in [8], where the authors make a conjecture on the minimal time needed to achieve uniform controllability. Then, the estimates on this minimal time are improved in [19] with a complex analytic method. The result of [8] was also generalized in several space dimensions and for nonconstant transport speed in [21]. Such uniform control properties in singular limits are also addressed for vanishing dispersion in [17] and for vanishing dispersion and viscosity in [18]. All these articles deal with singular perturbations of linear transport equations.

Concerning nonlinear control problems in vanishing viscosity, the only result to the best of our knowledge has been stated by Glass and Guerrero [16] for the Burgers equation, i.e., in the case $f(u) = \frac{u^2}{2}$. Theorems 1.1, 1.2, and 1.3 generalize this uniform controllability theorem for a large range of flux functions.

Our first result is concerned with convex flux functions f.

THEOREM 1.1. *There exists $\alpha_0 \geq 1$ satisfying the following property. For every flux function satisfying $f \in W^{2,\infty}_{loc}(\mathbb{R})$, $f'' \geq 0$ a.e., and*
(A+)

$$\liminf_{A \to +\infty} A^{-\gamma} f''(A) > 0; \quad \limsup_{A \to +\infty} e^{-A^{2\gamma+1-\delta}} f''(A) < +\infty \text{ for some } \gamma > -\frac{1}{2}, \delta > 0,$$

(A−) $\left(resp., \liminf_{A \to -\infty} |A|^{-\gamma} f''(A) > 0; \right.$

$$\left. \limsup_{A \to -\infty} e^{-|A|^{2\gamma+1-\delta}} f''(A) < +\infty \text{ for some } \gamma > -\frac{1}{2}, \delta > 0 \right),$$

there exists a constant $C > 0$ such that for all M satisfying $f'(M) > 0$ (resp., $f'(M) < 0$), there exists $\varepsilon_0 = \varepsilon_0(M) > 0$ such that for any $u_0 \in L^\infty(0, L)$, any time $T > \alpha_0 \frac{L}{|f'(M)|}$, and any $\varepsilon \in (0, \varepsilon_0)$, there exist two control functions g_0^ε and g_L^ε satisfying

$$\|g_0^\varepsilon\|_{L^\infty(0,T)} + \|g_L^\varepsilon\|_{L^\infty(0,T)} \leq C \left(\|u_0\|_{L^\infty(0,L)} + |M| \right),$$

such that the solution of (1), (2), and (3) associated to g_0^ε and g_L^ε satisfies $u|_{t=T} = M$ on $(0, L)$.

This theorem is the direct generalization of [16, Theorem 1.1]. Note that we shall see that one can take $\alpha_0 = 6$, or, as found numerically, $\alpha_0 = 5.3$. Even in the case of a Burgers equation, this improves the minimal control time found in [16] (which is $\alpha_0 = 9$ or, as found numerically, $\alpha_0 = 6.3$). Having a minimal controllability time is natural here since the inviscid system (for $\varepsilon = 0$) has a finite propagation speed (see also Remark 1.4 below).

In this theorem, the second part of assumptions (A+) and (A−) is due to a technical argument and does not seem to be necessary (see also Remarks 3.2 and 3.9 below). Under more natural (and weaker) assumptions, we prove the following (weaker) result for convex flux functions.

THEOREM 1.2. *There exists $\alpha_0 \geq 1$ satisfying the following property. For every flux function satisfying $f \in W^{2,\infty}_{loc}(\mathbb{R})$, $f'' \geq 0$ a.e., and $f'(u) \to +\infty$ as $u \to +\infty$ (resp., $f'(u) \to -\infty$ as $u \to -\infty$), there exists a constant $C > 0$ such that for all $R_0 > 0$ and all M satisfying $f'(M) > 0$ (resp., $f'(M) < 0$), there exists $\varepsilon_0 = \varepsilon_0(R_0, M) > 0$ such that for any $u_0 \in L^\infty(0, L)$ with $\|u_0\|_{L^\infty(0,L)} \leq R_0$, any time $T > \alpha_0 \frac{L}{|f'(M)|}$, and any $\varepsilon \in (0, \varepsilon_0)$, there exist two control functions g_0^ε and g_L^ε satisfying*

$$\|g_0^\varepsilon\|_{L^\infty(0,T)} + \|g_L^\varepsilon\|_{L^\infty(0,T)} \leq C \left(R_0 + |M| \right),$$

such that the solution of (1), (2), and (3) associated to g_0^ε and g_L^ε satisfies $u|_{t=T} = M$ on $(0, L)$.

The only difference from the result of Theorem 1.1 is that here ε_0 also depends on the norm of the initial datum u_0. The question of whether the result of Theorem 1.1 still holds only with the assumptions of Theorem 1.2 remains open. In particular, Theorem 1.1 does not apply for flux functions satisfying $f(u) \sim_{u \to +\infty} u^\rho$ with $1 < \rho \le \frac{3}{2}$, whereas Theorem 1.2 does.

We also prove a result for nonconvex flux functions, that is, for general nonlinear transport equations. In this setting, however, we do not estimate the time needed to control, and our result is less precise. Before stating it, let us define $\sigma(A, B)$ as the shock speed between two constant states A and B, given by the Rankine–Hugoniot condition, i.e., the slope of the chord of f between A and B,

$$(4) \qquad \sigma(A, B) = \frac{f(B) - f(A)}{B - A} \text{ if } A \neq B, \quad \text{and} \quad \sigma(A, A) = f'(A).$$

On the interval $(P, N) \subset \mathbb{R}$, the strict Oleinik admissibility conditions for the flux function f read

$$\text{(SOC+)} \qquad \qquad \sigma(P, N) < \sigma(A, N) \quad \text{ for all } A \in (P, N),$$

meaning that, on the interval (P, N), the graph of f is below the chord between P and N, or

$$\text{(SOC−)} \qquad \qquad \sigma(P, N) > \sigma(A, N) \quad \text{ for all } A \in (P, N),$$

meaning that, on the interval (P, N), the graph of f is above the chord between P and N. Note that (SOC+) (resp., (SOC−)) is equivalent to having $\sigma(A, P) < \sigma(P, N)$ (resp., $\sigma(A, P) > \sigma(P, N)$) for all $A \in (P, N)$. Conditions (SOC+) and (SOC−) ensure the existence of an admissible shock wave between P and N (see [12, section 8.6]).

We can now state the result concerning nonconvex flux functions.

THEOREM 1.3. *Suppose that $f \in \mathcal{C}^2(\mathbb{R})$ and $u_0 \in L^\infty(0, L)$ satisfy the following two conditions:*
   (i) *$(f'')^{-1}(\{0\}) \cap K$ has a finite number of connected components for any compact interval $K \subset \mathbb{R}$.*
   (ii) *There exists an open bounded interval $I \subset \mathbb{R}$ such that $[\text{ess inf } u_0, \text{ess sup } u_0] \subset I$ and f satisfies* (SOC+) *or* (SOC−) *on I.*
Then, for all M satisfying $f'(M) \neq 0$, there exist $C_0 > 0$, $T_0 > 0$, and $\varepsilon_0 > 0$ (depending only on I and M) such that for any time $T > T_0$ and any $\varepsilon \in (0, \varepsilon_0)$, there exist two control functions g_0^ε and g_L^ε satisfying

$$\|g_0^\varepsilon\|_{L^\infty(0,T)} + \|g_L^\varepsilon\|_{L^\infty(0,T)} \le C_0,$$

such that the solution of (1), (2), *and* (3) *associated to g_0^ε and g_L^ε satisfies $u|_{t=T} = M$ on $(0, L)$.*

Note that condition (i) actually means that f'' vanishes on a finite union of points and (closed) intervals on each bounded subset of $\mathbb{R}$. This is generally satisfied except for some pathological examples (for instance, $f(u) = u^5 \cos(\frac{1}{u}) + u$ if $u \neq 0$, $f(0) = 0$, and the target is 0). More precisely, this condition is generically satisfied in $\mathcal{C}^3(\mathbb{R})$ since, for any compact $K \subset \mathbb{R}$, the set

$$A = \{f \in \mathcal{C}^3(K; \mathbb{R}), (f''(x), f'''(x)) \neq (0, 0) \text{ for all } x \in K\}$$

is open and dense in $\mathcal{C}^3(K; \mathbb{R})$. However, note that this genericity property does not hold in the space $\mathcal{C}^2(\mathbb{R})$ (see [3]).

Condition (ii) concerning the initial datum means that there exists an admissible shock wave allowing us to get rid of the initial condition; condition (ii) is more restrictive. For instance, it is not satisfied if $f(u) = \cos(u)$ and $(\text{ess inf } u_0, \text{ess sup } u_0)$ contains an interval of the form $[k\pi, (k+1)\pi]$ for some $k \in \mathbb{Z}$. Yet, it is satisfied in a large range of situations, including, for any $u_0 \in L^\infty(0, L)$, the case where f satisfies $\lim_{u \to +\infty} f'(u) = \pm\infty$.

Since we are interested in the properties of uniform controllability as $\varepsilon \to 0^+$, it seems natural to refer to the results that are known both for the viscous problem (for $\varepsilon > 0$, fixed) and for the inviscid problem ($\varepsilon = 0$).

First, the controllability of the viscous equation (1) for fixed $\varepsilon > 0$ has mainly been considered for the Burgers equation. Two different types of control results have been proved. The local exact controllability to trajectories for this equation has been established in [15]. In this work, the authors also prove that the local exact controllability does not hold as long as one controls in a subinterval of $(0, L)$, which is equivalent to controlling at one endpoint. Concerning the global exact controllability for the viscous Burgers equation, it is proved in [20] that it does not hold even if the control is acting on both sides of the domain. However, in [11] the author proves a global controllability result from 0 to constant states. More precisely, he states that for $u_0 = 0$ and for any $T > 0$, one can drive the solution of (1), (2), (3) to any constant M provided that $|M|$ is sufficiently large with respect to T. This result is improved (as a corollary of the uniform controllability result) in [16], allowing any $u_0 \in L^\infty(0, L)$ and giving a precise condition on the target M and the minimal control time. Finally, adding a third control globally distributed on $(0, L)$, and independent on x, the author of [5] establishes the global controllability of the viscous Burgers equation for any $T > 0$. Note that this last result is proved by using both controllability properties of the nonviscous equation (see below) and a local result.

Second, concerning the inviscid problem

$$(5) \qquad\qquad u_t + [f(u)]_x = 0,$$

and in the context of entropy solutions, controllability questions have been addressed by Ancona and Marson for general strictly convex flux functions f in [1]. In this work the controllability problem is posed in the half real line with null initial condition, and the set of attainable states is completely described. For the problem on a bounded interval and with a general initial datum, the controllability of the nonviscous Burgers equation $(f(u) = \frac{u^2}{2})$ was studied in [22], where some conditions are given on the final state in order to ensure this property. Finally, in the context of classical solutions, and adding a third control globally distributed on $(0, L)$, independent on x, Chapouly [4, 5] proved global controllability in arbitrary small time for the nonviscous Burgers equation. This result was recently generalized in [28] to convex flux functions and entropy solutions.

We recall that for conservation laws such as (5), classical solutions starting out from smooth initial data generally develop discontinuities in finite time. As a consequence, only weak solutions may exist for large times. In the context of weak solutions, however, uniqueness is lost. To overcome the obstacle of nonuniqueness, restrictions need to be imposed to select the physically relevant weak solution. One criterion for such a selection is to require that the admissible solution satisfy an entropy condition, which reads as follows (see [12, Chapter 6], for instance): for any smooth convex function $\eta : \mathbb{R} \to \mathbb{R}$ and associated entropy flux $q(u) = \int^u \eta'(\omega) f'(\omega) d\omega$, the following

holds in the sense of measures:

$$\eta(u)_t + q(u)_x \leq 0.$$

Another selection principle is to require that the admissible solution be the limit of a family of solutions of equations containing a diffusive term, such as the one considered here. One can prove (see [25] or [12, Chapter 6]) that both definitions coincide, so that entropy solutions are those that can be obtained by vanishing viscosity. One can summarize the situation by saying that the viscosity has disappeared from the equation and is only effective for the selection of admissible discontinuities. The Cauchy problem together with the convergence of vanishing viscosity approximations to the entropy solutions of a general scalar conservation law were studied in the seminal work of Kružkov [25].

It is therefore very natural, when considering control problems for conservation laws, to study the cost of the viscosity, that is, to determine if known controllability properties for the hyperbolic equation are still valid for the model with small viscosity, and how the size of the control evolves as the viscosity approaches 0.

Another important motivation for studying the singular limit in control problems is the search for controllability properties for the perturbated system itself. This is well illustrated by the papers [10], [7], and [6], where the authors investigate the Navier–Stokes system with Navier slip boundary conditions. They use a global controllability result for the inviscid equation (in this case the Euler equation) to deduce global approximate controllability of the the Navier–Stokes system. Note that in [10] and [6], global exact controllability is then deduced by a local controllability result obtained by a Carleman inequality. The strategy we use to prove Theorem 1.1 is very close to that used in these works. We here also provide a controllability result for (1) for a fixed viscosity (see Proposition 1.7 below).

1.2. Some remarks and further results. Here, we make some remarks concerning the set of uniformly attainable states and state two propositions concerning the inviscid system ($\varepsilon = 0$) and the viscous system with $\varepsilon = 1$.

Remark 1.4. In general, entropy solutions of (5) cannot reach a state u_T (starting, for instance, from $u_0 = 0$) in a time less than $\frac{L}{\inf |f'(u_T)|}$. In particular, the states u_T satisfying $f'(u_T) = 0$ cannot be reached unless one has $u_0 = u_T$. This can be proved by considering generalized backward characteristics (see [1]). Hence the minimal control time $\alpha_0 \frac{L}{|f'(M)|}$ in Theorem 1.1 is not surprising. Note that even in the cases of the linear transport equation at speed $M \in \mathbb{R}$ or the Burgers equation, the uniform controllability results [8], [19], [18], [21], and [16] consider a time of control of the form $C \frac{L}{|M|}$, $C > 1$.

Remark 1.5. Here, we are looking for the set $\mathcal{E}_T$ of states that are reachable uniformly in the asymptotics $\varepsilon \to 0^+$ at time $t = T$. This implies, in particular, that $\mathcal{E}_T$ is contained in the reachable set $\mathcal{E}_T^0$ for the nonviscous equation (5) and in the reachable set $\mathcal{E}_T^\varepsilon$ for the viscous equation (1) for any $\varepsilon \in (0, \varepsilon_0)$: $\mathcal{E}_T \subset \bigcap_{0 < \varepsilon < \varepsilon_0} \mathcal{E}_T^\varepsilon \cap \mathcal{E}_T^0$. In general, this intersection seems difficult to describe since the solutions of (1) are very regular, whereas the solutions of (5) can have discontinuities. We thus restrict ourselves to equilibrium points of the system, which are the most interesting states to control. Let $u_T(x)$ be a uniformly controllable stationary state as $\varepsilon \to 0^+$. It satisfies both

$$f(u_T)_x = 0 \quad \text{and} \quad f(u_T)_x - \varepsilon u_{T,xx} = 0 \quad \text{on } (0, L),$$

so that we have, for some constants c and d,

$$u_T(x) = cx + d \quad \text{and} \quad f'(cx + d)c = 0 \quad \text{on } (0, L).$$

As a consequence, either $c = 0$ and u_T is constant, or $c \neq 0$, $u_T(x) = cx + d$, and $f'(u_T) = 0$ on $(0, L)$. Referring to Remark 1.4, we see that states satisfying $f'(u_T) = 0$ are not attainable for the inviscid system (5), and necessarily $u_T = d$ is a constant. Finally, the set of uniformly attainable stationary states for (1) is exactly the set of constant states with nonzero speed.

In the vanishing viscosity limit $\varepsilon \to 0^+$, (the proof of) Theorem 1.3 gives a controllability result to constant states for entropy solutions of (5), which is new as well.

COROLLARY 1.6. *Suppose that $f \in \mathcal{C}^2(\mathbb{R})$ and $u_0 \in L^\infty(0, L)$ satisfy conditions* (i) *and* (ii) *of Theorem* 1.3. *Then, for all M satisfying $f'(M) \neq 0$, there exist $T_0 > 0$ and an entropy solution $u \in \mathcal{C}^0(\mathbb{R}_+; L^1_{loc}(\mathbb{R}))$ of*

$$u_t + [f(u)]_x = 0 \quad in \ \mathbb{R}_+ \times \mathbb{R},$$

such that

$$u|_{t=0} = u_0 \quad and \quad u|_{t=T} = M \quad on \ (0, L)$$

for any time $T > T_0$.

Even if this result can be viewed as a consequence of Theorem 1.3, it can also be proved directly constructing entropy solutions. Such a proof follows the construction of section 2.4 using shock waves in place of traveling waves and rarefaction waves in place of viscous ones. This type of direct proof of Corollary 1.6 would already contain all the ideas and the difficulties of Theorem 1.3 since the vanishing viscosity problem is addressed separately in section 3.

Note that this proposition can be seen as a boundary control result for conservation laws on the interval $(0, L)$. However, in this case, one has to take care of the sense we give to boundary conditions. Indeed, they must not be understood in the sense of Dirichlet, which is not the adapted notion for conservation laws, but rather in the sense of [2] or [1].

Note also that even in the case of a constant viscosity, the analogue of Theorem 1.3 provides a new global controllability result to constant states in large time for semilinear heat equations.

PROPOSITION 1.7. *Suppose that $f \in \mathcal{C}^2(\mathbb{R})$ and $u_0 \in L^\infty(0, L)$ satisfy conditions* (i) *and* (ii) *of Theorem* 1.3. *Then, for all M satisfying $f'(M) \neq 0$, there exists $T_0 > 0$ such that for any time $T > T_0$, there exist two control functions $g_0, g_L \in L^\infty(0, T)$ such that the solution of*

$$\begin{cases} u_t - u_{xx} + [f(u)]_x = 0 & in \ (0, T) \times (0, L), \\ u|_{x=0} = g_0(t) \ \ and \ \ u|_{x=L} = g_L(t) & in \ (0, T), \\ u|_{t=0} = u_0 & in \ (0, L) \end{cases}$$

satisfies $u|_{t=T} = M$ in $(0, L)$.

To prove this proposition, it suffices to follow the proof of Theorem 1.3 line by line and replace the argument "ε small" by "T large." It works since all the constants we obtain in the approximate controllability arguments are of the form $e^{-K\frac{T}{\varepsilon}}$ (see also Remark 3.5 below).

1.3. Structure of the paper and idea of the proofs. The main idea for proving Theorems 1.1, 1.2, and 1.3 is to combine global approximate controllability results relying on the hyperbolic nature of the problem and local exact controllability relying on the parabolic perturbation term.

The proofs of Theorems 1.1 and 1.2 follow the strategy of the article [16]. One of the main ingredients is the use of the return method of Coron, which consists in finding

a particular trajectory of the system which moves far away from the initial state to get back to the final state afterward (see [9, Chapter 6] for a general presentation). This strategy to prove global controllability results is, for instance, very close to that used in [10], [7], and [6] for the Navier–Stokes equations and in [5] for the Burgers equation. In the situation of Theorems 1.1 and 1.2, we steer the system to a large constant state N (such that $f'(N)$ has the same sign as $f'(M)$), and then we get back to the constant target M. The first step (reaching N) can be done as fast as needed, taking N sufficiently large.

