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Controllability of Two Coupled Wave Equations on a Compact Manifold

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Abstract

We consider the exact controllability problem on a compact manifold Ω for two coupled wave equations, with a control function acting on one of them only. Action on the second wave equation is obtained through a coupling term. First, when the two waves propagate with the same speed, we introduce the time $T_{\omega\to\mathcal{O}\to\omega}$ for which all geodesics traveling in Ω go through the control region ω, then through the coupling region $\mathcal{O}$, and finally come back in ω. We prove that the system is controllable if and only if both ω and $\mathcal{O}$ satisfy the Geometric Control Condition and the control time is larger than $T_{\omega\to\mathcal{O}\to\omega}$. Second, we prove that the associated HUM control operator is a pseudodifferential operator and we exhibit its principal symbol. Finally, if the two waves propagate with different speeds, we give sharp sufficient controllability conditions on the functional spaces, the geometry of the sets ω and $\mathcal{O}$, and the minimal time.

Contents

1. Introduction and Main Result

1.1. Setting and Motivation

Let (Ω, g) be a $\mathscr{C}^\infty$ compact connected n-dimensional Riemannian manifold without boundary. We denote by Δ the (negative) Laplace–Beltrami operator on Ω for the metric g, and $P = P(x, \partial_t, \partial_x) = \partial_t^2 - \Delta$ denotes the d'Alembert operator (or wave operator) on the manifold $\mathbb{R} \times \Omega$. We take two smooth functions b_ω and b on Ω. We consider the controllability problem for the system of coupled wave equations

$$\begin{cases} Pu_1 + b(x)\, u_2 = 0 & \text{in } (0, T) \times \Omega, \\ Pu_2 = b_\omega(x)\, f & \text{in } (0, T) \times \Omega. \end{cases} \tag{1.1}$$

Here, the state of the system is $(u_1, u_2, \partial_t u_1, \partial_t u_2)$ and f is our control function, with possible action on the set $\{b_\omega \neq 0\}$. Taking zero initial data, together with a forcing term $f \in L^2((0, T) \times \Omega))$, the associated solution of (1.1) lies for any time in the space $H^2(\Omega) \times H^1(\Omega) \times H^1(\Omega) \times L^2(\Omega)$ as $u_2 \in L^2(0, T; H^1(\Omega))$. Hence, there is a gain of regularity for the uncontrolled variable u_1 (see also [1,3,4]).

In this context, the following definition states the adapted control problems, viz. exact controllability, null-controllability, and controllability from zero. Note that because of the linearity and the time reversibility of the system we consider, the three statements are in fact equivalent.

Definition 1.1. We say that System (1.1) is controllable in time $T > 0$ if one of the following (equivalent) assertions is satisfied:

- (Exact controllability) For any initial data $(u_1^0, u_2^0, u_1^1, u_2^1) \in H^2(\Omega) \times H^1(\Omega) \times H^1(\Omega) \times L^2(\Omega)$ and any target $(\tilde{u}_1^0, \tilde{u}_2^0, \tilde{u}_1^1, \tilde{u}_2^1) \in H^2(\Omega) \times H^1(\Omega) \times H^1(\Omega) \times L^2(\Omega)$ there exists a control function $f \in L^2((0, T) \times \Omega)$ such that the solution of (1.1) issued from $(u_1, u_2, \partial_t u_1, \partial_t u_2)|_{t=0} = (u_1^0, u_2^0, u_1^1, u_2^1)$, satisfies $(u_1, u_2, \partial_t u_1, \partial_t u_2)|_{t=T} = (\tilde{u}_1^0, \tilde{u}_2^0, \tilde{u}_1^1, \tilde{u}_2^1)$;
- (Null-controllability) For any initial data $(u_1^0, u_2^0, u_1^1, u_2^1) \in H^2(\Omega) \times H^1(\Omega) \times H^1(\Omega) \times L^2(\Omega)$, there exists a control function $f \in L^2((0, T) \times \Omega)$ such that the solution of (1.1) associated to the initial data $(u_1, u_2, \partial_t u_1, \partial_t u_2)|_{t=0} = (u_1^0, u_2^0, u_1^1, u_2^1)$ satisfies $(u_1, u_2, \partial_t u_1, \partial_t u_2)|_{t=T} = (0, 0, 0, 0)$;
- (Controllability from zero) For any target $(\tilde{u}_1^0, \tilde{u}_2^0, \tilde{u}_1^1, \tilde{u}_2^1) \in H^2(\Omega) \times H^1(\Omega) \times H^1(\Omega) \times L^2(\Omega)$, there exists a control function $f \in L^2((0, T) \times \Omega)$ such that the solution of (1.1) starting from rest $(u_1, u_2, \partial_t u_1, \partial_t u_2)|_{t=0} = (0, 0, 0, 0)$ satisfies $(u_1, u_2, \partial_t u_1, \partial_t u_2)|_{t=T} = (\tilde{u}_1^0, \tilde{u}_2^0, \tilde{u}_1^1, \tilde{u}_2^1)$.