The main difference between our proofs of Theorems 1.1 and 1.2 and that in [16] is concerned with the global approximate controllability results (see sections 3.1 and 3.2). The proofs given in [16] for the Burgers equation rely on the Hopf formula, which gives an explicit expression of the solution of the viscous Burgers equation on the real line. To the best of our knowledge, this formula does not exist for general flux functions, and so we have to develop different arguments. We hence have to study the convergence rate of some particular solutions of viscous conservation laws (namely, viscous shock waves and viscous rarefaction waves) in the vanishing viscosity limit $\varepsilon \to 0^+$. This is done in section 3.

Then, the proof of Theorem 1.3 relies on an iterative version of the proof of Theorem 1.2. In a first step, we use assumption (ii) to get rid of the initial condition thanks to a viscous shock wave (see a definition in the next paragraph). After that, under assumption (i), there is between the initial datum and the target a finite number of zones on which $f'' \geq 0$ or $f'' \leq 0$. In each of these intervals we can develop the same type of arguments as in the proof of Theorem 1.2. We also have to study how to pass from one zone to another. Note that in this proof, the difficulty does not come from the uniformity with respect to ε. The problem here is handling the nonmonotonicity of the speeds (i.e., the nonconvex flux function) and is hence of hyperbolic nature. The inviscid framework (for $\varepsilon = 0$) already contains the difficulties encountered here, and the proof still holds in this case (see Corollary 1.6 above).

The outline of the paper is the following. In section 2, we prove the three main theorems (Theorems 1.1, 1.2, and 1.3) assuming three key propositions (stated in in section 2.1). The first two intermediate propositions are concerned with two different global approximate controllability results, corresponding to the two control phases used in Theorems 1.1 and 1.2—global approximate controllability using a traveling wave and global approximate controllability using a rarefaction wave. The third intermediate proposition deals with the local exact controllability argument, which will be used systematically after both approximate controllability results. Note that the local exact controllability result is proved using a fixed point argument for which we need to have a small parameter (ε in the three theorems) or a large parameter (T in Proposition 1.7 or N in the first step of the proof of Theorem 1.1). In section 2.2, we combine these arguments and give the global strategy allowing us to conclude the proofs of Theorems 1.1 and 1.2 in section 2.3 and of Theorem 1.3 in section 2.4. Finally, in section 3, we prove the three key propositions.

In the appendix, we collect some technical or classical results used throughout the paper.

To conclude this section, let us introduce the traveling wave (or viscous shock wave) solutions of (1) and recall some of their basic properties (see [12, section 8.6]). In the following, we shall make intensive use of these solutions. Searching a solution $\check{u}$ of (1) on whole $\mathbb{R}$, under the form

$$\check{u}(t,x) = U\left(\frac{x - st}{\varepsilon}\right), \quad (t,x) \in \mathbb{R}_+ \times \mathbb{R},$$

that approximates as $\varepsilon \to 0^+$ a shock wave between N and P leads us to consider the ordinary differential equation

$$(6) \qquad \dot{U} = f(U) - f(N) - s(U - N),$$

$$(7) \qquad s = \sigma(P, N),$$

$$(8) \qquad \lim_{\xi \to +\infty} U(\xi) = P, \quad \text{and} \quad \lim_{\xi \to -\infty} U(\xi) = N,$$

once having replaced $\check{u}$ by U in (1) and integrated. Here $\dot{U}$ denotes the derivative with respect to ξ, $U = U(\xi)$, $\xi \in \mathbb{R}$ is the wave profile, and s is the speed of the wave. This speed is exactly the speed of the associated shock wave and is prescribed by the Rankine–Hugoniot condition (7). Under the assumptions (SOC+) and $P < N$ (resp., (SOC−) and $P > N$), system (6)–(8) admits a solution (see [12, section 8.6]) that moreover has the following properties:

- U is decreasing (resp., increasing) from N to P, since the Rankine–Hugoniot condition (7) together with the fact that $P < U(\xi) < N$ (resp., $N < U(\xi) < P$) for all $\xi \in \mathbb{R}$ implies that the vector field in the right-hand side of (6) is always negative (resp., positive);
- $\lim_{\xi \to \pm\infty} \dot{U}(\xi) = 0$ as a consequence of (6)–(8);
- for any $\xi_0 \in \mathbb{R}$, $U_{\xi_0} = U(\cdot - \xi_0)$ is still a solution of (6)–(8) since (6) is autonomous;
- $U \in W^{3,\infty}(\mathbb{R})$ for $f \in W^{2,\infty}_{\text{loc}}(\mathbb{R})$ by a bootstrap argument in (6).

In the following, we shall say that a solution U of (6)–(8) is a traveling wave "from P to N" if the vector field in the right-hand side of (6) is oriented from P to N. More precisely, if (SOC+) is satisfied between P and N, we shall say that U is a traveling wave "from P to N" if $P < N$ and $s > 0$ or if $P > N$ and $s < 0$. Similarly, if (SOC−) is satisfied between P and N, we shall say that U is a traveling wave "from P to N" if $P < N$ and $s < 0$ or if $P > N$ and $s > 0$.

Remark 1.8. In what follows, during proofs, C will denote a generic positive constant, whose value may change from line to line. Writing $C = C(p, \beta, \dots)$ means that this constant depends on the parameters $p, \beta, \dots$.

2. Proofs of the three theorems.

2.1. Three intermediate propositions. We first formulate the three intermediate key propositions, proved in section 3.

First, we give an approximate controllability result to a large state N, using a traveling wave. Given $A, B \in \mathbb{R}$, let us recall that $\sigma(A, B)$ is defined in (4) and denotes the slope of the chord of f between A and B. It represents the speed of the shock wave between A and B, if it is admissible. Since $f \in W^{2,\infty}_{\text{loc}}(\mathbb{R})$, we have $\sigma \in W^{1,\infty}_{\text{loc}}(\mathbb{R}^2)$. On the interval $(P, N] \subset \mathbb{R}$, we shall use the following particular version of the Oleinik condition for the flux function f, which is a sufficient admissibility condition for a shock wave between P and N:

$$(\text{POC}) \qquad \sigma(P, N) < \sigma(A, N) \quad \text{for all } A \in (P, N].$$

This is the strict Oleinik condition (SOC+) with the additional assumption $\sigma(P, N) < f'(N)$. Note that (POC) implies in particular the existence of a traveling wave (with speed $\sigma(P, N)$) between P and N (which is equivalent to the usual nonstrict Oleinik condition; see [12, section 8.6]). Under the additional assumption $f(N) > f(P)$, the speed of this traveling wave is positive. According to the convention described at the end of section 1.3, it is a traveling wave from P to N.

PROPOSITION 2.1. *For all $\varepsilon > 0$ and all constant states P and N such that*
- $P < \text{ess inf } u_0 \leq \text{ess sup } u_0 < N$,
- (POC) *holds on* $(P, N]$, *and*
- $f(N) > f(P)$,

there exist control functions g_0 and g_L with

$$(9) \qquad \|g_0\|_{L^\infty(0,+\infty)} \leq \max(|N|, |P|) \quad and \quad \|g_L\|_{L^\infty(0,+\infty)} \leq \max(|N|, |P|),$$

such that the solution u of (1), (2), (3) *satisfies, for any $\kappa > 0$, $\nu \in (0,1)$ and $t^* > \frac{L+\kappa}{(1-\nu)s}$,*

$$(10) \qquad \|u(t^*, \cdot) - N\|_{H^1(0,L)} \leq D^* \, e^{-\frac{\nu}{\varepsilon}s(\sigma_m - s)t^*},$$

where $s = \sigma(P, N)$, $\sigma_m = \min\{\sigma(A, N), A \in [\text{ess inf } u_0, N]\}$, and D^* is a constant $D^* = D^*(f, \varepsilon, t^*, N, P, \kappa) > 0$. Moreover, for all $\varepsilon_0 > 0$, there exists $C = C(f, N, P, \kappa)$ such that

$$D^* \leq \frac{C}{\varepsilon^{\frac{5}{2}}}(1 + t^{*6})$$

for all $\varepsilon \in (0, \varepsilon_0)$. The explicit value of the constant D^* is given in (31).

This proposition is proved in section 3.1.

Remark 2.2. Note that $s > 0$ and $\sigma_m > s$ under assumptions (POC) and $f(N) > f(P)$ so that u converges exponentially to N on the interval considered as $\varepsilon \to 0^+$ or $t^* \to +\infty$. In the case where f is convex, we have $\sigma_m = \sigma(\text{ess inf } u_0, N)$ since $\sigma(\cdot, N)$ is nondecreasing.

For a general flux function f, if we replace (POC) by (SOC+) in the assumptions of Proposition 2.1, then $\sigma_m = \inf\{\sigma(A, N), A \in [\text{ess inf } u_0, N)\}$, and $\sigma_m = s$ can occur if $s = f'(N)$. In this case, our proof fails at the very beginning (see the proof of Lemma 3.1 below): it is known that the convergence of the traveling wave to the state N is no longer exponential but polynomial (see [27, Lemma 1] or [24, Proposition 2.1]). The obstruction to exponential convergence is hence natural in this situation.

This justifies the introduction of the particular version of the Oleinik condition (POC).

Second, we give an approximate controllability result from a large state N to the state $M < N$, thanks to a rarefaction wave.

PROPOSITION 2.3. *Suppose that $f'' \geq 0$ on the interval (M, N) and $f'(M) > 0$. Then, for all $\varepsilon_0 > 0$, $\kappa > 0$, and $t^* > \frac{L+\kappa}{f'(M)}$, there exists a constant $\delta(t^*) = \delta(f, t^*, \kappa, \varepsilon_0, M, N) > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$, there exist control functions g_0 and g_L with*

$$(11) \qquad \|g_0\|_{L^\infty(0,t^*)} \leq \max(|M|, |N|) \quad and \quad \|g_L\|_{L^\infty(0,t^*)} \leq \max(|M|, |N|),$$

such that the solution u of (1), (2) *with initial condition $u|_{t=0} = N$ satisfies*

$$(12) \qquad \|u(t^*, \cdot) - M\|_{H^1(0,L)} \leq \frac{\delta(t^*)}{\varepsilon^{\frac{3}{2}}} \exp\left(-\frac{1}{4\varepsilon t^*}(f'(M)t^* - L - \kappa)^2\right).$$

This proposition is proved in section 3.2.

Finally, we give a local controllability result. We suppose that the initial condition u_0 is H^1-exponentially close (in terms of ε) to the constant target, say, N, and we want to reach it exactly. This will be done after both the "shock phase" and the

"rarefaction phase," i.e., for an initial datum that satisfies the estimate (10) or (12). More precisely, we prove the following proposition, where we assume that $f'(N) > 0$ for simplicity (the case $f'(N) < 0$ follows the same procedure).

PROPOSITION 2.4. *There exists $\alpha_1 > 0$ satisfying the following property. For every $f \in W_{loc}^{2,\infty}(\mathbb{R})$, $N \in \mathbb{R}$, and $u_0 \in H^1(0, L)$ such that $f'(N) > 0$ and*

$$\|u_0 - N\|_{H^1(0,L)} \leq e^{-\frac{K_0}{\varepsilon}}$$

for some $K_0 > 0$, there exist $\varepsilon_0 > 0$ such that for all $T \geq \alpha_1 \frac{L}{f'(N)}$ and $0 < \varepsilon < \varepsilon_0$, there exist two control functions g_0 and g_L, with

$$\|g_0\|_{L^\infty(0,T)} \leq 2|N| \quad and \quad \|g_L\|_{L^\infty(0,T)} \leq 2|N|,$$

such that the solution u of (1), (2), *and* (3) *satisfies*

$$u|_{t=T} = N \quad in \ (0, L).$$

This proposition is proved in section 3.3. A variation of this result is given in Proposition 3.8 below, where the local exact controllability is achieved by taking the state N large instead of ε small. The statement of this proposition, used in the proof of Theorem 1.1, is very close to that of Proposition 2.4, and we postpone it to section 3.3.

2.2. Combining approximate global and exact local controllability results. With the use of Propositions 2.1, 2.3, and 2.4 (resp., 3.8), we are now able to prove Theorems 1.2 and 1.3 (resp., 1.1). Before that, we combine the global approximate controllability results and the local one to provide two different global exact controllability results using a traveling wave (see Proposition 2.5 below) or a rarefaction wave (see Proposition 2.6 below). After that, we can conclude the proofs of the theorems.

PROPOSITION 2.5. *Let α_1 be the constant given by Proposition 2.4. Suppose that there exist constant states P and N, such that one of these conditions is satisfied:*

  (a) $P < \mathrm{ess\,inf}\, u_0 \leq \mathrm{ess\,sup}\, u_0 < N$, $f(N) > f(P)$, *and* (SOC+) *holds on* (P, N);
  (b) $N < \mathrm{ess\,inf}\, u_0 \leq \mathrm{ess\,sup}\, u_0 < P$, $f(N) > f(P)$, *and* (SOC+) *holds on* (N, P);
  (c) $P < \mathrm{ess\,inf}\, u_0 \leq \mathrm{ess\,sup}\, u_0 < N$, $f(N) < f(P)$, *and* (SOC−) *holds on* (P, N);
  (d) $N < \mathrm{ess\,inf}\, u_0 \leq \mathrm{ess\,sup}\, u_0 < P$, $f(N) < f(P)$, *and* (SOC−) *holds on* (N, P).

Then, setting $s = \sigma(P, N)$, there exist $C > 0$ depending only on f and $\varepsilon_0 > 0$ such that for all $T > \alpha_1 \frac{L}{|f'(N)|} + \frac{L}{|s|}$ and $0 < \varepsilon < \varepsilon_0$, there exist two control functions g_0 and g_L, with

$$\|g_0\|_{L^\infty(0,T)} \leq C(|N| + |P|) \quad and \quad \|g_L\|_{L^\infty(0,T)} \leq C(|N| + |P|),$$

such that the solution u of (1), (2), *and* (3) *satisfies*

$$u|_{t=T} = N \quad in \ (0, L).$$

Proof. Here, we prove the proposition only in case (a). The proofs of cases (b)–(d) follow that of (a) for a modified Proposition 2.1, since the conditions that we write are the conditions of existence of a traveling wave from P to N.

Since P and N satisfy the strict condition (SOC+) and $f(N) > f(P)$, one can find P' close to P, and N' close to N, such that the stronger condition (POC) is satisfied on (P', N'), together with $P' < \operatorname{ess\,inf} u_0 \leq \operatorname{ess\,sup} u_0 < N'$ and $f(N') > f(P')$. As a consequence, Proposition 2.1 applies, and hence one can find control functions that drive u_0 to a state u_1 which is H^1-exponentially close to N in any time $T_1 > \frac{L}{|s|}$. After that, the assumptions of Proposition 2.4 are satisfied, and thus u_1 can be steered exactly to N in any time $T_2 \geq \alpha_1 \frac{L}{|f'(N)|}$. Finally, the result of Proposition 2.5 holds for any time $T \geq T_1 + T_2$.   □

PROPOSITION 2.6. *Let α_1 be the constant given by Proposition 2.4. Suppose that one of these conditions is satisfied:*
  (a) *$M < N$, $f'' \geq 0$ on the interval (M, N) and $f'(M) > 0$;*
  (b) *$N < M$, $f'' \geq 0$ on the interval (N, M) and $f'(M) < 0$;*
  (c) *$M < N$, $f'' \leq 0$ on the interval (M, N) and $f'(M) < 0$;*
  (d) *$N < M$, $f'' \leq 0$ on the interval (N, M) and $f'(M) > 0$.*
Then, there exist $C > 0$ only depending on f and $\varepsilon_0 > 0$ such that for all $T > (\alpha_1 + 1)\frac{L}{|f'(M)|}$ and $0 < \varepsilon < \varepsilon_0$, there exist two control functions g_0 and g_L, with

$$(13) \qquad \|g_0\|_{L^\infty(0,T)} \leq C(|M| + |N|) \quad and \quad \|g_L\|_{L^\infty(0,T)} \leq C(|M| + |N|),$$

such that the solution u of (1), (2) with initial condition $u|_{t=0} = N$ satisfies

$$u|_{t=T} = M \quad in \ (0, L).$$

Proof. Here, we prove the proposition only in case (a). The proofs of cases (b)–(d) follow that of (a) for a modified Proposition 2.3, since the conditions we write are the conditions of existence of a rarefaction wave steering N to M, having a speed of nonvanishing fixed sign.

In case (a), the assumptions of Proposition 2.3 are satisfied, and hence one can find control functions that drive u_0 to a state u_1 which is H^1-exponentially close to M in any time $T_1 > \frac{L}{|f'(M)|}$. After that, the assumptions of Proposition 2.4 are satisfied, and thus u_1 can be steered exactly to N in a time $T_2 \geq \alpha_1 \frac{L}{|f'(M)|}$. Finally, the result of Proposition 2.5 holds for any time $T \geq T_1 + T_2$.   □

2.3. Proof of Theorems 1.1 and 1.2: The convex case. In this section, we suppose $f'' \geq 0$. Let us first prove Theorem 1.2, using Propositions 2.5 and 2.6.

Proof of Theorem 1.2. We first suppose that $f'(M) > 0$ and choose some $P < \operatorname{ess\,inf} u_0$. Since $\lim_{u \to +\infty} f'(u) = +\infty$, and setting $s = \sigma(N, P)$, we have $\lim_{N \to +\infty} s = +\infty$. As a consequence, for any time $T_1 > 0$ there exists $N > \operatorname{ess\,sup} u_0$ sufficiently large so that $f(N) > f(P)$ and $0 < \alpha_1 \frac{L}{f'(N)} + \frac{L}{s} < T_1$. As a consequence of Proposition 2.5, one can reach N exactly, uniformly with ε in time T_1. Then Proposition 2.6 holds between N and M, and for any $T_2 > \alpha_0 \frac{L}{f'(M)}$ (with $\alpha_0 = \alpha_1 + 1$), there exist two control functions bounded uniformly with respect to ε such that $u|_{t=T_1+T_2} = M$. Finally, the result of Theorem 1.1 holds for any time $T \geq T_1 + T_2$.

In the case $f'(M) < 0$, exactly the same proof still holds using $\lim_{u \to -\infty} f'(u) = -\infty$, together with Proposition 2.5 (b) and Proposition 2.6 (b).   □

In Theorem 1.1, however, we do not want ε_0 to depend on the initial datum u_0. For this, we use the following proposition instead of Proposition 2.5.

PROPOSITION 2.7. *Suppose that condition (A+) holds. Then, for all $\varepsilon_0 > 0$ and $T > 0$, there exists a state $N_0 > 0$, such that for all $\varepsilon \in (0, \varepsilon_0)$ and $N > N_0$, there*
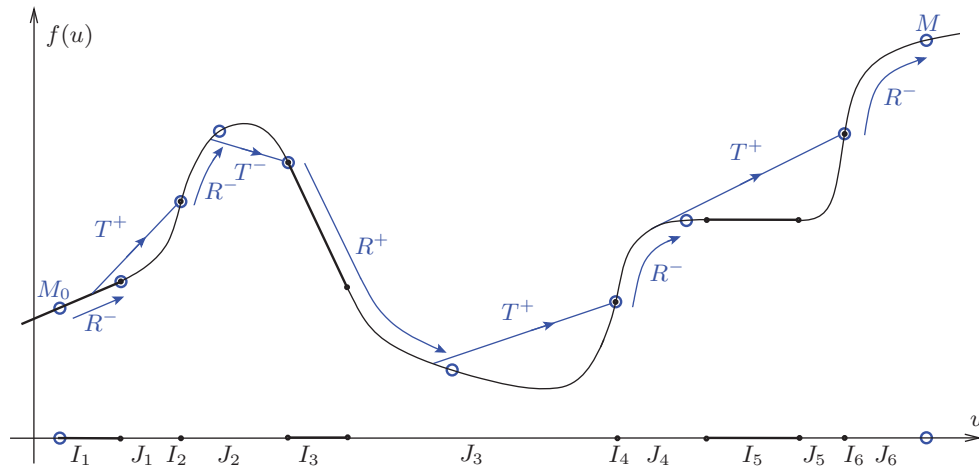
FIG. 1. *Global control strategy for nonconvex f. R^+: Convex rarefaction wave. R^-: Concave rarefaction wave. T^+: Convex traveling wave. T^-: Concave traveling wave.*

exist control functions g_0 and g_L with

$$\|g_0\|_{L^\infty(0,T)} \leq \|u_0\|_{L^\infty(0,L)} + 2|N| \quad and \quad \|g_L\|_{L^\infty(0,T)} \leq \|u_0\|_{L^\infty(0,L)} + 2|N|,$$

such that the solution u of (1), (2), *and* (3) *satisfies*

$$u|_{t=T} = N \quad in \ (0,L).$$

The proof of this proposition combines the approximate controllability result to the large state N of Proposition 2.1 and the local exact controllability result of Proposition 3.8. Both results hold for any $\varepsilon > 0$, for N sufficiently large, and for control times $T > \frac{L}{s}$ and $T \geq \alpha_1 \frac{L}{f'(N)}$, both vanishing as $N \to +\infty$.