For most results proved in this article, we shall assume that the function b is non-negative on Ω, and denote by $\omega = \{b_\omega \neq 0\}$ the control set and by $\mathcal{O} = \{b \neq 0\}$ the coupling set (which is the indirect control set for the first equation in (1.1)).

A natural necessary and sufficient condition to obtain controllability for wave equations is to assume that the control set satisfies the Geometric Control Condition (GCC) defined in [8,38]. For $\omega \subset \Omega$ and $T > 0$, we shall say that (ω, T) satisfies GCC if every geodesic traveling at speed one in Ω meets ω in a time $t < T$. We say that ω satisfies GCC if there exists $T > 0$ such that (ω, T) satisfies GCC. We also set $T_\omega = \inf\{T > 0, (\omega, T) \text{ satisfies GCC}\}$.

Note that in the situation of System (1.1), a necessary condition is that both sets ω and $\mathcal{O}$ satisfy GCC (otherwise one of the two equations is not controllable). If ω does not satisfy GCC, even the second equation of (1.1) is not controllable (see [8,9] for a single wave equation). If $\mathcal{O}$ does not satisfy GCC, the first equation of (1.1) is not controllable for the same reason.

The controllability problem for systems like (1.1) has already been addressed in [3,4,39]. In the first two articles, and in the context of *symmetric* systems, it is proved that controllability holds in large time under optimal geometric conditions on the sets ω and $\mathcal{O}$ (with a smallness assumption on some coupling coefficients). However, the minimal time given in these articles depends upon all parameters of the problem (that is b and b_ω). In the situation of System (1.1), it seems natural that the control time should depend only on the geometry of the sets Ω, ω and $\mathcal{O}$, as it is the case for a single wave equation. In [39], the authors study System (1.1) in the one dimensional torus. Following [15], they obtain a sharper estimate on the control time than in [3,4] (in particular, it depends only on the geometry of Ω, ω and $\mathcal{O}$, and yet in general this estimate is not optimal. We can also quote the work of [16, Chapter 7] where the authors study the simultaneous control (by a single control function) of a system of uncoupled wave equations in one dimension.

We provide some motivations for considering control systems like (1.1).

Controllability of physical systems. Several physical systems can be described by coupled partial differential equations: elasticity, thermoelasticity, elecromagnetism, plate systems, etc.. The property of exact controllability for those types of systems is not fully understood yet.

System (1.1) can be seen as a toy model for such systems. Its study is an attempt to understand the phenomena governing the exact controllability process.

Controllability of parabolic systems. The controllability of parabolic systems has been intensively studied in the last decade (see for instance the review article [5]). One of the challenging questions in this area is to understand the optimal geometric conditions on the control set ω and the coupling set $\mathcal{O}$ needed for null-controllability. The first positive result concerns the case where $\omega \cap \mathcal{O} \neq \emptyset$ (see [5] or [26]). As for the case $\omega \cap \mathcal{O} = \emptyset$, little is known. The idea of [3,4] was to make use of the transmutation method to reduce the parabolic problem to a system of coupled wave equations. This allowed these authors to establish null-controllability of symmetric systems under the only condition that both ω and $\mathcal{O}$ satisfy GCC. In particular, this includes several situations where $\omega \cap \mathcal{O} = \emptyset$ (see [3,4] and the figures therein). However, in such results, ω and $\mathcal{O}$ both need to satisfy GCC, whereas for parabolic systems we expect a null controllability result to hold without any geometric assumptions on these two subsets. Concerning cascade heat equations, the only result (to our knowledge) is proved in one space dimension with the same strategy in [39]. The present work provides an extension of this result in general n-dimensional compact manifolds under geometric conditions.

Insensitizing controls for the wave equation. The question of insensitizing control for a wave equation, introduced by J.-L. Lions [32] and addressed in [15,46] is the following. We consider the controlled wave equation

$$\begin{cases} Pu = b_\omega(x)\, f & \text{in } (0,T)\times\Omega, \\ u|_{t=0} = u_0 + \tau_0 z_0 & \text{in } \Omega, \\ \partial_t u|_{t=0} = u_1 + \tau_1 z_1 & \text{in } \Omega, \end{cases} \tag{1.2}$$

where the data $(u_0, u_1) \in H^1(\Omega) \times L^2(\Omega)$ are fixed, and $\tau_0 z_0$, $\tau_1 z_1$ represent unknown noises, with

$$\|z_0\|_{H^1(\Omega)} = \|z_1\|_{L^2(\Omega)} = 1, \tag{1.3}$$

and $\tau_0, \tau_1 \in \mathbb{R}$. A control function $f \in L^2((0,T)\times\Omega)$ is said to insensitize the cost functional

$$\Phi(u) = \frac{1}{2}\int_0^T \int_\Omega b(x)|u(t,x)|^2 \,\mathrm{d}x\,\mathrm{d}t, \tag{1.4}$$

if for any pair (z_0, z_1) satisfying (1.3), the corresponding solution of (1.2) satisfies

$$\frac{\mathrm{d}}{d\tau_0}\Phi(u)\Big|_{\tau_0=\tau_1=0} = \frac{\mathrm{d}}{d\tau_1}\Phi(u)\Big|_{\tau_0=\tau_1=0} = 0.$$

This basically means that for this particular control function f, the cost functional (that is the local L^2 norm of the solution on $\mathcal{O}$) is insensitive to small variations of the initial data. This problem can be recast as a constrained coupled control problem of the form (1.1), to which our results will apply.

On simultaneous control. By introducing $v_1 = u_1 + u_2$ and $v_2 = u_2 - u_1$ in (1.1) we obtain the following system

$$\begin{cases} Pv_1 + b(x)\,(v_1+v_2)/2 = b_\omega(x)\, f & \text{in } (0,T)\times\Omega, \\ Pv_2 - b(x)\,(v_1+v_2)/2 = b_\omega(x)\, f & \text{in } (0,T)\times\Omega. \end{cases}$$

The control problem considered here thus corresponds to a case of simultaneous control as introduced by Lions [31]. Yet using this form by mixing the components u_1 and u_2 the additional regularity of one of them, u_1, becomes invisible. The form (1.1) is thus more adapted for the analysis we carry out here.

The main purposes of this article are to prove controllability for System (1.1), to find an explicit expression of the minimal control time in the simple situation where Ω is a compact manifold without boundary, and to describe precisely the microlocal properties of the optimal control operator that is yielding the control function of minimal L^2-norm.

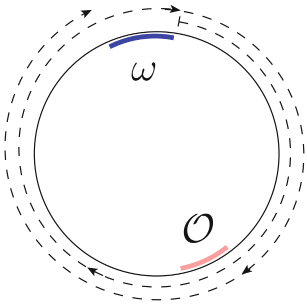

Fig. 1. A one-dimensional example with a particular geodesic associated with the minimal control time $T_{\omega\to\mathcal{O}\to\omega} = 2L - |\omega|$ on $\Omega = \mathbb{R}/L\mathbb{Z}$ equipped with the flat metric

1.2. Main Results

Our main results are threefold. First, we give a necessary and sufficient condition for the controllability of System (1.1). Second, we give a precise description of the optimal control operator associated with System (1.1). Third, we give sharp sufficient conditions for the controllability of similar systems when the two waves propagate with different speeds.