The proof of Theorem 1.1 then exactly follows that of Theorem 1.2, using both Proposition 2.7 to reach the large constant N and Proposition 2.6 to come back to the state M with a rarefaction wave.

2.4. Proof of Theorem 1.3: The nonconvex case. For the sake of brevity in the following proof, we do not mention that every control step is done uniformly with respect to ε, as a consequence of Propositions 2.5 and 2.6.

Proof of Theorem 1.3. The first step is to get rid of the initial condition u_0, and that is the only role of assumption (ii). This assumption, together with Proposition 2.5, gives the controllability from u_0 to some constant state M_0 with $f'(M_0) \neq 0$. The entire proof is now reduced to a controllability problem between two constant states (M_0 and M) and is illustrated in Figure 1.

Let us suppose that $M_0 < M$ (the case $M_0 > M$ follows from the same arguments). From assumption (i), $(f'')^{-1}(\{0\}) \cap [M_0, M]$ can be written as $\bigcup_{j=1}^m I_j$, where the union is disjoint and ordered. Note that each I_j is a segment (possibly reduced to a point) since f'' is supposed to be continuous here. Let us denote $(J_k)_{1 \leq k \leq p}$ as the connected components of $(M_0, M) \setminus \bigcup_{j=1}^m I_j$, so that $(M_0, M) \setminus \bigcup_{j=1}^m I_j = \bigcup_{k=1}^p J_k$, where this last union is disjoint and ordered. As a consequence, J_k is an open interval on which either $f'' > 0$ or $f'' < 0$.

1. *Driving A to B with $A, B \in \overline{J}_k$.* We first prove that, for any $A \in \overline{J}_k$ and $B \in \overline{J}_k$ satisfying $f'(A) \neq 0$ and $f'(B) \neq 0$, one can steer A to B in a finite time. We

suppose that $f'' > 0$ on J_k (the case $f'' < 0$ follows from the same arguments) and denote by L_k (resp., R_k) the left (resp., right) endpoint of $\overline{J}_k$.

• If $f'(L_k) > 0$, then, as a consequence of Lemma 2.8 below, and using the fact that f is strictly convex on (L_k, R_k), there exists $L'_k < L_k$ such that f satisfies (SOC+) on (L'_k, R_k). Thus, there exists a traveling wave from L'_k to R_k, and, according to Proposition 2.5 (a), A can be driven to R_k in finite time (see the situation in J_1 in Figure 1). Then, we are in the situation of Proposition 2.6 (a), and a rarefaction wave drives R_k to B in finite time. Note that if $f'(L_k) = 0$, we have $L_k < A$, and the same strategy holds.

• If $f'(R_k) \leq 0$, f has no local minimum on J_k, and this case follows the same reasoning.

• If $f'(L_k) < 0$ and $f'(R_k) > 0$, there exists $C_k \in J_k$ satisfying $f'(C_k) = 0$. The situation $f'(A)f'(B) > 0$ has already been treated, so we have only to consider the case $f'(A) < 0$ and $f'(B) > 0$ (the cases $f'(A) > 0$ and $f'(B) < 0$ follow exactly the same arguments). According to Proposition 2.6 (b), A can be brought to A' as close to C_k as needed, by means of a rarefaction wave (see the situation in J_3 in Figure 1). It suffices to take A' so that $f(A') < f(B)$, and as a consequence of Proposition 2.5 (a), we can steer A' to B thanks to a traveling wave (f is convex on J_k and hence satisfies (SOC+) on this interval).

2. *Driving A to B with $A, B \in I_j$.* Next, we prove that, if $f' \neq 0$ on I_j for some j, then for any $A, B \in I_j$, one can steer A to B in finite time. Recall that $f'' = 0$ on I_j. If $f' > 0$ on I_j and $B < A$, we are in the context of Proposition 2.6 (a), and a "convex rarefaction wave" drives A to B (see the situation in I_3 in Figure 1). If $f' > 0$ on I_j and $A < B$, Proposition 2.6 (d) applies, and a "concave rarefaction wave" drives A to B (see the situation in I_1 in Figure 1). We can do the same if $f' < 0$ on I_j using Proposition 2.6 (b) and (c). Note that in this case, (1) is only a linear transport equation with constant speed, which we control on both endpoints of the interval, and the result is also a consequence of [8].

3. *Driving A to B with $A \in J_k$ and $B \in J_{k+1}$ when f' vanishes between J_k and J_{k+1}.* The last case to consider is the case where $J_k = (L_k, L_j)$, $I_j = [L_j, R_j]$, $J_{k+1} = (R_j, R_{k+1})$, $f' = 0$ on I_j, $A \in J_k$, $B \in J_{k+1}$, and we want to steer A to B. Suppose that $f'(A) > 0$ (the case $f'(A) < 0$ follows the same reasoning). Then, according to Proposition 2.6 (d), a "concave rarefaction wave" can bring A to some A' as close as we want from L_j (see the situation in J_4 in Figure 1).

• If $f'(B) < 0$, we can choose $A' < L_j$ such that $f(A') > f(B)$, so that Proposition 2.5 (c) holds, and we can steer A' to B by means of a concave traveling wave.

• If $f'(B) > 0$, the continuity of f' yields that we can choose $A' < L_j$ such that $0 < f'(A') < \sigma(A', B)$ so that (SOC+) holds on (A', B), and, according to Proposition 2.5 (a), we can steer A' to B by means of a "convex traveling wave" (see the situation in I_5 in Figure 1).

Therefore, using iteratively the three arguments above, one can pass from each zone I_j to the neighboring zone J_k (case 2) and to each J_k to the neighboring I_j (case 1) since $I_j \cap \overline{J}_k \neq \emptyset$, provided that $f' \neq 0$ on I_j. If $f' = 0$ on I_j, one can jump from J_k before I_j to J_{k+1} after I_j (case 3). Finally, from assumption (i), there is a finite number of I_j and J_k between M_0 and M, so that we can steer M_0 to M with a finite number of such iterations, i.e., in finite time. This global strategy is illustrated in Figure 1.   ☐

Note that this proof does not provide an estimate on the minimal time T_0 needed to control. Moreover, the strategy developed here is clearly not optimal in time since it is mainly a local strategy. For instance, in Figure 1, we see that one can steer the

point $I_3 \cap \overline{J}_3$ directly to a point in J_6 with a single traveling wave, a strategy which is much faster than that described in the proof. Optimizing this strategy in time seems to be an interesting open problem.

We end this section by proving a lemma used in the Proof of Theorem 1.3.

LEMMA 2.8. *Suppose that $f \in \mathcal{C}^1(\mathbb{R})$ satisfies* (SOC+) *on the interval* (L, R) *and that $f'(L) < \sigma(L, R)$. Then, there exists $\rho_0 > 0$ such that for all $\rho \in (0, \rho_0)$, the function f satisfies* (SOC+) *on* $(L - \rho, R)$.

Proof. First, we have

$$\frac{\partial}{\partial u} \sigma(u, R)|_{u=L} = \frac{1}{R - L} \left(\sigma(L, R) - f'(L) \right) > 0.$$

Hence, there exists $\rho_1 > 0$ such that for all $\rho \in (0, \rho_1)$, $\sigma(L - \rho, R) < \sigma(L, R)$. Since (SOC+) reads $\sigma(L, R) < \sigma(u, R)$ for all $u \in (L, R)$, we now have

$$(14) \qquad \sigma(L - \rho, R) < \sigma(u, R) \quad \text{for all } \rho \in (0, \rho_1), u \in (L, R).$$

Second, the assumption $f'(L) < \sigma(L, R)$ together with the continuity of f' and $\sigma(\cdot, R)$ gives the existence of $\rho_2 > 0$ such that for all $\rho \in (0, \rho_2)$ and all $v \in (L - \rho, L)$, we have $f'(v) < \sigma(L - \rho, R)$. Integrating this inequality with respect to v on $(L - \rho, u)$ for $u \in (L - \rho, L)$ and dividing by $(u - (L - \rho))$ yields $\sigma(L - \rho, u) < \sigma(L - \rho, R)$. This is equivalent to having

$$(15) \qquad \sigma(L - \rho, R) < \sigma(u, R) \quad \text{for all } \rho \in (0, \rho_2), u \in (L - \rho, L).$$

Finally, setting $\rho_0 = \min\{\rho_1, \rho_2\}$ and combining (14) and (15) concludes the proof of the lemma. $\quad\square$

3. Proofs of the three intermediate propositions.

3.1. Proof of Proposition 2.1: Approximate controllability using a traveling wave. The solution u in Proposition 2.1 is constructed as the restriction to $(0, t^*) \times (0, L)$ of the solution defined on the whole $\mathbb{R}_+ \times \mathbb{R}$ (still denoted u) of the following problem (see Figure 2):

$$(16) \qquad \begin{cases} u_t + [f(u)]_x - \varepsilon u_{xx} = 0 & \text{in } \mathbb{R}_+ \times \mathbb{R}, \\ u|_{t=0} = \tilde{u}_0 & \text{in } \mathbb{R}, \end{cases}$$

with

$$(17) \qquad \tilde{u}_0(x) = \begin{cases} N & \text{if } x \leq 0, \\ u_0 & \text{if } x \in (0, L), \\ \text{ess inf } u_0 & \text{if } x \geq L. \end{cases}$$

Then, the control functions g_0 and g_L are obtained by taking the trace of u along the lines $(0, t^*) \times \{0\}$ and $(0, t^*) \times \{L\}$. The estimates on the control cost (9) follow directly from the comparison principle [12, Theorem 6.3.2], and, hence, we have only to prove (10).

The first step of the proof (see Lemma 3.1 below) consists in giving an L^∞ estimate of the convergence rate of traveling waves (viscous shock waves) to the associated nonviscous shock wave as $\varepsilon \to 0^+$. Then a comparison principle gives an L^∞ estimate of the convergence rate of u to N. This strategy is illustrated in Figure 2. To conclude the proof of Proposition 2.1, we use a bootstrap argument relying on the parabolic nature of (16) to estimate the H^1 norm in terms of the L^∞ norm (losing a polynomial factor of ε).
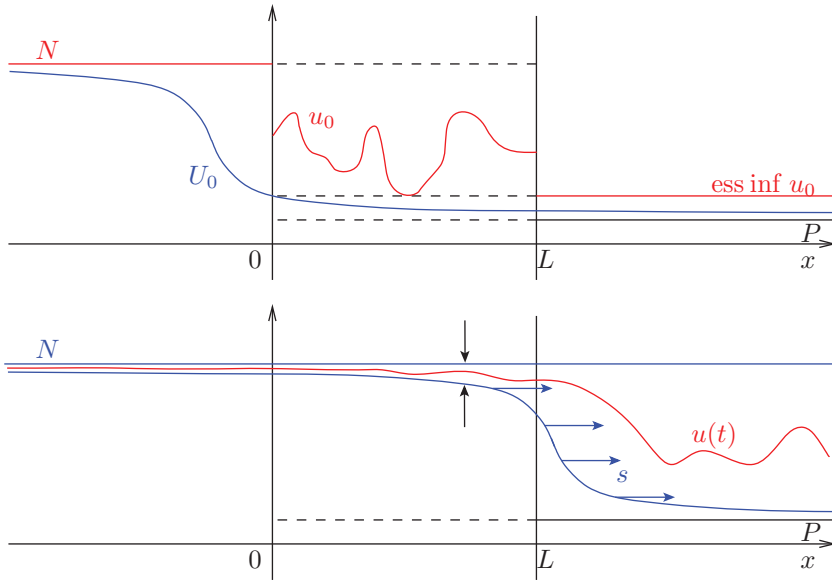
FIG. 2. *Comparison principle during the traveling wave. Initial data and solution after a time* $t > \frac{L}{s}$ *has gone by.*

Note that the exponential convergence of traveling waves at infinity (under condition (POC)) is a well-known phenomenon (see, for instance, [27, Lemma 1] or [24, Proposition 2.1]).

Similarly, the problem of the convergence of solutions toward traveling waves, initiated by Il'in and Oleinik [23], has now been widely studied. Results in this area (see [23, 27, 24] and the references therein) are of the following type: suppose that the initial datum $u|_{t=0}$ is sufficiently close to the traveling wave $U(\frac{x}{\varepsilon})$ (especially as $x \to \pm\infty$); then the associated solution of (16) converges toward the traveling wave $U(\frac{x-st}{\varepsilon})$ as $t \to +\infty$. The rate of convergence is also studied and is in general exponential (with respect to the time t). These results are stated for a for fixed viscosity ε.

For our purposes, all these results are not precise enough (no explicit decay rate, no explicit minimal time, etc.) and much too general (here, the initial condition $\tilde{u}_0$ we use, given in (17), is very specific). In particular, in Proposition 2.1, we have to give explicit constants (especially for the decay rate and the minimal time for (10) to hold) and hence keep track of the dependencies with respect to all the parameters of our problem. To summarize, the result of Lemma 3.1 is much more precise than the convergence results of [23, 27, 24] in a much more restrictive setting.

LEMMA 3.1. *Suppose that there exist constant states* P *and* N, *with* $P < $ ess inf $u_0 \leq$ ess sup $u_0 < N$, *such that* (POC) *holds on* $(P, N]$, *and* $f(N) > f(P)$. *Then, for every* $\varepsilon > 0$ *and* $\beta \in \mathbb{R}$, *the solution* u *of* (16), (17) *satisfies*

$$(18) \qquad \|N - u(t, \cdot)\|_{L^\infty(-\infty, \beta)} \leq (N - \text{ess inf } u_0)e^{-\frac{s}{\varepsilon}(\sigma_m - s)(t - \frac{\beta}{s})} \quad \text{for } t > \frac{\beta}{s},$$

where $s = \sigma(P, N)$ *and* $\sigma_m = \min\{\sigma(A, N), A \in [\text{ess inf } u_0, N]\}$.

Proof. We introduce the family of traveling waves from P to N, given by

$$\check{u}(t, x) = U\left(\frac{x - st}{\varepsilon}\right), \quad (t, x) \in \mathbb{R}_+ \times \mathbb{R},$$

whose existence is ensured by assumption (POC) (see [12, section 8.6]). Here, $U = U(\xi)$, $\xi \in \mathbb{R}$, is the wave profile, satisfying (6), (8), and $s = \sigma(P, N) > 0$ is the speed of the wave. The ordinary differential equation (6) satisfied by U can be reformulated as

$$\frac{d}{d\xi}(U - N) = (-s + \sigma(U, N))(U - N),$$

which yields, for any $\xi, \xi_0 \in \mathbb{R}$,

$$(U(\xi) - N) = (U(\xi_0) - N) \exp\left(\int_{\xi_0}^{\xi}(-s + \sigma(U(\tau), N))d\tau\right).$$

From now on, we select the traveling wave that satisfies

$$U_0(0) = \operatorname{ess\,inf} u_0 > P$$

(which is unique since it now solves a Cauchy problem), so that

$$(19) \qquad 0 < N - U_0(\xi) = (N - \operatorname{ess\,inf} u_0) \exp\left(\int_0^{\xi}(-s + \sigma(U_0(\tau), N))d\tau\right).$$

Note also that $U(\xi) \to P$ when $\xi \to +\infty$, and the solution U_0 can also be characterized by

$$\int_0^{+\infty}(-s + \sigma(U_0(\tau), N))d\tau = \log\left(\frac{N - P}{N - \operatorname{ess\,inf} u_0}\right).$$

For $\xi < 0$ and $\tau \in [\xi, 0]$, we have $U_0(\tau) \in [U_0(0), U_0(\xi)] \subset [U_0(0), N]$. On the compact interval $[U_0(0), N]$, the function $A \mapsto \sigma(A, N)$ is continuous and satisfies $\sigma(A, N) > s$, so that

$$-s + \sigma(U_0(\tau), N) \geq \sigma_m - s > 0 \quad \text{for all } \xi < 0, \ \tau \in [\xi, 0].$$

As a consequence,

$$\int_0^{\xi} -s + \sigma(U_0(\tau), N)d\tau \leq \xi(\sigma_m - s) \quad \text{for all } \xi < 0,$$

so that (19) now yields

$$0 < N - U_0(\xi) \leq (N - \operatorname{ess\,inf} u_0)e^{\xi(\sigma_m - s)} \quad \text{for all } \xi < 0.$$

Coming back to the variables t, x, we obtain for $x \in (-\infty, \beta)$

$$(20) \qquad 0 < N - U_0\left(\frac{x - st}{\varepsilon}\right) \leq (N - \operatorname{ess\,inf} u_0)e^{\frac{1}{\varepsilon}(\sigma_m - s)(\beta - st)} \quad \text{for } t > \frac{\beta}{s}.$$

We conclude the proof by comparing the traveling wave U_0 and the solution u of (16)–(17). From the choice of $U_0(0)$, we have at time $t = 0$

$$U_0\left(\frac{x}{\varepsilon}\right) \leq \tilde{u}_0(x) \leq N \quad \text{for all } x \in \mathbb{R}, \ \varepsilon > 0.$$

The comparison principle [12, Theorem 6.3.2] then implies that for every $(t, x) \in \mathbb{R}_+ \times \mathbb{R}$,

$$U_0\left(\frac{x - st}{\varepsilon}\right) \leq u(t, x) \leq N,$$

so that estimate (20) now yields

$$\|N - u(t, \cdot)\|_{L^\infty(-\infty, \beta)} \leq (N - \operatorname{ess\,inf} u_0) e^{-\frac{s}{\varepsilon}(\sigma_m - s)(t - \frac{\beta}{s})} \quad \text{for } t > \frac{\beta}{s},$$

and the lemma is proved. □

Remark 3.2. If f is convex on (P, N), the decay rate obtained does not seem to be optimal. In this case, under the assumptions of Lemma 3.1, one can prove that for any $\theta < 1$, there exist $\xi_0 = \xi_0(N) > 0$ and $C > 0$ such that

$$\|N - u(t, \cdot)\|_{L^\infty(-\infty, \beta)} \leq C e^{-\frac{\theta}{\varepsilon} s(f'(N) - s)(t - \frac{\beta}{s})} \quad \text{for } t > \frac{\beta + \xi_0 \varepsilon}{s}.$$

The convergence rate that we can expect is thus in the convex case of the form $\theta s(f'(N) - s)$. This plays an important role when taking N large, as in the first part of the proof of Theorem 1.1, since $\theta s(f'(N) - s)$ is much larger than $s(\sigma_m - s)$. For the Burgers equation [16], for instance, $\theta s(f'(N) - s) \sim \theta N^2/4$, whereas $s(\sigma_m - s) \sim (\operatorname{ess\,inf} u_0 - P)N$ as $N \to +\infty$. The problem for general convex functions f is to give the asymptotic behavior of $\xi_0(N)$. If $\xi_0(N)/s \to 0$, then assumption (A+) can be replaced by the more general (and somehow more natural) assumption

$$\liminf_{A \to +\infty} A^{-\gamma} f''(A) > 0; \quad \limsup_{A \to +\infty} e^{-A^{2\gamma + 2 - \delta}} f''(A) < +\infty \text{ for some } \gamma > -1, \delta > 0.$$

Such a condition would include in particular flux functions satisfying $f(u) \sim_{u \to +\infty} u^\rho$ with $1 < \rho \leq \frac{3}{2}$, for which assumption (A+) does not hold.

We can now conclude the proof of Proposition 2.1 by a bootstrap argument.

End of the proof of Proposition 2.1. We study the evolution of (16), (17) for $t \in (0, t^*)$, where t^* is the time at which we want to obtain estimate (10). First, we set

$$v(t, x) = (u - N) \circ \pi(t, x), \qquad \pi(t, x) = (t, x + f'(N)t),$$

so that, for any $\beta \in \mathbb{R}$,

$$\|v(t, \cdot)\|_{L^\infty(-\infty, \beta)} = \|u(t, \cdot) - N\|_{L^\infty(-\infty, \beta + f'(N)t)}.$$

Estimate (18) of Lemma 3.1 gives

$$(21) \quad \|v(t, \cdot)\|_{L^\infty(-\infty, \beta)} \leq (N - \operatorname{ess\,inf} u_0) \exp\left(-\frac{s}{\varepsilon}(\sigma_m - s)\left(t - \frac{\beta + f'(N)t}{s}\right)\right)$$

for $t > \frac{\beta + f'(N)t}{s}$, i.e., $t < \frac{-\beta}{f'(N) - s}$. For some $\mu > 1$ that will be chosen later on, we fix $\beta = \beta^* = -(f'(N) - s)\mu t^* < 0$, so that estimate (21) holds for every $t \in (0, \mu t^*)$. Estimate (21) now becomes

$$(22) \quad \|v(t, \cdot)\|_{L^\infty(-\infty, \beta^*)} \leq (N - \operatorname{ess\,inf} u_0) \exp\left(-\frac{1}{\varepsilon}(\sigma_m - s)(f'(N) - s)(\mu t^* - t)\right)$$

for all $t < \mu t^*$.

We denote by $\Omega_1 = (a_1, b_1)$ a bounded open interval of $(-\infty, \beta^*)$ and $\chi_1 \in \mathcal{C}_c^\infty(\Omega_1)$ a cut-off function satisfying $\chi_1 = 1$ on some Ω_2, with $\overline{\Omega}_2 \subset \Omega_1$. The function $w_1(t, x) = \chi_1(x)v(t, x)$ satisfies

$$\begin{cases} w_{1,t} - \varepsilon w_{1,xx} = [f'(N) - f'(u \circ \pi)](w_{1,x} - \chi_1'v) - \varepsilon(\chi_1''v + 2\chi_1'v_x) & \text{in } (0, t^*) \times \Omega_1, \\ w_1 = 0 & \text{on } \partial\Omega_1, \\ w_1(0, x) = 0 & \text{in } \Omega_1. \end{cases}$$

The parabolic regularizing effect (see Lemma A.2 for $m = 0$) gives for this system

$$(23) \quad \varepsilon \int_0^{t^*} \|w_1\|_{H_0^1(\Omega_1)}^2 dt$$
$$\leq \frac{1}{\varepsilon} \int_0^{t^*} \|[f'(N) - f'(u \circ \pi)](w_{1,x} - \chi_1'v) - \varepsilon(\chi_1''v + 2\chi_1'v_x)\|_{H^{-1}(\Omega_1)}^2 dt.$$

Let us now estimate each of the terms on the right-hand side. Using the Poincaré inequality (86), together with estimate (89) of section A.1 (where a definition of the H^{-1} norm is given), the first term can be dominated as

$$(24)$$
$$\int_0^{t^*} \|[f'(N) - f'(u \circ \pi)]w_{1,x}(t, \cdot)\|_{H^{-1}(\Omega_1)}^2 dt$$
$$\leq C \int_0^{t^*} \left(\|f'\|_{L^\infty(P,N)}^2 + (1 + |\Omega_1|^2)\|(u_x \circ \pi)f''(u \circ \pi)\|_{L^2(\Omega_1)}^2 \right) \|w_{1,x}(t, \cdot)\|_{H^{-1}(\Omega_1)}^2 dt$$
$$\leq C \int_0^{t^*} \left(\|f'\|_{L^\infty(P,N)}^2 + (1 + |\Omega_1|^2)\|f''\|_{L^\infty(P,N)}^2 \|u_x \circ \pi\|_{L^2(\Omega_1)}^2 \right) \|w_1(t, \cdot)\|_{L^2(\Omega_1)}^2 dt$$
$$\leq C \int_0^{t^*} \left(\|f'\|_{L^\infty(P,N)}^2 + (1 + |\Omega_1|^2)\|f''\|_{L^\infty(P,N)}^2 \|u_x \circ \pi\|_{L^2(\Omega_1)}^2 \right) |\Omega_1| \|v\|_{L^\infty(\Omega_1)}^2 dt.$$

It remains only to estimate $\|u_x \circ \pi(t, \cdot)\|_{L^2(\Omega_1)}^2$. For this, we consider another bounded open set $\tilde{\Omega}_1 = (a_1 - 1, b_1 + f'(N)t^* + 1)$, so that $\Omega_1 = (a_1, b_1) \subset [a_1, b_1 + f'(N)t^*] \subset \tilde{\Omega}_1$. We take $\tilde{\chi}_1 \in \mathcal{C}_c^\infty(\tilde{\Omega}_1)$ such that $\tilde{\chi}_1 = 1$ on $[a_1, b_1 + f'(N)t^*]$ and set $y_1(t, x) = \tilde{\chi}_1(x)u(t, x)$. Since y_1 satisfies

$$y_{1,t} - \varepsilon y_{1,xx} = -\tilde{\chi}_1[f(u)]_x - \varepsilon(\tilde{\chi}_1''u + 2\tilde{\chi}_1'u_x),$$

the parabolic regularizing effect (see Lemma A.2 for $m = 0$) gives

$$\varepsilon^2 \int_0^{t^*} \|y_1\|_{H_0^1(\tilde{\Omega}_1)}^2 dt \leq \varepsilon \|u_0\|_{L^2(\tilde{\Omega}_1)}^2 + C \int_0^{t^*} \|\tilde{\chi}_1[f(u)]_x\|_{H^{-1}(\tilde{\Omega}_1)}^2 dt$$
$$+ C\varepsilon^2 \int_0^{t^*} \|\tilde{\chi}_1''u + 2\tilde{\chi}_1'u_x\|_{H^{-1}(\tilde{\Omega}_1)}^2 dt.$$

Using then (86) and (89), we obtain

$$\varepsilon^2 \int_0^{t^*} \|y_1\|_{H_0^1(\tilde{\Omega}_1)}^2 dt \leq \varepsilon \max(|P|^2, |N|^2)$$
$$+ C(1 + |\tilde{\Omega}_1|^2) \left(\int_0^{t^*} \|f(u)\|_{L^2(\tilde{\Omega}_1)}^2 dt + \varepsilon^2 \int_0^{t^*} \|u\|_{L^2(\tilde{\Omega}_1)}^2 dt \right),$$

and hence, since $P \le u(t,x) \le N$,

$$\int_0^{t^*} \|u_x \circ \pi\|^2_{L^2(\Omega_1)} dt \le \left(\frac{1}{\varepsilon} + C(|\tilde{\Omega}_1| + |\tilde{\Omega}_1|^3)t^* \right) \max(|P|^2, |N|^2)$$
$$+ \frac{C}{\varepsilon^2}(|\tilde{\Omega}_1| + |\tilde{\Omega}_1|^3)t^* \|f\|^2_{L^\infty(P,N)}.$$

Coming back to (24) and using estimate (22) on v, this yields

$$(25) \quad \int_0^{t^*} \|[f'(N) - f'(u)]w_{1,x}(t,\cdot)\|^2_{H^{-1}(\Omega_1)} dt$$
$$\le C D_1 |\Omega_1|(N-P)^2 e^{-\frac{2}{\varepsilon}(\sigma_m - s)(f'(N)-s)(\mu-1)t^*},$$

with

$$D_1 = D_1(f, P, N, |\Omega_1|, |\tilde{\Omega}_1|, t^*, \varepsilon) = \|f'\|^2_{L^\infty(P,N)}$$
$$+ \frac{1}{\varepsilon}(1 + |\Omega_1|^2)\|f''\|^2_{L^\infty(P,N)} \max(|P|^2, |N|^2)$$
$$+ (|\tilde{\Omega}_1| + |\tilde{\Omega}_1|^5)\|f''\|^2_{L^\infty(P,N)} t^* \left(\max(|P|^2, |N|^2) + \frac{1}{\varepsilon^2}\|f\|^2_{L^\infty(P,N)} \right).$$

Concerning the other terms in (23), we simply have

$$(26)$$
$$\int_0^{t^*} \|[f'(N) - f'(u \circ \pi)]\chi_1' v - \varepsilon(\chi_1'' v + 2\chi_1' v_x)\|^2_{H^{-1}(\Omega_1)} dt$$
$$\le C \int_0^{t^*} \left(|\Omega_1|^2 \|f'\|^2_{L^\infty(P,N)} + \varepsilon^2(1 + |\Omega_1|^2) \right) \|v\|^2_{L^2(\Omega_1)} dt$$
$$\le C|\Omega_1| \left(|\Omega_1|^2 \|f'\|^2_{L^\infty(P,N)} + \varepsilon^2(1 + |\Omega_1|^2) \right) t^*(N-P)^2 e^{-\frac{2}{\varepsilon}(\sigma_m - s)(f'(N)-s)(\mu-1)t^*}$$

after using estimate (22) on v.

Now, replacing (25) and (26) in (23), we obtain

$$(27) \quad \int_0^{t^*} \|w_1\|^2_{H_0^1(\Omega_1)} dt \le \frac{C}{\varepsilon^2} \left(D_1 + \left(|\Omega_1|^2 \|f'\|^2_{L^\infty(P,N)} + \varepsilon^2(1 + |\Omega_1|^2) \right)t^* \right)$$
$$\times |\Omega_1|(N-P)^2 e^{-\frac{2}{\varepsilon}(\sigma_m - s)(f'(N)-s)(\mu-1)t^*}.$$

We now take $\chi_2 \in \mathcal{C}_c^\infty(\Omega_2)$ and set $w_2(t,x) = \chi_2(x)w_1(t,x) = \chi_2(x)v(t,x)$, which satisfies

$$\begin{cases} w_{2,t} - \varepsilon w_{2,xx} = [f'(N) - f'(u)]\chi_2 w_{1,x} - \varepsilon(\chi_2'' w_1 + 2\chi_2' w_{1,x}) & \text{in } (0,t^*) \times \Omega_1, \\ w_2 = 0 & \text{on } \partial\Omega_1, \\ w_2(0,x) = 0 & \text{in } \Omega_1. \end{cases}$$

The parabolic regularizing effect (see Lemma A.2 for $m = 1$) gives for this system

$$\|w_2(t^*,\cdot)\|^2_{H_0^1(\Omega_1)} \le \frac{1}{\varepsilon} \int_0^{t^*} \|[f'(N) - f'(u)]\chi_2 w_{1,x} - \varepsilon(\chi_2'' w_1 + 2\chi_2' w_{1,x})\|^2_{L^2(\Omega_1)} dt,$$

which directly yields

$$\|w_2(t^*,\cdot)\|^2_{H^1_0(\Omega_1)} \le C\left(\frac{1}{\varepsilon}\|f'\|^2_{L^\infty(P,N)} + \varepsilon(1+|\Omega_1|^2)\right)\int_0^{t^*}\|w_1\|^2_{H^1_0(\Omega_1)}dt.$$

As a consequence of (27), we thus have, for $t^* > 0$,

$$(28) \qquad \|w_2(t^*,\cdot)\|^2_{H^1(\Omega_2)} \le CD_2 e^{-\frac{2}{\varepsilon}(\sigma_m - s)(f'(N)-s)(\mu-1)t^*},$$

with

$$
\begin{aligned}
D_2 &= D_2(f, P, N, |\Omega_1|, |\tilde\Omega_1|, t^*, \varepsilon)\\
&= \frac{|\Omega_1|}{\varepsilon^2}(N-P)^2(1+|\Omega_1|^2)\left(\frac{1}{\varepsilon}\|f'\|^2_{L^\infty(P,N)} + \varepsilon(1+|\Omega_1|^2)\right)\\
&\quad \times \Bigg\{\|f'\|^2_{L^\infty(P,N)} + \frac{1}{\varepsilon}(1+|\Omega_1|^2)\|f''\|^2_{L^\infty(P,N)}\max(|P|^2,|N|^2)\\
&\quad + (|\tilde\Omega_1| + |\tilde\Omega_1|^5)\|f''\|^2_{L^\infty(P,N)}t^*\left(\max(|P|^2,|N|^2) + \frac{1}{\varepsilon^2}\|f\|^2_{L^\infty(P,N)}\right)\\
(29)&\quad + \left(|\Omega_1|^2\|f'\|^2_{L^\infty(P,N)} + \varepsilon^2(1+|\Omega_1|^2)\right)t^*\Bigg\}.
\end{aligned}
$$

The constant C in (28) depends only on the derivatives of the cut-off functions $\|\chi_1^{(j)}\|_{L^\infty}$, $\|\tilde\chi_1^{(j)}\|_{L^\infty}$, and $\|\chi_2^{(j)}\|_{L^\infty}$ for $j \le 2$.

Note that we could have proved the same type of estimate for the H^2 norm but not more since we only supposed $f \in W^{2,\infty}_{\text{loc}}$. However, if f is more regular, we can prove the estimate in higher regularity spaces.

Now, to come back to u, we choose the sets Ω_1, Ω_2 and the function χ_2 such that $\chi_2 = 1$ on $(-f'(N)t^*, \beta^* - \kappa) \subset \Omega_2 \subset \Omega_1 = (-f'(N)t^* - \kappa, \beta^*)$ for some constant $\kappa > 0$ and for $t^* > 0$ satisfying $\beta^* - \kappa > -f'(N)t^*$ (i.e., $[\mu s + (1-\mu)f'(N)]t^* > \kappa$). Note that $|\Omega_1| = \beta^* + \kappa + f'(N)t^* = \kappa + [\mu s + (1-\mu)f'(N)]t^*$ depends on t^*, on $f'(N)$, and on s. Similarly, $|\tilde\Omega_1| = f'(N)t^* + 2 + |\Omega_1|$. Estimate (28) now yields for all $\kappa > 0$ and $t^* > \frac{\kappa}{\mu s + (1-\mu)f'(N)}$

$$\|v(t^*,\cdot)\|_{H^1(-f'(N)t^*, \beta^*-\kappa)} \le D\ e^{-\frac{1}{\varepsilon}(\sigma_m - s)(f'(N)-s)(\mu-1)t^*},$$

with $D = C_\kappa\sqrt{D_2}$, where C_κ depends only on κ, and D_2 is defined in (29). Recalling the expression of β^* and v, we obtain

$$(30) \qquad
\begin{aligned}
\|u(t^*,\cdot) - N\|_{H^1(0, -\mu(f'(N)-s)t^* + f'(N)t^* - \kappa)} &\le D\ e^{-\frac{1}{\varepsilon}(\sigma_m - s)(f'(N)-s)(\mu-1)t^*}\\
\|u(t^*,\cdot) - N\|_{H^1(0, [(1-\mu)f'(N)+\mu s]t^* - \kappa)} &\le D\ e^{-\frac{1}{\varepsilon}(\sigma_m - s)(f'(N)-s)(\mu-1)t^*}.
\end{aligned}
$$

We now choose $\mu = 1 + \nu\frac{s}{f'(N)-s}$ with $\nu \in (0,1)$, so that

$$
\begin{cases}
(\mu-1) = \nu\frac{s}{f'(N)-s} > 0,\\
(1-\mu)f'(N) + \mu s = (1-\nu)s > 0.
\end{cases}
$$

Replacing this in (30) gives for any $\kappa > 0$, $\nu \in (0,1)$, and $t^* > \frac{\kappa}{(1-\nu)s}$

$$\|u(t^*,\cdot) - N\|_{H^1(0, (1-\nu)st^* - \kappa)} \le D\ e^{-\frac{\nu}{\varepsilon}s(\sigma_m - s)t^*}.$$
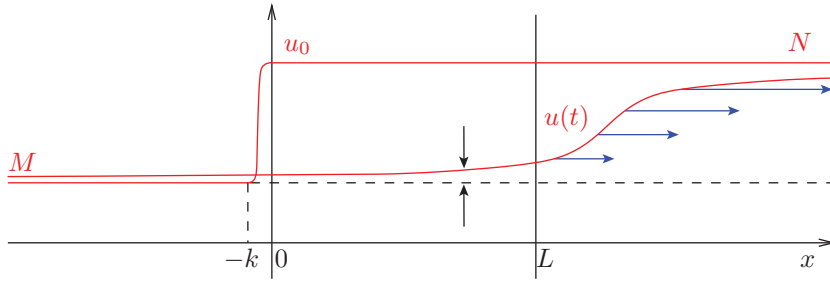
FIG. 3. *Rarefaction wave. Initial data and solution after a time $t > \frac{L}{f'(M)}$ has gone by.*

Finally, on $(0, L)$ we obtain, for any $\kappa > 0$, $\nu \in (0, 1)$, and $t^* > \frac{L + \kappa}{(1 - \nu)s}$,

$$\|u(t^*, \cdot) - N\|_{H^1(0,L)} \leq D \; e^{-\frac{\nu}{\varepsilon} s(\sigma_m - s)t^*}.$$

It remains only to estimate D to conclude the proof. As a consequence of (29) and the choice of the different parameters, we have

$$
\begin{aligned}
D \leq D^* := {} & C_\kappa \frac{N - P}{\varepsilon} \Big(1 + (\|f'\|_{L^\infty(P,N)} t^*)^{\frac{3}{2}} \Big) \\
& \times \Big(\frac{1}{\varepsilon^{\frac{1}{2}}} \|f'\|_{L^\infty(P,N)} + \varepsilon^{\frac{1}{2}} (1 + \|f'\|_{L^\infty(P,N)} t^*) \Big) \\
& \times \Big\{ \|f'\|_{L^\infty(P,N)} + \frac{1}{\varepsilon^{\frac{1}{2}}} (1 + \|f'\|_{L^\infty(P,N)} t^*) \|f''\|_{L^\infty(P,N)} \max(|P|, |N|) \\
& \quad + \big(1 + (\|f'\|_{L^\infty(P,N)} t^*)^{\frac{5}{2}} \big) \|f''\|_{L^\infty(P,N)} (t^*)^{\frac{1}{2}} \\
& \qquad \times \Big(\max(|P|, |N|) + \frac{1}{\varepsilon} \|f\|_{L^\infty(P,N)} \Big) \\
& \quad + \big((1 + \|f'\|_{L^\infty(P,N)} t^*) \|f'\|_{L^\infty(P,N)} + \varepsilon(1 + \|f'\|_{L^\infty(P,N)} t^*) \big) (t^*)^{\frac{1}{2}} \Big\}.
\end{aligned}
$$

(31)

In particular, we have $D \leq \frac{C}{\varepsilon^{\frac{5}{2}}} (1 + t^{*6})$ for all $\varepsilon \leq \varepsilon_0$ and all $t^* \geq 0$.   □

3.2. Proof of Proposition 2.3: Approximate controllability using a rarefaction wave. As for the proof of Proposition 2.3, the solution u is obtained by taking the restriction to $(0, t^*) \times (0, L)$ of the solution defined on whole $\mathbb{R}_+ \times \mathbb{R}$ (still denoted u) of the following problem (see Figure 3):

(32)
$$
\begin{cases}
u_t + [f(u)]_x - \varepsilon u_{xx} = 0 & \text{in } \mathbb{R}_+ \times \mathbb{R}, \\
u|_{t=0} = u_0 & \text{in } \mathbb{R},
\end{cases}
$$

with

(33)
$$
u_0(x) \begin{cases}
= M & \text{if } x \leq -k, \\
= N & \text{if } x \geq 0, \\
\in \mathcal{C}^\infty([-k, 0]), & \text{increasing},
\end{cases}
$$

for some (small) $k > 0$.