1.2.1. Controllability of System (1.1) To state our first main result, we introduce the adapted control time.

Definition 1.2. Given two sets ω and $\mathcal{O}$ both satisfying GCC, we set $T_{\omega\to\mathcal{O}\to\omega}$ to be the infimum of times $T > 0$ for which the following assertion is satisfied:

every geodesic traveling at speed one in Ω meets ω in a time $t_0 < T$, meets $\mathcal{O}$ in a time $t_1 \in (t_0, T)$ and meets ω again in a time $t_2 \in (t_1, T)$.

Note that in general $T_{\omega\to\mathcal{O}\to\omega} \neq T_{\mathcal{O}\to\omega\to\mathcal{O}}$, and that we have the estimate

$$\max(T_{\mathcal{O}}, T_\omega) \leqq T_{\omega\to\mathcal{O}\to\omega} \leqq 2T_\omega + T_{\mathcal{O}}.$$

Fig. 1 illustrates the value of the control time in a simple one-dimensional setting.

We can now state our main controllability result (in the sense of Definition 1.1).

Theorem 1.3. *Suppose that $b \geqq 0$ on Ω, and that both sets ω and $\mathcal{O}$ satisfy GCC. Then, System* (1.1) *is controllable if $T > T_{\omega\to\mathcal{O}\to\omega}$ and is not controllable if $T < T_{\omega\to\mathcal{O}\to\omega}$.*

In particular this result holds without any assumption on the smallness of the coupling coefficient b as opposed to [3,4].

According to the Hilbert Uniqueness Method (HUM) of J.-L. Lions [31], which we discuss in Section 5.1 for the system we consider, the controllability property of Theorem 1.3 is equivalent to an observability inequality for the adjoint system.

More precisely, System (1.1) is exactly controllable in time T if and only if the inequality

$$E_{-1}(v_1(0)) + E_0(v_2(0)) \leqq C \int_0^T \int_\Omega |b_\omega v_2|^2 \, dx \, dt \tag{1.5}$$

holds for every $(v_1, v_2) \in \mathscr{C}^0([0,T]; H^{-1}(\Omega) \times L^2(\Omega)) \cap \mathscr{C}^1([0,T]; H^{-2}(\Omega) \times H^{-1}(\Omega))$ solutions of

$$\begin{cases} Pv_1 = 0 & \text{in } (0,T)\times\Omega, \\ Pv_2 = -b(x)\, v_1 & \text{in } (0,T)\times\Omega. \end{cases} \tag{1.6}$$

In the observability inequality (1.5), we use the notation

$$E_k(v) = \|v\|^2_{H^k(\Omega)} + \|\partial_t v\|^2_{H^{k-1}(\Omega)}, \quad k \in \mathbb{Z},$$

where the space $H^s(\Omega)$ is endowed with the norm

$$\|v\|_{H^s(\Omega)} = \|(1-\Delta)^{\frac{s}{2}} v\|_{L^2(\Omega)}, \quad s \in \mathbb{R},$$

and the associated inner product.

The proof of the observability inequality (1.5) is based on a contradiction argument, inspired by that of [27]. Similarly, the key tools involved are microlocal defect measures introduced by P. Gérard [21] and L. Tartar [43], and used to solve control problems in [9,11,27].

1.2.2. Hilbert Uniqueness Method and Description of the Control An important feature of the Hilbert Uniqueness Method, as presented by Lions [31], lays in the following two facts: the control one obtains, f_{HUM}, minimizes the cost functional $\|f\|^2_{L^2((0,T)\times\Omega)}$ among all $f \in L^2((0,T)\times\Omega)$ realizing a control for System (1.1) (see Section 5); it is the optimal L^2-control. Moreover, it is itself a solution of the adjoint system (for instance System (2.12) in our situation) for appropriate initial data, say W^0.