Then, the control functions g_0 and g_L are obtained by taking the traces of u along the lines $(0, t^*) \times \{0\}$ and $(0, t^*) \times \{L\}$. As in section 3.1, (11) follows directly from the comparison principle [12, Theorem 6.3.2], and we hence have only to prove (12).

set (see the definition in (43)). In particular, this yields $\Phi \in \mathcal{C}^0(\mathbb{R}_+; W^{1,\infty}(\mathbb{R}))$ and $\Phi v \in \mathcal{C}^0(\mathbb{R}_+; H^1(\mathbb{R}))$ since $v \in \mathcal{C}^0(\mathbb{R}_+; H^1(\mathbb{R}))$ from (37). As a consequence of (37), Φv satisfies the following equation:

$$(\Phi v)_t + f'(u)(\Phi v)_x - \varepsilon(\Phi v)_{xx} = v(\Phi_t + f'(u)\Phi_x) - \varepsilon v \Phi_{xx} - 2\varepsilon v_x \Phi_x + X.$$

Since Φ is constant along flow lines, the first term on the right-hand side vanishes. Next, taking the inner product of this equation with Φv yields

(41)
$$\frac{1}{2}\frac{d}{dt}\|\Phi v\|_{L^2}^2 + \int_{\mathbb{R}} f'(u)(\Phi v)_x \Phi v + \varepsilon\|(\Phi v)_x\|_{L^2}^2 = -\varepsilon \int_{\mathbb{R}} v^2 \Phi_{xx}\Phi - 2\varepsilon \int_{\mathbb{R}} v_x \Phi_x \Phi v + \int_{\mathbb{R}} X\Phi v.$$

In this expression, we have, after integrations by parts,

$$\int_{\mathbb{R}} f'(u)(\Phi v)_x \Phi v = -\frac{1}{2}\int_{\mathbb{R}} (f'(u))_x (\Phi v)^2$$

and

$$-\varepsilon \int_{\mathbb{R}} v^2 \Phi_{xx}\Phi - 2\varepsilon \int_{\mathbb{R}} v_x \Phi_x \Phi v = 2\varepsilon \int_{\mathbb{R}} v_x \Phi_x \Phi v + \varepsilon \int_{\mathbb{R}} (v\Phi_x)^2 - 2\varepsilon \int_{\mathbb{R}} v_x \Phi_x \Phi v$$

$$= \varepsilon \int_{\mathbb{R}} (v\Phi_x)^2.$$

Now (41) yields

(42)
$$\frac{1}{2}\frac{d}{dt}\|\Phi v\|_{L^2}^2 \leq \frac{1}{2}\int_{\mathbb{R}} (f'(u))_x (\Phi v)^2 + \varepsilon\|v\Phi_x\|_{L^2}^2 + \int_{\mathbb{R}} X\Phi v.$$

We choose the function g in (40) as

(43)
$$g(x) = \min\{R, d(x, [-k, 0])\} \in W^{1,\infty}(\mathbb{R})$$

for some constant $R > 0$. Notice that we have $\|g'\|_{L^\infty(\mathbb{R})} = 1$. Then, $\Phi_x = \alpha(\psi^{-1})_x g' \circ \psi^{-1} \Phi$ can be estimated thanks to the following lemma.

LEMMA 3.4. *Let u be the solution of* (32), (33), *and let ψ^{-1} be the backward flow associated with the vector field $f'(u)$. Then we have, for any $t \geq 0$ and $\varepsilon > 0$,*

(44)
$$0 \leq u_x(t, x) \leq \|u_{0,x}\|_{L^\infty(\mathbb{R})}$$

and

(45)
$$\|(\psi^{-1})_x(t, \cdot)\|_{L^\infty(\mathbb{R})} \leq 1.$$

Note that the estimates (44) and (45) are consequences of the fact that the solution of (32), (33) is a rarefaction wave. The proof of this lemma is postponed to the end of this proof.

As a consequence of Lemma 3.4, we now have

(46)
$$\|(f'(u))_x\|_{L^\infty(\mathbb{R})} \leq \|f''\|_{L^\infty(M,N)}\|u_{0,x}\|_{L^\infty(\mathbb{R})}$$

from (44) and

(47)
$$\|\alpha (\psi^{-1})_x g' \circ \psi^{-1}\|_{L^\infty(\mathbb{R}_+ \times \mathbb{R})} \leq \alpha\|(\psi^{-1})_x\|_{L^\infty}\|g'\|_{L^\infty} \leq \alpha$$

from (45). Coming back to (42), estimates (46) and (47) yield

(48)
$$\frac{1}{2}\frac{d}{dt}\|\Phi v\|_{L_x^2}^2 \le \frac{\|f''\|_{L^\infty(M,N)}\|u_{0,x}\|_{L^\infty(\mathbb{R})}}{2}\|\Phi v\|_{L_x^2}^2 + \varepsilon\alpha^2\|\Phi v\|_{L_x^2}^2 + \frac{1}{2}\|X\|_{L_x^2}^2 + \frac{1}{2}\|\Phi v\|_{L_x^2}^2.$$

Now, using Gronwall's lemma in

$$\frac{d}{dt}\|\Phi v\|_{L_x^2}^2 \le \left(2\varepsilon\alpha^2 + \|f''\|_{L^\infty(M,N)}\|u_{0,x}\|_{L^\infty(\mathbb{R})} + 1\right)\|\Phi v\|_{L_x^2}^2 + \|X\|_{L_x^2}^2$$

yields, for any $t^* > 0$,

(49)
$$\|\Phi v(t^*,\cdot)\|_{L^2}^2 \le \|X\|_{L^2(0,t^*;L^2(\mathbb{R}))}^2 \exp\left[\left(2\varepsilon\alpha^2 + \|f''\|_{L^\infty(M,N)}\|u_{0,x}\|_{L^\infty(\mathbb{R})} + 1\right)t^*\right],$$

since $(\Phi v)|_{t=0} = 0$. We define $\gamma(t^*)$ as in (35) so that (49) and (38) read

(50)
$$\|\Phi v(t^*,\cdot)\|_{L^2}^2 \le \gamma(t^*)^2 e^{2\varepsilon\alpha^2 t^*}.$$

Note that $\gamma(t^*)$ does not depend on ε, α, but only on ε_0 and k through the initial condition u_0 and w.

Let us now take $\eta \in (0, R)$. If $x \in \psi\left(t^*, (-\infty, -k-\eta) \cup (\eta, +\infty)\right)$, then

$$\psi^{-1}(t^*, x) \in (-\infty, -k-\eta) \cup (\eta, +\infty),$$

and $g \circ \psi^{-1}(t^*, x) \ge \eta$. Thus, $\Phi(t^*, x) = e^{\alpha g \circ \psi^{-1}(t^*,x)} \ge e^{\alpha\eta}$ so that

$$\|\Phi v(t^*,\cdot)\|_{L^2(\mathbb{R})} \ge e^{\alpha\eta}\|v(t^*,\cdot)\|_{L^2(\psi(t^*,(-\infty,-k-\eta)\cup(\eta,+\infty)))}.$$

This together with (50) gives

(51)
$$\|v(t^*,\cdot)\|_{L^2(\psi(t^*,(-\infty,-k-\eta)\cup(\eta,+\infty)))} \le \gamma(t^*)e^{\varepsilon\alpha^2 t^* - \alpha\eta},$$
$$\|v(t^*,\cdot)\|_{L^2((-\infty,\psi(t^*,-k-\eta))\cup(\psi(t^*,\eta),+\infty))} \le \gamma(t^*)e^{-\frac{\eta^2}{4\varepsilon t^*}}$$

after having chosen $\alpha = \frac{\eta}{2\varepsilon t^*}$. This inequality does not depend on R, so making $R \to +\infty$, we see that (51) holds for any $\eta > 0$.

It remains only to prove that $(-\infty, -k-\eta+f'(M)t^*) \cup (\eta+f'(N)t^*, +\infty) \subset (-\infty, \psi(t^*, -k-\eta)) \cup (\psi(t^*, \eta), +\infty))$. Actually, this is a direct consequence of

$$\psi(t^*,\eta) = \eta + \int_0^{t^*} f'(u)(t, \psi(t,\eta))dt \le \eta + f'(N)t^*,$$

$$\psi(t^*,-k-\eta) = -k-\eta + \int_0^{t^*} f'(u)(t, \psi(t,-k-\eta))dt \ge -k-\eta + f'(M)t^*,$$

where we have used the convexity of f and the comparison principle [12, Theorem 6.3.2] for the solutions of viscous conservation laws. This concludes the proof of Lemma 3.3. □

We now have to prove Lemma 3.4.

Proof of Lemma 3.4. First, we check that for any $\varepsilon > 0$ and $t \ge 0$, the speed $f'(u)$ is nondecreasing, i.e., $(f'(u))_x \ge 0$. Since $f'' \ge 0$, we have only to prove that u_x is nonnegative. The function $y = u_x$ is solution of

(52)
$$\begin{cases} y_t - \varepsilon y_{xx} + f'(u)y_x + f''(u)y^2 = 0 & \text{in } \mathbb{R}_+ \times \mathbb{R}, \\ y|_{t=0} = u_x|_{t=0} \ge 0 & \text{in } \mathbb{R}. \end{cases}$$

As a consequence of the weak maximum principle for parabolic equations (see, for instance, [14, p. 368]), we have $u_x(t,x) = y(t,x) \geq 0$ in $\mathbb{R}_+ \times \mathbb{R}$ and

(53) $$(f'(u))_x(t,x) \geq 0, \quad (t,x) \in \mathbb{R}_+ \times \mathbb{R}.$$

Second, since $f'' \geq 0$, (52) yields

$$\begin{cases} y_t - \varepsilon y_{xx} + f'(u)y_x \leq 0 & \text{in } \mathbb{R}_+ \times \mathbb{R}, \\ y|_{t=0} = u_x|_{t=0} \in L^\infty(\mathbb{R}). \end{cases}$$

For this system, the same maximum principle gives

$$u_x(t,x) = y(t,x) \leq \|y|_{t=0}\|_{L^\infty(\mathbb{R})} = \|u_x|_{t=0}\|_{L^\infty(\mathbb{R})}.$$

Hence, we have

$$0 \leq u_x(t,x) \leq \|u_x|_{t=0}\|_{L^\infty(\mathbb{R})}, \quad (t,x) \in \mathbb{R}_+ \times \mathbb{R},$$

and (44) is proved.

Finally, to prove (45), we recall that the backward flow ψ^{-1} is given by

$$\psi^{-1}(t,x) = x - \int_0^t f'(u)(s,\psi^{-1}(s,x))ds, \quad (t,x) \in \mathbb{R}_+ \times \mathbb{R},$$

which yields, taking the derivative with respect to x and t,

$$\begin{cases} \dfrac{d}{dt}(\psi^{-1})_x(t,x) = -(\psi^{-1})_x(t,x) \cdot (f'(u))_x(t,\psi^{-1}(t,x)), & (t,x) \in \mathbb{R}_+ \times \mathbb{R}, \\ (\psi^{-1})_x(0,x) = 1. \end{cases}$$

This can be rewritten under the form

$$(\psi^{-1})_x(t,x) = \exp\left(-\int_0^t (f'(u))_x(s,\psi^{-1}(s,x))ds\right),$$

which, thanks to (53), yields

$$0 \leq (\psi^{-1})_x(t,x) \leq 1, \quad (t,x) \in \mathbb{R}_+ \times \mathbb{R},$$

and Lemma 3.4 is proved. □

As a consequence of Lemma 3.3 and using a bootstrap argument, we are now able to prove the central result of this section.

Proof of Proposition 2.3. We study the evolution of (32), (33) for $t \in (0, t^*)$. First, we set

$$v(t,x) = (u - M) \circ \pi(t,x), \qquad \pi(t,x) = (t, x + f'(M)t),$$

for which estimate (34) of Lemma 3.3 yields, for any $\eta > 0$,

(54) $$\|v(t,\cdot)\|_{L^2(-\infty,-k-\eta)} \leq \gamma(t)e^{-\frac{\eta^2}{4\varepsilon t}}, \quad t \geq 0.$$

As in the proof of Proposition 2.1, we denote by Ω_1 a bounded open interval of $(-\infty, -k-\eta)$ and by $\chi_1 \in \mathcal{C}_c^\infty(\Omega_1)$ a cut-off function satisfying $\chi_1 = 1$ on Ω_2 with $\overline{\Omega_2} \subset \Omega_1$. The function $w_1(t,x) = \chi_1(x)v(t,x)$ satisfies

$$\begin{cases} w_{1,t} - \varepsilon w_{1,xx} = [f'(M) - f'(u \circ \pi)]\chi_1 v_x - \varepsilon(\chi_1'' v + 2\chi_1' v_x) & \text{in } (0,t^*) \times \Omega_1, \\ w_1 = 0 & \text{on } \partial\Omega_1, \\ w_1(0,x) = 0 & \text{in } \Omega_1. \end{cases}$$

The parabolic regularizing effect (see Lemma A.2 for $m = 0$) gives for this system

(55)
$$\varepsilon \int_0^{t^*} \|w_1\|_{H_0^1(\Omega_1)}^2 dt \le \frac{1}{\varepsilon} \int_0^{t^*} \|[f'(M) - f'(u \circ \pi)]\chi_1 v_x - \varepsilon(\chi_1'' v + 2\chi_1' v_x)\|_{H^{-1}(\Omega_1)}^2 dt.$$

Let us now estimate each of the terms on the right-hand side as in the proof of Proposition 2.1. Here, however, we see that thanks to Lemma 3.4, it is not necessary to perform a preliminary H^1 estimate on u, as opposed to the proof of Proposition 2.1. Using Estimate (89), the first term is

$$\|[f'(M) - f'(u \circ \pi)]\chi_1 v_x\|_{H^{-1}(\Omega_1)} \le C\Big(\|f'\|_{L^\infty(M,N)}$$
$$+ (1 + |\Omega_1|^{\frac{3}{2}})\|(f'(u \circ \pi))_x\|_{L^\infty(\Omega_1)}\Big)\|v\|_{L^2(\Omega_1)}.$$

As a consequence of Lemma 3.4, we have

$$\|(f'(u \circ \pi))_x\|_{L^\infty(\mathbb{R})} \le \|f''\|_{L^\infty(M,N)}\|u_{0,x}\|_{L^\infty(\mathbb{R})},$$

so that

$$\|[f'(M) - f'(u \circ \pi)]\chi_1 v_x\|_{H^{-1}(\Omega_1)} \le C(1 + |\Omega_1|^{\frac{3}{2}})\|v\|_{L^2(\Omega_1)},$$

where the constant $C > 0$ depends only on f, $u_{0,x}$, M, and N. Concerning the other terms, we have, using (86),

$$\varepsilon\|\chi_1'' v + 2\chi_1' v_x\|_{H^{-1}(\Omega_1)} \le \varepsilon C(|\Omega_1| + 1)\|v\|_{L^2(\Omega_1)},$$

where the constants denoted by C do not depend on Ω_1. Coming back to (55) and using (54), we now have, for $\varepsilon \in (0, \varepsilon_0)$,

(56)
$$\int_0^{t^*} \|w_1\|_{H_0^1(\Omega_1)}^2 dt \le \frac{C}{\varepsilon^2}\left(1 + |\Omega_1|^3\right) \int_0^{t^*} \|v\|_{L^2(\Omega_1)}^2 dt \le \frac{C}{\varepsilon^2}\left(1 + |\Omega_1|^3\right) t^* \gamma(t^*)^2 e^{-\frac{\eta^2}{2\varepsilon t^*}},$$

since $\Omega_1 \subset (-\infty, -k - \eta)$, and the functions $\gamma(t)$ and $e^{-\frac{\eta^2}{2\varepsilon t}}$ are nondecreasing with respect to t.

We now take $\chi_2 \in \mathcal{C}_c^\infty(\Omega_2)$ and set $w_2(t, x) = \chi_2(x)w_1(t, x) = \chi_2(x)v(t, x)$, which satisfies

$$\begin{cases} w_{2,t} - \varepsilon w_{2,xx} = [f'(M) - f'(u \circ \pi)]\chi_2 w_{1,x} - \varepsilon(\chi_2'' w_1 + 2\chi_2' w_{1,x}) & \text{in } (0, t^*) \times \Omega_1, \\ w_2 = 0 & \text{on } \partial\Omega_1, \\ w_2(0, x) = 0 & \text{in } \Omega_1. \end{cases}$$

The parabolic regularizing effect (see Lemma A.2 for $m = 1$) directly yields, for $\varepsilon \in (0, \varepsilon_0)$,

$$\|w_2(t^*, \cdot)\|_{H_0^1(\Omega_1)}^2 \le \frac{C}{\varepsilon} \int_0^{t^*} \|w_{1,x}\|_{L^2(\Omega_1)}^2 + \varepsilon^2\|w_1\|_{L^2(\Omega_1)}^2 dt$$
$$\le C\left(\frac{1}{\varepsilon} + |\Omega_1|^2\right) \int_0^{t^*} \|w_1\|_{H_0^1(\Omega_1)}^2 dt.$$

As a consequence of (56), we thus have, for $\varepsilon \in (0, \varepsilon_0)$,

(57)
$$\|w_2(t^*, \cdot)\|_{H_0^1(\Omega_1)}^2 \le \frac{C}{\varepsilon^3}\left(1 + |\Omega_1|^5\right) t^* \gamma(t^*)^2 e^{-\frac{\eta^2}{2\varepsilon t^*}}.$$

Now, to come back to u, we choose the sets Ω_1, Ω_2 and the function χ_2 such that $\chi_2 = 1$ on $(-f'(M)t^*, -\eta - 2k) \subset \Omega_2 \subset \Omega_1 = (-f'(M)t^* - k, -\eta - k) \subset (-\infty, -\eta - k)$. With this choice, we have $|\Omega_1| = f'(M)t^* - \eta$, and estimate (57) yields, for any $t^* > 0$ and $0 < \eta < f'(M)t^*$,

$$\|u(t^*, \cdot) - M\|_{H^1(0, -\eta - 2k + f'(M)t^*)} \leq \frac{C}{\varepsilon^{\frac{3}{2}}} \left(1 + |f'(M)t^* - \eta|^{\frac{5}{2}}\right) \sqrt{t^*} \gamma(t^*) e^{-\frac{\eta^2}{4\varepsilon t^*}}.$$

It remains to choose η so that $-\eta - 2k + f'(M)t^* = L$, that is, $\eta = f'(M)t^* - L - 2k$, which is positive as soon as $t^* > \frac{L+2k}{f'(M)}$. Finally, we have for any $t^* > \frac{L+2k}{f'(M)}$

$$\|u(t^*, \cdot) - M\|_{H^1(0, L)}$$
$$\leq \frac{C}{\varepsilon^{\frac{3}{2}}} \left(1 + (L + 2k)^{\frac{5}{2}}\right) \sqrt{t^*} \gamma(t^*) \exp\left(-\frac{1}{4\varepsilon t^*}(f'(M)t^* - L - 2k)^2\right),$$

and Proposition 2.3 is proved, setting $\kappa = 2k$ and $\delta(t^*) = C(1 + (L + 2k)^{\frac{5}{2}})\sqrt{t^*} \gamma(t^*)$. $\square$

Remark 3.5. This proposition and its proof need slight modifications when proving Proposition 1.7, since the right-hand side of (12) needs to be exponentially decreasing as $t^* \to +\infty$. For this, we first replace $\frac{1}{2}\|X\|_{L_x^2}^2 + \frac{1}{2}\|\Phi v\|_{L_x^2}^2$ in estimate (48) with $\frac{1}{2\mu}\|X\|_{L_x^2}^2 + \frac{\mu}{2}\|\Phi v\|_{L_x^2}^2$ for all $\mu > 0$. Choosing u_0 such that $\|u_{0,x}\|_{L^\infty(\mathbb{R})} = \frac{C}{k}$, estimate (12) for $\varepsilon = 1$ now reads, for all $k, \mu > 0$,

$$\|u(t^*, \cdot) - M\|_{H^1(0, L)}$$
$$\leq C(L + 2k)\sqrt{\frac{t^*}{\mu}} \|X\|_{L^2(0, t^*; L^2(\mathbb{R}))} e^{\left(\frac{C}{2k}\|f''\|_{L^\infty(M,N)} + \frac{\mu}{2}\right)t^* - \frac{1}{4t^*}(f'(M)t^* - L - 2k)^2}.$$

Noting that $\|X\|_{L^2(0, t^*; L^2(\mathbb{R}))}$ increases at most linearly in t^* and fixing k large enough and μ small enough so that $\left(\frac{C}{2k}\|f''\|_{L^\infty(M,N)} + \frac{\mu}{2}\right) < \frac{f'(M)^2}{4}$, we obtain the sought exponential decay as $t^* \to +\infty$.

3.3. Proof of Proposition 2.4: Local exact controllability. The proof of Proposition 2.4 follows the steps of [16]. When doing this, we shall see that one can take $\alpha_1 = 5$ (or, as found numerically, $\alpha_1 = 4.3$).

We first set $y(t, x) = u(t, x) - N$ so that y satisfies

(58) $$\begin{cases} y_t + [f(N + y)]_x - \varepsilon y_{xx} = 0 & \text{in } (0, T) \times (0, L), \\ y|_{t=0} = y_0 = u_0 - N & \text{in } (0, L), \\ \|y_0\|_{H^1(0, L)} \leq e^{-\frac{K_0}{\varepsilon}}. \end{cases}$$

Now, our objective is to find boundary controls $y|_{x=0} = g_0(t) - N$ and $y|_{x=L} = g_L(t) - N$ such that $y|_{t=T} = 0$ and

$$\|g_0 - N\|_{L^\infty(0, T)} \leq |N| \quad \text{and} \quad \|g_L - N\|_{L^\infty(0, T)} \leq |N|.$$

More precisely, we prove the existence of a controlled solution y, satisfying (58) and $y|_{t=T} = 0$ and then take the traces of y on $(0, T) \times \{0\}$ and $(0, T) \times \{L\}$ to obtain the controls. The existence of such a controlled solution is proved by means of a fixed point argument. For this, let us first consider the following linearization of system (58)

for some $z \in L^1(0,T;W^{1,\infty}(0,L)) \cap L^\infty((0,T) \times (0,L))$:

$$(59) \quad \begin{cases} y_t - \varepsilon y_{xx} + [\sigma(N + z(t,x),N)y]_x = 0 & \text{in } (0,T) \times (0,L), \\ y|_{x=0} = \tilde{g}_0, \quad y|_{x=L} = \tilde{g}_L & \text{in } (0,T), \\ y|_{t=0} = y_0 & \text{in } (0,L), \end{cases}$$

where we have denoted $\sigma(N+z,N) = \frac{f(N+z)-f(N)}{z}$, with $\sigma(N+\cdot,N) \in W^{1,\infty}_{loc}(\mathbb{R})$. Note that formally, a fixed point of a map $z \mapsto y$, where y is a solution of (59) associated to some controls $\tilde{g}_0, \tilde{g}_L$, is a solution of the problem (58). It will be convenient to extend this control problem to (a,b) for some $a < 0$ and $b > L$ and introduce $\tilde{y}_0$ and $\tilde{z}$ smooth extensions of y_0 and z, satisfying

$$(60) \quad \begin{array}{l} \tilde{y}_0 = y_0 \text{ on } (0,L), \quad \tilde{y}_0(a) = \tilde{y}_0(b) = 0, \text{ and } \quad \|\tilde{y}_0\|_{H^1_0(a,b)} \le C_E \|y_0\|_{H^1(0,L)}, \\ \tilde{z} = z \text{ on } (0,T) \times (0,L), \text{ and } \quad \|\tilde{z}\|_{L^1W^{1,\infty} \cap L^\infty L^\infty} \le C_E \|z\|_{L^1W^{1,\infty} \cap L^\infty L^\infty} \end{array}$$

(see, for instance, [14, section 5.4]). We now consider the extended linear system

$$(61) \quad \begin{cases} \tilde{y}_t - \varepsilon \tilde{y}_{xx} + [(f'(N) + \zeta(t,x))\tilde{y}]_x = 0 & \text{in } (0,T) \times (a,b), \\ \tilde{y}|_{x=a} = \tilde{g}(t), \quad \tilde{y}|_{x=b} = 0 & \text{in } (0,T), \\ \tilde{y}|_{t=0} = \tilde{y}_0 & \text{in } (a,b), \end{cases}$$

where ζ is defined by

$$\zeta(t,x) = \sigma(N + \tilde{z}(t,x),N) - f'(N).$$

To prove the null-controllability of this system, we shall prove an observability estimate for its adjoint. We set

$$(62) \quad \lambda = b - a > L.$$

We have the following controllability lemma.

LEMMA 3.6. *There exist $\alpha_1 > 0$ and $\varepsilon_0 > 0$ such that for all*

$$\zeta \in L^1(0,T;W^{1,\infty}(0,L)) \cap L^\infty((0,T) \times (0,L))$$

satisfying

$$(63) \quad \|\zeta_x\|_{L^1L^\infty} + \|\zeta\|_{L^\infty L^\infty} \le C_E\, e^{-\frac{K_0}{2\varepsilon}}$$

(where K_0 is introduced in Proposition 2.4 and C_E is the norm of the extension operator introduced in (60)), all $T \ge \alpha_1 \frac{\lambda}{f'(N)}$, all $\varepsilon \in (0,\varepsilon_0)$, and $\tilde{y}_0 \in H^1_0(a,b)$ satisfying $\|\tilde{y}_0\|_{H^1_0(a,b)} \le C_E\, e^{-\frac{K_0}{\varepsilon}}$, there exists a control function $\tilde{g} \in L^2(0,T)$ with $\|\tilde{g}\|_{L^2(0,T)} \le e^{-\frac{K_0}{\varepsilon}}$ such that the associated solution to (61) satisfies

$$\tilde{y}|_{t=T} = 0 \quad on \; (a,b).$$

Note that the constant α_1 here is the same as the one in Proposition 2.4. In the course of the proof, we shall see that one can take $\alpha_1 = 5$ or $\alpha_1 = 4.3$, as claimed before.

Proof. For this linear control problem (61), we use the classical approach, which consists in obtaining a suitable observability inequality for the adjoint system of (61), which reads

$$(64) \quad \begin{cases} -\varphi_t - \varepsilon \varphi_{xx} - (f'(N) + \zeta(t,x))\varphi_x = 0 & \text{in } (0,T) \times (a,b), \\ \varphi|_{x=a} = 0, \quad \varphi|_{x=b} = 0 & \text{in } (0,T), \\ \varphi|_{t=T} = \varphi_T & \text{in } (a,b), \end{cases}$$

where $\varphi_T \in L^2(a,b)$ is the final condition of this backward problem. We aim to prove the following observability inequality for the solutions of (64):

$$\tag{65} \|\varphi|_{t=0}\|_{L^2(a,b)} \leq K(T,\varepsilon)\|\varphi_x|_{x=a}\|_{L^2(0,T)}.$$

Then, classical duality arguments give the null-controllability of system (61) with a control function $\tilde{g}$ whose L^2 norm is bounded by

$$\frac{K(T,\varepsilon)}{\varepsilon}\|\tilde{y}_0\|_{H_0^1(a,b)} \leq C_E \frac{K(T,\varepsilon)}{\varepsilon} e^{-\frac{\kappa_0}{\varepsilon}}.$$

To prove (65), we mostly follow [16] and use two of their technical estimates. More precisely, once rescaled with respect to the parameters, the dissipation estimate and the Carleman estimate read as follows (λ is defined in (62)).

Dissipation result. For every $t \in (\frac{\lambda}{f'(N)-\|\zeta\|_{L^\infty L^\infty}}, T)$, we have

$$\tag{66} \|\varphi(0,\cdot)\|_{L^2(a,b)}^2$$

$$\leq \exp\left\{\frac{\lambda^2\|\zeta_x\|_{L^1 L^\infty}}{4} - \frac{((f'(N)-\|\zeta\|_{L^\infty L^\infty})t-\lambda)^2}{2\varepsilon t}e^{-4\lambda^2\|\zeta_x\|_{L^1 L^\infty}}\right\}\|\varphi(t,\cdot)\|_{L^2(a,b)}^2.$$

Note that $f'(N) - \|\zeta\|_{L^\infty L^\infty} > 0$ for ε sufficiently small (or $f'(N)$ sufficiently large in the proof of Proposition 3.8 below).

Carleman inequality. Assume that ζ satisfies (63) and $T > \frac{3}{2}\frac{\lambda}{f'(N)-\|\zeta\|_{L^\infty L^\infty}}$. Then, we have

$$\tag{67} \int_a^b \int_{2T/3}^{5T/6} |\varphi|^2 dt\, dx \leq Ce^{\frac{\kappa\lambda^2}{\varepsilon T}}\left(\int_0^T |\varphi_x(t,a)|^2 dt + \int_a^b |\varphi(0,x)|^2 dx\right).$$

These two estimates are proved in [16, section 4], with $\kappa = 4$ or $\kappa = 2.61$ as found numerically.

To obtain the observability inequality (65), we suppose that

$$T > \frac{3}{2}\frac{\lambda}{f'(N)-\|\zeta\|_{L^\infty L^\infty}}.$$

As a consequence (66) holds on $(\frac{2T}{3}, \frac{5T}{6}) \subset (\frac{\lambda}{f'(N)-\|\zeta\|_{L^\infty L^\infty}}, T)$. Integrating (66) on $(\frac{2T}{3}, \frac{5T}{6})$, we hence obtain

$$\frac{T}{6}\|\varphi(0,\cdot)\|_{L^2(a,b)}^2 \leq \exp\left\{\frac{\lambda^2\|\zeta_x\|_{L^1 L^\infty}}{4}\right\}$$

$$\times \int_{2T/3}^{5T/6} \exp\left\{-\frac{((f'(N)-\|\zeta\|_{L^\infty L^\infty})t-\lambda)^2}{2\varepsilon t}e^{-4\lambda^2\|\zeta_x\|_{L^1 L^\infty}}\right\}\|\varphi(t,\cdot)\|_{L^2(a,b)}^2 dt.$$

Using the fact that

$$t \mapsto \frac{((f'(N)-\|\zeta\|_{L^\infty L^\infty})t-\lambda)^2}{2\varepsilon t}$$

is an increasing function as soon as $t > \frac{\lambda}{f'(N)-\|\zeta\|_{L^\infty L^\infty}}$, together with (67), we have

$$\tag{68} \|\varphi(0,\cdot)\|_{L^2(a,b)}^2 \leq Ce^{-\frac{D(\varepsilon,T,\lambda,\zeta)}{\varepsilon T}}\left(\int_0^T |\varphi_x(t,a)|^2 dt + \int_a^b |\varphi(0,x)|^2 dx\right),$$

with

$$D(\varepsilon, T, \lambda, \zeta) = \frac{3e^{-4\lambda^2 \|\zeta_x\|_{L^1 L^\infty}}}{4} \left((f'(N) - \|\zeta\|_{L^\infty L^\infty}) \frac{2T}{3} - \lambda \right)^2 - \kappa \lambda^2.$$

Now we see that for T sufficiently large, i.e., $T \geq \alpha_1 \frac{\lambda}{f'(N)}$, we have $D(\varepsilon, T, \lambda, \zeta) > 0$ and we can absorb the last term in (68) by the left-hand side, taking $\varepsilon_0 = \varepsilon_0(\alpha_1)$ sufficiently small so that it works with $T = \alpha_1 \frac{\lambda}{f'(N)}$. We also notice that we can take $\alpha_1 = 5$ if $\kappa = 4$ or $\alpha_1 = 4.30$ if $\kappa = 2.61$. Finally, we obtain the observability inequality (65) with $K(T, \varepsilon) = Ce^{-\frac{D(\varepsilon, T, \lambda, \zeta)}{\varepsilon T}}$. This concludes the proof of Lemma 3.6.   □

Now given this result for the linearized system, we are able to implement a fixed point strategy to conclude the proof of Proposition 2.4.

Proof of Proposition 2.4. We first recall Kakutani's fixed point theorem as presented in [29, Theorem 9.2.2].

THEOREM 3.7. *Let $\mathcal{Z}$ be a Banach space, E a subset of $\mathcal{Z}$, and $\Lambda : E \to 2^{\mathcal{Z}}$ a multivalued mapping. Suppose the following:*
  (i) *E is compact, convex, and nonempty, and for every $z \in E$, $\Lambda(z) \subset E$.*
  (ii) *For every $z \in E$, $\Lambda(z)$ is a compact, convex, and nonempty subset of $\mathcal{Z}$.*
  (iii) *Λ is "upper-semicontinuous"; i.e., if $z_n \to z$ in E and $y_n \in \Lambda(z_n)$ satisfies $y_n \to y$ in $\mathcal{Z}$, then $y \in \Lambda(z)$.*
Then Λ has a fixed point in E; i.e., there exists $z \in E$ such that $z \in \Lambda(z)$.

Let us now define the appropriate space $\mathcal{Z}$, subset E, and mapping Λ. We choose

(69) $$\mathcal{Z} = H^{\frac{3}{4}}(0, T; L^2(0, L)) \cap L^2(0, T; H^1(0, L))$$

for the Banach space and

$$E^\varepsilon = \left\{ z \in H^1(0, T; L^2(0, L)) \cap L^2(0, T; H^2(0, L)), \ \|z\|_{H^1 L^2 \cap L^2 H^2} \leq e^{-\frac{K_0}{2\varepsilon}} \right\} \subset \mathcal{Z}.$$

Given a fixed $y_0 \in H^1(0, L)$ such that $\|y_0\|_{H^1(0, L)} \leq e^{-\frac{K_0}{\varepsilon}}$, we set

$$\Lambda(z) = \left\{ y \in H^1(0, T; L^2(0, L)) \cap L^2(0, T; H^2(0, L)) \text{ satisfying } (70)\text{--}(73) \right\}$$

with

(70) $$y_t - \varepsilon y_{xx} = -[\sigma(N + z, N)y]_x \quad \text{in } (0, T) \times (0, L),$$

(71) $$y|_{t=0} = y_0 \quad \text{in } (0, L),$$

(72) $$y|_{t=T} = 0 \quad \text{in } (0, L),$$

(73) $$\|y\|_{H^1 L^2 \cap L^2 H^2} \leq e^{-\frac{K_0}{2\varepsilon}}$$

and check that Kakutani's theorem applies with $\mathcal{Z}$, E^ε, and Λ for $\varepsilon > 0$ sufficiently small.

To prove (i), note that the compact injection of $H^1 L^2 \cap L^2 H^2$ in $\mathcal{Z}$ gives the compactness of E^ε in $\mathcal{Z}$, and the fact that it is a ball yields its convexity. Moreover, E^ε is nonempty since it contains the null function, and $\Lambda(z) \subset E^\varepsilon$ if $z \in E^\varepsilon$, as a consequence of their definition.

To prove (ii), notice first that $\Lambda(z)$ is convex since the conditions (70)–(72) are linear and (73) is convex. To prove that it is closed (and hence compact), let us consider a sequence $(y_n)_{n \in \mathbb{N}} \subset \Lambda(z)$ converging to y in $\mathcal{Z}$. Since $\mathcal{Z} \subset \mathcal{C}^0([0, T]; L^2(0, L))$,

conditions (71) and (72) are still valid for the limit y. Since the right-hand side of (73) does not depend on n, this estimate also holds for y. Moreover, y_n converges to y in $\mathcal{D}'((0,T) \times (0,L))$, and, hence, the linear equation (70) is satisfied by y in $\mathcal{D}'((0,T) \times (0,L))$. As a consequence of (73), $y \in H^1 L^2 \cap L^2 H^2$, so that y satisfies (70) in $L^2 L^2$, and $\Lambda(z)$ is closed.

Let us now prove that $\Lambda(z)$ is nonempty if $z \in E^\varepsilon$. We denote by $\tilde{z}$ and $\tilde{y}_0$ extensions of z and y_0 on (a,b) satisfying (60). Denoting $\tilde{y}$ as the associated solution of (61) (for any control $\tilde{g}$), we see that $\tilde{y}|_{(0,L)}$ solves (70)–(71). Moreover, we have $\|\tilde{y}_0\|_{H_0^1(a,b)} \le Ce^{-\frac{K_0}{\varepsilon}}$ and

$$\|\tilde{z}\|_{L^1 W^{1,\infty}(a,b) \cap L^\infty L^\infty(a,b)} \le C\|\tilde{z}\|_{L^1 W^{1,\infty}(0,L) \cap L^\infty L^\infty(0,L)}$$
$$\le C\|z\|_{H^1 L^2(0,L) \cap L^2 H^2(0,L)} \le Ce^{-\frac{K_0}{2\varepsilon}}.$$

Hence, denoting $\zeta(t,x) = \sigma(N + \tilde{z}, N) - f'(N)$, we have

$$(74) \qquad \|\zeta\|_{L^\infty L^\infty} \le \frac{\|f''\|_{L^\infty(K)}}{2}\|\tilde{z}\|_{L^\infty L^\infty} \le Ce^{-\frac{K_0}{2\varepsilon}}$$

for some compact $K \subset \mathbb{R}$ containing N, and

$$(75) \qquad \|\zeta_x\|_{L^1 L^\infty} = \|[\sigma(N + \tilde{z}, N)]_x\|_{L^1 L^\infty} \le \|f''\|_{L^\infty(K)}\|\tilde{z}_x\|_{L^1 L^\infty} \le Ce^{-\frac{K_0}{2\varepsilon}}.$$

As a consequence of (74) and (75), estimate (63) holds, and Lemma 3.6 applies as soon as $T \ge \alpha_1 \frac{\lambda}{f'(N)}$ (which we shall suppose in the following). In particular, for $\varepsilon < \varepsilon_0$, there exists a control function $\tilde{g}$ such that $\|\tilde{g}\|_{L^2(0,T)} \le Ce^{-\frac{K_0}{\varepsilon}}$ and the associated solution of (61) satisfies $\tilde{y}|_{t=T} = 0$, and thus $\tilde{y}|_{(0,L)}$ fulfills (72). Moreover, $\tilde{y}$ is defined as a transposition solution of (61) so that the regularity estimate (96) of Lemma A.3 gives, for some $C > 0$ independent of ε,

$$(76) \qquad \|\tilde{y}\|_{L^2 L^2(a,b)} \le \frac{C}{\sqrt{\varepsilon}}\left(\|\tilde{g}\|_{L^2(0,T)} + \|\tilde{y}_0\|_{L^2(a,b)}\right).$$

We now take open intervals Ω_1 and Ω_2 such that $[0,L] \subset \Omega_2 \subset \overline{\Omega}_2 \subset \Omega_1 \subset \overline{\Omega}_1 \subset (a,b)$ and a cut-off function $\chi_1 \in \mathcal{C}_c^\infty(\Omega_1)$ defined as before. The function $w_1 = \chi_1 \tilde{y}$ satisfies

$$\begin{cases} w_{1,t} - \varepsilon w_{1,xx} = -\chi_1[\sigma(N+\tilde{z},N)\tilde{y}]_x - 2\varepsilon\chi_1'\tilde{y}_x - \varepsilon\chi_1''\tilde{y} & \text{in } (0,T) \times \Omega_1, \\ w_1 = 0 & \text{in } (0,T) \times \partial\Omega_1, \\ w_1|_{t=0} = \chi_1\tilde{y}_0 & \in H_0^1(\Omega_1), \end{cases}$$

so that the parabolic regularity result of Lemma A.2, taken for $m = 0$, gives

$$\|w_1\|_{L^2 H_0^1(\Omega_1)} \le \frac{C}{\varepsilon}\Big(\|\chi_1\tilde{y}_0\|_{L^2(\Omega_1)} + \|\chi_1[\sigma(N+\tilde{z},N)\tilde{y}]_x$$
$$+ 2\varepsilon\chi_1'\tilde{y}_x + \varepsilon\chi_1''\tilde{y}\|_{L^2 H^{-1}(\Omega_1)}\Big).$$

The last two terms on the right-hand side can be estimated by $C\|\tilde{y}\|_{L^2 L^2(\Omega_1)}$, and we have

$$\|\chi_1[\sigma(N+\tilde{z},N)\tilde{y}]_x\|_{L^2 H^{-1}(\Omega_1)} \le C\|\sigma(N+\tilde{z},N)\tilde{y}\|_{L^2 L^2(\Omega_1)} \le C\|\tilde{y}\|_{L^2 L^2(a,b)},$$

since $\sigma(N + \cdot, N) \in L^\infty$ and $\tilde{z} \in L^\infty L^\infty$. As a consequence of (76), we now obtain

$$(77) \qquad \|w_1\|_{H^1 H^{-1}(\Omega_1) \cap L^2 H_0^1(\Omega_1) \cap L^\infty L^2(\Omega_1)} \le \frac{C}{\varepsilon^{\frac{3}{2}}}\left(\|\tilde{g}\|_{L^2(0,T)} + \|\tilde{y}_0\|_{L^2(a,b)}\right).$$

Taking another cut-off function $\chi_2 \in \mathcal{C}_c^\infty(\Omega_2)$ as before, the function $w_2 = \chi_2 w_1 = \chi_2 \tilde{y}$ satisfies

$$\begin{cases} w_{2,t} - \varepsilon w_{2,xx} = -\chi_2[\sigma(N+\tilde{z},N)w_1]_x - 2\varepsilon\chi_2' w_{1,x} - \varepsilon\chi_2'' w_1 & \text{in } (0,T)\times\Omega_2, \\ w_2 = 0 & \text{in } (0,T)\times\partial\Omega_2, \\ w_2|_{t=0} = \chi_2 w_1 & \in H_0^1(\Omega_2), \end{cases}$$

so that the parabolic regularity result of Lemma A.2, taken for $m = 1$, gives

$$(78) \quad \|w_2\|_{L^2(H^2\cap H_0^1(\Omega_2))\cap H^1 L^2(\Omega_2)}$$
$$\leq \frac{C}{\varepsilon}\left(\|\chi_2 w_1\|_{H_0^1(\Omega_2)} + \|\chi_2[\sigma(N+\tilde{z},N)w_1]_x + 2\varepsilon\chi_2' w_{0,x} + \varepsilon\chi_2'' w_1\|_{L^2 L^2(\Omega_2)}\right).$$

In this expression, we can estimate

$$\|\chi_2[\sigma(N+\tilde{z},N)w_1]_x\|_{L^2 L^2(\Omega_2)} \leq \|\sigma(N+\tilde{z},N)w_{0,x}\|_{L^2 L^2(\Omega_1)}$$
$$+ \|[\sigma(N+\tilde{z},N)]_x w_1\|_{L^2 L^2(\Omega_1)}$$
$$\leq \|\sigma(N+\tilde{z},N)\|_{L^\infty L^\infty}\|w_1\|_{L^2 H_0^1(\Omega_1)}$$
$$+ \|\sigma(N+\tilde{z},N)\|_{L^2 W^{1,\infty}}\|w_1\|_{L^\infty L^2(\Omega_1)},$$

where all the terms in the right-hand side are finite, since $\sigma(N+\cdot,N) \in L^\infty$, $\tilde{z} \in L^\infty L^\infty \cap L^2 W^{1,\infty}$, and w_1 satisfies (77). Replacing this in (78) and using (77) yields

$$\|w_2\|_{L^2(H^2\cap H_0^1(\Omega_2))\cap H^1 L^2(\Omega_2)} \leq \frac{C}{\varepsilon^{\frac{5}{2}}}\left(\|\tilde{g}\|_{L^2(0,T)} + \|\tilde{y}_0\|_{L^2(a,b)}\right),$$

and thus

$$\|\tilde{y}|_{(0,L)}\|_{L^2 H^2(0,L)\cap H^1 L^2(0,L)} \leq \|w_2\|_{L^2(H^2\cap H_0^1(\Omega_2))\cap H^1 L^2(\Omega_2)}$$
$$\leq \frac{C}{\varepsilon^{\frac{5}{2}}}\left(2e^{-\frac{K_0}{\varepsilon}} + e^{-\frac{K_0}{\varepsilon}}\right) \leq e^{-\frac{K_0}{2\varepsilon}}$$

for $0 < \varepsilon < \varepsilon_0$. This implies that $\tilde{y}|_{(0,L)}$ satisfies (73) for $0 < \varepsilon < \varepsilon_0$ and hence all conditions (70)–(73), and $\Lambda(z)$ is nonempty.

To prove (iii), let us consider two sequences, $z_n \to z$ in E^ε for the topology of $H^1 L^2 \cap L^2 H^2$, and $y_n \in \Lambda(z_n)$ satisfying $y_n \to y$ in $\mathcal{Z}$, and check that $y \in \Lambda(z)$. The three assertions (71)–(73) still hold for y since (71)–(73) do not depend on z_n, and in particular $y \in H^1 L^2 \cap L^2 H^2$. Let us denote R_n as the right-hand side of (70) for z_n and y_n and R as its counterpart for z and y. We have

$$R_n - R = [\sigma(N+z,N)]_x y + \sigma(N+z,N)y_x - [\sigma(N+z_n,N)]_x y_n - \sigma(N+z_n,N)y_{n,x},$$

and hence $\|R_n - R\|_{L^2 L^2} \leq A_n + B_n$, with

$$A_n = \|[\sigma(N+z,N)]_x y - [\sigma(N+z,N)]_x y_n\|_{L^2 L^2}$$
$$+ \|[\sigma(N+z,N)]_x y_n - [\sigma(N+z_n,N)]_x y_n\|_{L^2 L^2}$$
$$\leq \|[\sigma(N+z,N)]_x\|_{L^2 L^\infty}\|y - y_n\|_{L^\infty L^2}$$
$$+ \|[\sigma(N+z,N)]_x - [\sigma(N+z_n,N)]_x\|_{L^2 L^\infty}\|y_n\|_{L^\infty L^2}$$

and

$$B_n = \|\sigma(N+z,N)y_x - \sigma(N+z,N)y_{n,x}\|_{L^2 L^2}$$
$$+ \|\sigma(N+z,N)y_{n,x} - \sigma(N+z_n,N)y_{n,x}\|_{L^2 L^2}$$
$$\leq \|\sigma(N+z,N)\|_{L^\infty L^\infty}\|y_x - y_{n,x}\|_{L^2 L^2}$$
$$+ \|\sigma(N+z,N) - \sigma(N+z_n,N)\|_{L^\infty L^\infty}\|y_{n,x}\|_{L^2 L^2}.$$

Recalling that $z_n \to z$ in E^ε and $y_n \to y$ in $\mathcal{Z}$, we deduce $\lim_{n \to +\infty} A_n = 0$ since $y_n \to y$ in $H^{\frac{3}{4}} L^2 \subset L^\infty L^2$ and $\sigma(N + z_n, N) \to \sigma(N + z, N)$ in $L^2 W^{1,\infty}$. Moreover, $\lim_{n \to +\infty} B_n = 0$ since $y_n \to y$ in $L^2 H^1$ and $\sigma(N + z_n, N) \to \sigma(N + z, N)$ in $L^\infty L^\infty$. We can now write (70) for z_n and y_n as

$$(79) \qquad\qquad y_{n,t} - \varepsilon y_{n,xx} = R_n,$$

where the right-hand side R_n satisfies $R_n \to R$ in $L^2 L^2$. This together with (73) implies that when $n \to +\infty$ we have

$$(80) \qquad\qquad y_t - \varepsilon y_{xx} = R \quad \text{in } L^2 L^2,$$

so that (70) holds for the limit y. Finally, $y \in \Lambda(z)$, and condition (iii) is fulfilled.

Consequently, Kakutani's theorem applies, and there exists $y \in \Lambda(y)$; that is, for y_0 satisfying $\|y_0\|_{H^1(0,L)} \le e^{-\frac{K_0}{\varepsilon}}$, we have found a function y satisfying

$$
\begin{cases}
y_t - \varepsilon y_{xx} + [\sigma(N + y, N)y]_x = 0 & \text{in } (0,T) \times (0,L), \\
y|_{t=0} = y_0 & \text{in } (0,L), \\
y|_{t=T} = 0 & \text{in } (0,L), \\
\|y\|_{H^1 L^2 \cap L^2 H^2} \le e^{-\frac{K_0}{2\varepsilon}}.
\end{cases}
$$

It suffices now to take the control functions

$$g_0(t) := y|_{x=0}(t) + N \quad \text{and} \quad g_L(t) := y|_{x=L}(t) + N.$$

With this choice, we have $g_0, g_L \in L^\infty(0,T)$ since $H^1 L^2 \cap L^2 H^2 \subset H^{\frac{3}{5}} H^{\frac{4}{5}}$ and

$$\max(\|g_0\|_{L^\infty(0,T)}, \|g_L\|_{L^\infty(0,T)}) \le |N| + C e^{-\frac{K_0}{2\varepsilon}} \le 2|N|$$

for $0 < \varepsilon < \varepsilon_0$. In addition, this proof works for any time of control $T \ge \alpha_1 \frac{\lambda}{f'(N)}$ for any $\lambda > L$ (but all the constants depend on λ). This yields the result for any $T \ge \alpha_1 \frac{L}{f'(N)}$ and concludes the proof of Proposition 2.4. $\qquad\square$

To conclude this section, we give a slight modification of Proposition 2.4, adapted to the first phase of the proof of Theorem 1.1, which consists in reaching exactly a large state N, without assuming ε small. To this aim, the strategy adopted here is the same as that of Proposition 2.4 except that we take "N large" instead of "ε small" in the fixed point argument. To do so, we carefully keep track of the dependence with respect to the parameter N, as $N \to +\infty$.

PROPOSITION 3.8. *There exists $\alpha_1 > 0$ satisfying the following property. For every $f \in W^{2,\infty}_{loc}(\mathbb{R})$ satisfying assumption (A+), for all $\varepsilon_0 > 0$, there exist $N_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$, $N > N_0$, $T \ge \alpha_1 \frac{L}{f'(N)}$, and $u_0 \in H^1(0,L)$ satisfying for some $P < N_0$*

$$(81) \qquad\qquad \|u_0 - N\|_{H^1(0,L)} \le D^* \, e^{-K_0 s(\sigma_m - s)}$$

(where D^* is defined in (31)) with $s = \sigma(P, N)$, $\sigma_m = \sigma(E, N)$, and $P < E$, there exist two control functions g_0 and g_L such that the result of Proposition 2.4 holds.

To use this after Proposition 2.1, we take $E = \text{ess inf } u_0 > P$. We now sketch the proof of this proposition.

Proof. Here, we only replace the argument "ε small" in the previous proof by "N large." This is possible since assumption (A+) implies the existence of $Q \in \mathbb{R}$, $l_0, C > 0$, $\gamma > -\frac{1}{2}$, and $\delta > 0$ such that

$$f''(A) \ge l_0 A^\gamma \quad \text{and} \quad f''(A) \le C e^{A^{2\gamma+1-\delta}}$$

for all $A > Q$. A first consequence is that

$$\|f''\|_{L^\infty(P,N)} \le Ce^{N^{2\gamma+1-\delta}}; \ \|f'\|_{L^\infty(P,N)} \le Ce^{CN^{2\gamma+1-\delta}}; \ \|f\|_{L^\infty(P,N)} \le Ce^{CN^{2\gamma+1-\delta}}.$$

As the quantity D^* (defined in (31)) is bounded by some polynomial of N and $\|f^{(j)}\|_{L^\infty(P,N)}$, with $j = 0, 1, 2$, this yields the existence of a constant C such that

$$(82) \qquad\qquad\qquad\qquad D^* \le Ce^{CN^{2\gamma+1-\delta}}.$$

Moreover, since we have

$$f(N) = f(Q) + (N - Q)f'(Q) + \int_Q^N f''(\tau)(N - \tau)d\tau \ge \frac{l_0}{(\gamma+1)(\gamma+2)}N^{\gamma+2} + O(N),$$

as $N \to +\infty$, we can estimate

$$\sigma_m - s \ge (E-P)\frac{f(N)}{N^2} + O\left(\frac{1}{N}\right) \ge (E-P)\frac{l_0}{(\gamma+1)(\gamma+2)}N^\gamma + O\left(\frac{1}{N}\right), \quad N \to +\infty,$$

and

$$s \ge (E-P)\frac{f(N)}{2N} + O\left(\frac{1}{N}\right) \ge \frac{l_0}{2(\gamma+1)(\gamma+2)}N^{\gamma+1} + O(1), \quad N \to +\infty,$$

so that

$$(83) \qquad\qquad\qquad e^{-K_0 s(\sigma_m - s)} \le Ce^{-\frac{K_0 l_0^2 (E-P)}{2(\gamma+1)^2(\gamma+2)^2}N^{2\gamma+1}}.$$

Since we supposed $\gamma > -\frac{1}{2}$, this expression is exponentially decaying as $N \to +\infty$. It suffices now to follow the proof of Proposition 2.4, replacing the exponential decay $\|u_0 - N\|_{H^1(0,L)} \le e^{-\frac{K_0}{\varepsilon}}$ as $\varepsilon \to 0^+$, by the exponential decay

$$\|u_0 - N\|_{H^1(0,L)} \le D^* e^{-K_0 s(\sigma_m - s)} \le Ce^{CN^{2\gamma+1-\delta}} e^{-\frac{K_0 l_0^2 (E-P)}{2(\gamma+1)^2(\gamma+2)^2}N^{2\gamma+1}} \le e^{-K_1 N^{2\gamma+1}},$$

as $N \to +\infty$, for some $K_1 > 0$, as a consequence of assumption (A+) and estimates (82) and (83). We also have to systematically use $e^{-\frac{K_1}{2}N^{2\gamma+1}}$ in place of $e^{-\frac{K_0}{2\varepsilon}}$ and $N > N_0$ in place of $\varepsilon < \varepsilon_0$ in the proof Proposition 2.4. The analogues of estimates (74) and (75) are performed using the fact that, in these expressions, K is the ball centered on N of radius smaller than $e^{-K_1 N^{2\gamma+1}}$. ☐

Remark 3.9. This is the only place where we use the unnatural assumption (A+). Any other condition on f allowing us to perform the fixed point argument is sufficient for proving Theorem 1.1. Note that if we had proved a decay rate of the form $\theta s(f'(N) - s)$, as mentioned in Remark 3.2, estimate (83) would be replaced by $e^{-K_0 s(f'(N)-s)} \le Ce^{-CN^{2\gamma+2}}$. This explains the more general condition in Remark 3.2.

Appendix. Classical inequalities and parabolic regularity estimates.

A.1. Classical inequalities with explicit constants. In this section, we recall some definitions of norms and give classical inequalities with explicit constants in one space dimension that are used in the main part of the article.

Given that Ω is a bounded domain of $\mathbb{R}^n$ with smooth boundary, we denote by $(\omega_j^2, \phi_j)_{j \in \mathbb{N}^*}$ the (positive) eigenvalues and associated eigenfunctions of the Laplace–

Dirichlet operator $-\Delta_D$ on Ω, forming a Hilbert basis of $L^2(\Omega)$. We have $-\Delta\phi_j = \omega_j^2\phi_j$, $\phi_j|_{\partial\Omega} = 0$, and $(\phi_j, \phi_k)_{L^2(\Omega)} = \delta_{jk}$.

We denote by $H_0^1(\Omega)$ the closure of $\mathcal{C}_c^\infty(\Omega)$ in $H^1(\Omega)$, endowed with the norm

$$\text{(84)} \qquad \|v\|_{H_0^1(\Omega)} := \|(-\Delta_D)^{1/2}v\|_{L^2(\Omega)} = \|\nabla v\|_{L^2(\Omega)} = \left(\sum_{j\in\mathbb{N}^*}\omega_j^2|v_j|^2\right)^{1/2},$$

where $v_j = (v, \phi_j)_{L^2(\Omega)}$. We also denote by $H^{-1}(\Omega)$ the dual space of $H_0^1(\Omega)$, endowed with the norm

$$\|v\|_{H^{-1}(\Omega)} := \|(-\Delta_D)^{-1/2}v\|_{L^2(\Omega)} = \sup_{\|\varphi\|_{H_0^1(\Omega)}=1}\langle v, \varphi\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}$$

$$\text{(85)} \qquad = \left(\sum_{j\in\mathbb{N}^*}\omega_j^{-2}|v_j|^2\right)^{1/2},$$

where $v_j = \langle v, \phi_j\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}$.

PROPOSITION A.1. *Suppose that $\Omega = (a, b)$ with $a, b \in \mathbb{R}$, $a < b$, and set $|\Omega| = b - a$. Then, we have*

$$\text{(86)} \qquad \|v\|_{L^2(\Omega)}$$
$$\leq \frac{|\Omega|}{\pi}\|v\|_{H_0^1(\Omega)} \text{ for all } v \in H_0^1(\Omega) \quad \textit{(Poincaré inequality)},$$

$$\text{(87)} \qquad \|v\|_{L^\infty(\Omega)}^2$$
$$\leq \|v\|_{L^2}^2 + \|v\|_{H_0^1(\Omega)}^2 \leq \left(1 + \frac{|\Omega|^2}{\pi^2}\right)\|v\|_{H_0^1(\Omega)}^2 \text{ for all } v \in H_0^1(\Omega),$$

$$\text{(88)} \qquad \|uv\|_{H_0^1(\Omega)} \leq \left(\|u\|_{L^\infty(\Omega)} + \left(1 + \frac{|\Omega|}{\pi}\right)\|u_x\|_{L^2(\Omega)}\right)\|v\|_{H_0^1(\Omega)},$$
$$\textit{for all } u \in H^1(\Omega), v \in H_0^1(\Omega),$$

$$\text{(89)} \qquad \|uv\|_{H^{-1}(\Omega)} \leq \left(\|u\|_{L^\infty(\Omega)} + \left(1 + \frac{|\Omega|}{\pi}\right)\|u_x\|_{L^2(\Omega)}\right)\|v\|_{H^{-1}(\Omega)},$$
$$\textit{for all } u \in H^1(\Omega), v \in H^{-1}(\Omega).$$

Proof. Inequality (86) is a direct consequence of the explicit knowledge of the eigenvalues of the Laplace–Dirichlet operator in one dimension, namely, $\omega_j = \frac{j\pi}{|\Omega|}$, $j \in \mathbb{N}^*$. Inequality (87) is obtained by writing, for $v \in \mathcal{C}_c^1(a, b)$ and $x \in (a, b)$,

$$|v(x)|^2 = \int_a^x \partial_y(|v(y)|^2)dy = \int_a^x 2v(y)v_y(y)dy \leq \int_a^b |v(y)|^2 dy + \int_a^b |v_y(y)|^2 dy$$

and concluding with a density argument. Inequality (88) then directly follows

$$\|(uv)_x\|_{L^2}^2 = \int_a^b (u_xv + uv_x)^2$$
$$\leq \|u_x\|_{L^2}^2\|v\|_{L^\infty}^2 + \|u\|_{L^\infty}^2\|v\|_{H_0^1}^2 + 2\|u_x\|_{L^2}\|v\|_{L^\infty}\|u\|_{L^\infty}\|v\|_{H_0^1},$$

together with (87) and $H^1(\Omega) \hookrightarrow L^\infty(\Omega)$ in dimension one. Finally, (89) is a direct consequence of (88) and the definition (85) of H^{-1} as the dual space of H_0^1. $\quad\square$

A.2. Parabolic regularity estimates for classical solutions. For the sake of completeness, we recall here a classical regularity result for the heat equation that is used in the previous sections. However, we give a precise dependence with respect to ε and a simple proof. For $m \geq 0$, we denote by $\mathcal{H}^m = \mathcal{D}((-\Delta_D)^{\frac{m}{2}})$ the domain of the fractional Laplace–Dirichlet operator on $L^2(\Omega)$, and $\mathcal{H}^{-m} = (\mathcal{H}^m)'$. Here the duality is taken with respect to $\mathcal{H}^0 = L^2(\Omega)$. For instance, we have $\mathcal{H}^1 = H_0^1(\Omega)$, $\mathcal{H}^2 = H^2(\Omega) \cap H_0^1(\Omega)$, and $\mathcal{H}^{-1} = H^{-1}(\Omega) = (H_0^1(\Omega))'$. We have the following regularity result.

LEMMA A.2. *Let $u(t,x)$ be a classical solution of*

$$(90) \qquad \begin{cases} \partial_t u - \varepsilon \Delta u = f & in\ (0,t^*) \times \Omega, \\ u = 0 & on\ (0,t^*) \times \partial\Omega, \\ u|_{t=0} = u_0 & in\ \Omega, \end{cases}$$

and let $m \in \mathbb{R}$. Suppose that $u_0 \in \mathcal{H}^m$ and $f \in L^2(0,t^*;\mathcal{H}^{m-1})$; then

$$u \in \mathcal{C}^0(0,t^*;\mathcal{H}^m) \cap L^2(0,t^*;\mathcal{H}^{m+1}) \cap H^1(0,t^*;\mathcal{H}^{m-1}),$$

and we have, for all $t^* > 0$,

$$(91) \quad \varepsilon \|u(t^*)\|_{\mathcal{H}^m}^2 + \varepsilon^2 \int_0^{t^*} \|u(t)\|_{\mathcal{H}^{m+1}}^2 dt + \int_0^{t^*} \|u_t(t)\|_{\mathcal{H}^{m-1}}^2 dt$$
$$= \varepsilon \|u_0\|_{\mathcal{H}^m}^2 + \int_0^{t^*} \|f(t)\|_{\mathcal{H}^{m-1}}^2 dt.$$

Proof. We suppose that $m \geq 1$. In the case $m \leq 1$, we do the following computations with smooth functions f and u and conclude with a density argument. Taking the inner product of (90) with ϕ_j, we obtain

$$\begin{cases} u_{j,t} + \varepsilon \omega_j^2 u_j = f_j\ in\ (0,t^*), \\ u_j|_{t=0} = u_{j,0}, \end{cases}$$

where $u_j = (u,\phi_j)_{L^2}$, $u_{j,0} = (u_0,\phi_j)_{L^2}$ and $f_j = (f,\phi_j)_{L^2}$. Multiplying the first equation by ω_j^{m-1} and squaring it, we have

$$\omega_j^{2m-2} |u_{j,t}|^2 + \varepsilon^2 \omega_j^{2m+2} |u_j|^2 + 2\varepsilon \omega_j^{2m} u_j u_{j,t} = \omega_j^{2m-2} |f_j|^2.$$

Integrating this equation on $(0,t^*)$, we now obtain

$$(92) \quad \omega_j^{2m-2} \int_0^{t^*} |u_{j,t}|^2 dt + \varepsilon^2 \omega_j^{2m+2} \int_0^{t^*} |u_j|^2 dt + \varepsilon \omega_j^{2m} |u_j|^2(t^*)$$
$$= \varepsilon \omega_j^{2m} |u_{j,0}|^2 + \omega_j^{2m-2} \int_0^{t^*} |f_j|^2 dt.$$

The right-hand side is summable over $j \in \mathbb{N}^*$ since we supposed $\sum_{j \in \mathbb{N}^*} \omega_j^{2m} |u_{j,0}|^2 = \|u_0\|_{\mathcal{H}^m}^2 < +\infty$ and $\sum_{j \in \mathbb{N}^*} \omega_j^{2m-2} \int_0^{t^*} |f_j|^2 dt = \|f(t)\|_{L^2(0,t^*;\mathcal{H}^{m-1})}^2 < +\infty$. Hence, the left-hand side is summable. Since its three terms are nonnegative, they are all summable. Finally, summing (92) over $j \in \mathbb{N}^*$ yields (91), and the lemma is proved. ∎

A.3. Well-posedness of an initial-boundary value problem with low regularity. Here, we prove the well-posedness of problem (61) in some energy space and give the associated regularity estimates used in (76). More precisely, we consider the problem

(93)
$$\begin{cases} u_t - \varepsilon u_{xx} + [W(t,x)u]_x = 0 & \text{in } (0,T) \times (a,b), \\ u|_{x=a} = g(t), \quad u|_{x=b} = 0 & \text{in } (0,T), \\ u|_{t=0} = u_0 & \text{in } (a,b), \end{cases}$$

with W and g having low regularity, and show that (93) is well-posed in a weak sense, namely, in the sense of transposition solutions. We have the following result.

LEMMA A.3. *Suppose that* $W \in L^\infty(0,T; L^\infty(a,b)) \cap L^1(0,T; W^{1,\infty}(a,b))$. *Then for any* $g \in L^2(0,T)$ *and* $u_0 \in L^2(a,b)$, *there exists a unique* $u \in L^2(0,T; L^2(a,b))$ *such that*

(94)
$$\int_0^T \int_a^b u(t,x) F(t,x) dx\, dt = \int_a^b u_0(x)\varphi(0,x) dx + \varepsilon \int_0^T g(t)\varphi_x(t,a) dt$$

for every test function $F \in L^2(0,T; L^2(a,b))$ *and associated* φ *satisfying*

(95)
$$\begin{cases} -\varphi_t - \varepsilon\varphi_{xx} - W(t,x)\varphi_x = F(t,x) & \text{in } (0,T) \times (a,b), \\ \varphi|_{x=a} = 0, \quad \varphi|_{x=b} = 0 & \text{in } (0,T), \\ \varphi|_{t=T} = 0 & \text{in } (a,b) \end{cases}$$

in the classical sense. Moreover, there exists $C = C(T, \|W\|_{L^\infty L^\infty}, \|W_x\|_{L^1 L^\infty}) > 0$ *independent of* ε *such that*

(96)
$$\|u\|_{L^2(0,T;L^2(a,b))} \le \frac{C}{\sqrt{\varepsilon}} \left(\|g\|_{L^2(0,T)} + \|u_0\|_{L^2(a,b)} \right).$$

The solution u is called the transposition solution of (93). Note that u is in particular a solution of the first equation of (93) in the sense of distributions.

Proof. First notice that for any $W \in L^\infty(0,T; L^\infty(a,b)) \cap L^1(0,T; W^{1,\infty}(a,b))$ and $F \in L^2(0,T; L^2(a,b))$, the backward problem (95) is well-posed in the classical sense and a regularity estimate for its solution $\varphi \in H^1(0,T; L^2(a,b)) \cap L^2(0,T; H^2 \cap H_0^1(a,b))$ is given in Lemma A.4 below.

We now remark that, given $u_0 \in L^2(a,b)$ and $g \in L^2(0,T)$, the mapping

$$l : F \mapsto \int_a^b u_0(x)\varphi(0,x) dx + \varepsilon \int_0^T g(t)\varphi_x(t,a) dt$$

is linear. Furthermore, we have, using (97) below,

$$\begin{aligned} |l(F)| &\le \|u_0\|_{L^2(a,b)} \|\varphi(0,\cdot)\|_{L^2(a,b)} + \varepsilon\|g\|_{L^2(0,T)} \|\varphi_x(\cdot,a)\|_{L^2(0,T)} \\ &\le \|u_0\|_{L^2(a,b)} \|\varphi\|_{H^1(0,T;L^2(a,b))} + \varepsilon\|g\|_{L^2(0,T)} \|\varphi\|_{L^2(0,T;H^2(a,b))} \\ &\le \frac{C}{\sqrt{\varepsilon}} \left(\|u_0\|_{L^2(a,b)} + \|g\|_{L^2(0,T)} \right) \|F\|_{L^2 L^2}. \end{aligned}$$

As a consequence l is a continuous linear form on $L^2 L^2$, and the Riesz representation theorem gives the existence of a unique $u \in L^2 L^2$ satisfying (94) for every $F \in L^2 L^2$. In addition, we have

$$\|u\|_{L^2 L^2} \le \frac{C}{\sqrt{\varepsilon}} \left(\|u_0\|_{L^2(a,b)} + \|g\|_{L^2(0,T)} \right),$$

and the lemma is proved. □

Note that refining our estimates, we could have proved that $u \in L^2(0,T; H^{\frac{1}{2}-\delta}(a,b))$ for any $\delta > 0$ (which we do not need here).

LEMMA A.4. *Suppose that* $W \in L^\infty(0,T; L^\infty(a,b)) \cap L^1(0,T; W^{1,\infty}(a,b))$, $F \in L^2(0,T; L^2(a,b))$, *and* φ *is a solution of the backward problem* (95). *Then* $\varphi \in L^\infty(0,T; H_0^1) \cap L^2(0,T; H^2 \cap H_0^1) \cap H^1(0,T; L^2)$, *and we have for some* $C = C(T, \|W\|_{L^\infty L^\infty}, \|W_x\|_{L^1 L^\infty})$, *independent of* ε,

$$(97) \quad \varepsilon^2 \|\varphi\|_{L^\infty(0,T; H_0^1)}^2 + \varepsilon^3 \|\varphi\|_{L^2(0,T; H^2 \cap H_0^1)}^2 + \varepsilon \|\varphi_t\|_{L^2(0,T; L^2)}^2 \leq C \|F\|_{L^2(0,T; L^2)}^2.$$

Proof. Multiplying the first line of (95) by φ and integrating on (a,b) yields

$$-\frac{1}{2}\frac{d}{dt}\|\varphi(t,\cdot)\|_{L^2}^2 + \varepsilon\|\varphi_x(t,\cdot)\|_{L^2}^2 - \int_a^b W(t,x)\left(\frac{\varphi(t,x)^2}{2}\right)_x dx = \int_a^b F(t,x)\varphi(t,x)\,dx,$$

which, after an integration by parts, gives the estimate

$$(98) \qquad -\frac{1}{2}\frac{d}{dt}\|\varphi(t,\cdot)\|_{L^2}^2 + \varepsilon\|\varphi_x(t,\cdot)\|_{L^2}^2 \leq \frac{1}{2}\|F(t,\cdot)\|_{L^2}^2 + \frac{1}{2}\|\varphi(t,\cdot)\|_{L^2}^2$$
$$+ \|W_x(t,\cdot)\|_{L^\infty}\|\varphi(t,\cdot)\|_{L^2}^2.$$

Forgetting the term $\varepsilon\|\varphi_x(t,\cdot)\|_{L^2}^2$ and applying Gronwall's lemma backward in time yields

$$(99) \qquad \|\varphi\|_{L^\infty(0,T; L^2)}^2 \leq \frac{1}{2}\exp\left(\|W_x\|_{L^1(0,T; L^\infty)} + \frac{T}{2}\right)\|F(t,\cdot)\|_{L^2(0,T; L^2)}^2$$

since $\varphi|_{t=T} = 0$. Now integrating (98) on $(0,T)$ and using (99), we obtain

$$(100) \quad \varepsilon\|\varphi_x\|_{L^2(0,T; L^2)}^2$$
$$\leq \left[\frac{1}{2} + \left(\frac{T}{4} + \frac{\|W_x\|_{L^1(0,T; L^\infty)}}{2}\right)\exp\left(\|W_x\|_{L^1(0,T; L^\infty)} + \frac{T}{2}\right)\right]\|F\|_{L^2(0,T; L^2)}^2.$$

Next, we write the first equation of (95) as

$$-\varphi_t - \varepsilon\varphi_{xx} = W\varphi_x + F,$$

where the right-hand side is in $L^2(0,T; L^2(a,b))$ since $W \in L^\infty(0,T; L^\infty(a,b))$ and $\varphi_x \in L^2(0,T; L^2(a,b))$. The regularity estimate (91) of Lemma A.2 for the heat equation (taken backward in time) for $m = 1$ directly yields estimate (97) and concludes the proof of the lemma.   □

REFERENCES

[1] F. ANCONA AND A. MARSON, *On the attainable set for scalar nonlinear conservation laws with boundary control*, SIAM J. Control Optim., 36 (1998), pp. 290–312.
[2] C. BARDOS, A. LE ROUX, AND J.-C. NÉDÉLEC, *First order quasilinear equations with boundary conditions*, Comm. Partial Differential Equations, 4 (1979), pp. 1017–1034.

[3] A. Bruckner and K. Garg, *The level structure of a residual set of continuous functions*, Trans. Amer. Math. Soc., 232 (1977), pp. 307–321.

[4] M. Chapouly, *Global controllability of nonviscous Burgers type equations*, C. R. Math. Acad. Sci. Paris, 344 (2007), pp. 241–246.

[5] M. Chapouly, *Global controllability of nonviscous and viscous Burgers-type equations*, SIAM J. Control Optim., 48 (2009), pp. 1567–1599.

[6] M. Chapouly, *On the global null controllability of a Navier-Stokes system with Navier slip boundary conditions*, J. Differential Equations, 247 (2009), pp. 2094–2123.

[7] J.-M. Coron and A. Fursikov, *Global exact controllability of the 2-D Navier-Stokes equations on a manifold without boundary*, Russian J. Math. Phys., 4 (1996), pp. 429–448.

[8] J.-M. Coron and S. Guerrero, *Singular optimal control: A linear 1-D parabolic-hyperbolic example*, Asymptot. Anal., 44 (2005), pp. 237–257.

[9] J.-M. Coron, *Control and Nonlinearity*, Math. Surveys Monogr. 136, AMS, Providence, RI, 2007.

[10] J. Coron, *On the controllability of the 2-D incompressible Navier-Stokes equations with the Navier slip boundary conditions*, ESAIM Control Optim. Calc. Var., 1 (1996), pp. 35–75.

[11] J. Coron, *Some open problems on the control of nonlinear partial differential equations*, in Perspectives in Nonlinear Partial Differential Equations, Contemp. Math. 446, AMS, Providence, RI, 2007, pp. 215–243.

[12] C. Dafermos, *Hyperbolic Conservation Laws in Continuum Physics*, Springer-Verlag, Berlin, 2000.

[13] R. Danchin, *Poches de tourbillon visqueuses*, J. Math. Pures Appl., 76 (1997), pp. 609–647.

[14] L. Evans, *Partial Differential Equations*, Grad. Stud. Math., AMS, Providence, RI, 1998.

[15] A. V. Fursikov and O. Y. Imanuvilov, *Controllability of Evolution Equations*, Lecture Notes Ser. 34, Seoul National University Research Institute of Mathematics Global Analysis Research Center, Seoul, Korea, 1996.

[16] O. Glass and S. Guerrero, *On the uniform controllability of the Burgers equation*, SIAM J. Control Optim., 46 (2007), pp. 1211–1238.

[17] O. Glass and S. Guerrero, *Some exact controllability results for the linear KdV equation and uniform controllability in the zero-dispersion limit*, Asymptot. Anal., 60 (2008), pp. 61–100.

[18] O. Glass and S. Guerrero, *Uniform controllability of a transport equation in zero diffusion-dispersion limit*, Math. Models Methods Appl. Sci., 19 (2009), pp. 1567–1601.

[19] O. Glass, *A complex-analytic approach to the problem of uniform controllability of a transport equation in the vanishing viscosity limit*, J. Funct. Anal., 258 (2010), pp. 852–868.

[20] S. Guerrero and O. Imanuvilov, *Remarks on global controllability for the Burgers equation with two control forces*, Ann. Inst. H. Poincaré Anal. Non Linéaire, 24 (2007), pp. 897–906.

[21] S. Guerrero and G. Lebeau, *Singular optimal control for a transport-diffusion equation*, Comm. Partial Differential Equations, 32 (2007), pp. 1813–1836.

[22] T. Horsin, *On the controllability of the Burgers equation*, ESAIM Control Optim. Calc. Var., 3 (1998), pp. 83–95.

[23] A. M. Il'in and O. A. Oleĭnik, *Behavior of solutions of the Cauchy problem for certain quasilinear equations for unbounded increase of the time*, Dokl. Akad. Nauk SSSR, 120 (1958), pp. 25–28.

[24] S. Kawashima and A. Matsumura, *Stability of shock profiles in viscoelasticity with non-convex constitutive relations*, Comm. Pure Appl. Math., 47 (1994), pp. 1547–1569.

[25] S. Kružkov, *First order quasilinear equations with several independent variables*, Mat. Sb. (N.S.), 81 (1970), pp. 228–255 (in Russian).

[26] J.-L. Lions, *Contrôlabilité exacte, perturbations et stabilisation de systèmes distribués. Tome 2*, Recherches en Mathématiques Appliquées 9, Masson, Paris, 1988.

[27] A. Matsumura and K. Nishihara, *Asymptotic stability of traveling waves for scalar viscous conservation laws with non-convex nonlinearity*, Comm. Math. Phys., 165 (1994), pp. 83–96.

[28] V. Perrollaz, *Exact Controllability of Scalar Conservation Laws with an Additional Control in the Context of Entropy Solutions*, Laboratoire Jacques-Louis Lions, Paris, preprint, 2011.

[29] D. Smart, *Fixed Point Theorems*, Cambridge University Press, London, New York, 1974.