The Gramian operator $\mathcal{L}$ associated with Systems (1.1)–(1.6) is given by

$$\int_0^T \int_\Omega |b_\omega v_2|^2 \, dx \, dt = \langle \mathcal{L} V, V \rangle ,$$

where v_2 is the solution of (1.6) associated with the initial data $(v_1, v_2, \partial_t v_1, \partial_t v_2)|_{t=0} = V$. The duality bracket used above will be made precise in Section 5 (where Sobolev spaces will be shifted – see also Section 2.3). If the observability inequality (1.5) is satisfied, then the HUM control operator is the inverse of the mapping $\mathcal{L}$. From the initial data V to be controlled, the HUM operator maps the associated initial data W^0 for the adjoint system, giving rise to the control function f_{HUM}.

The second main goal of this article is to give an explicit representation of the HUM operator following the ideas of [18]. We prove the following result (see Theorem 5.5 and Corollary 5.6).

1. The Gramian operator is a matrix of pseudodifferential operators of order zero. The determinant of its principal symbol takes essentially the following form
$$\int_0^T \int_0^T (b_\omega^2 \circ \varphi_{t_1})(b_\omega^2 \circ \varphi_{t_2}) \left(\int_{t_1}^{t_2} b \circ \varphi_\sigma d\sigma \right)^2 \mathrm{d}t_1 \, \mathrm{d}t_2,$$
where φ_σ denotes the geodesic flow on $S^*\Omega$.
2. This operator is elliptic if and only if $T > T_{\omega\to\mathcal{O}\to\omega}$. This property provides a second proof of Theorem 1.3.
3. For $T > T_{\omega\to\mathcal{O}\to\omega}$, the HUM control operator is also a matrix of pseudodifferential operators of order zero.

A precise statement needs the introduction of some notation and will be given in Section 5.3. In particular, this result holds without any sign assumption on the function b. As a consequence, this method also provides a necessary and sufficient condition of (high-frequency) controllability for System 1.1 for any real-valued b, stated in Corollary 5.8.

The proof of this result follows the program elaborated in [18], in the case of the wave equation, and uses in an essential way the Egorov theorem. The information carried by microlocal defect measures is not sufficient to prove such a strong property of the HUM operator. Note that the third item above has several important consequences, as described in [18].

1.2.3. The Case of Different Speeds It also appears natural to consider the control problem for two coupled wave equations with different speeds. We thus set $P = \partial_t^2 - \Delta$ and $P_\gamma = \partial_t^2 - \gamma^2\Delta$ for some $\gamma > 0$, $\gamma \neq 1$.

Let $s \in \mathbb{R}$. Assume that we have $(u_1^0, u_1^1) \in H^{s+2}(\Omega) \times H^{s+1}(\Omega)$, $(u_2^0, u_2^1) \in H^{s+1}(\Omega) \times H^s(\Omega)$ and $F \in L^1(0, T; H^s(\Omega))$. Then, classically, there exists a unique solution to
$$\begin{cases} P u_1 + b(x)\, u_2 = 0 & \text{in } \mathbb{R} \times \Omega, \\ P_\gamma u_2 = F & \text{in } \mathbb{R} \times \Omega, \end{cases} \tag{1.7}$$
with the initial conditions,
$$(u_1, \partial_t u_1)|_{t=0} = (u_1^0, u_1^1), \quad (u_2, \partial_t u_2)|_{t=0} = (u_2^0, u_2^1),$$
satisfying
$$u_1 \in \mathscr{C}^0(\mathbb{R}, H^{s+2}(\Omega)) \cap \mathscr{C}^1(\mathbb{R}, H^{s+1}(\Omega)),$$
$$u_2 \in \mathscr{C}^0(\mathbb{R}, H^{s+1}(\Omega)) \cap \mathscr{C}^1(\mathbb{R}, H^{s}(\Omega)).$$

Our first result is a microlocal "hidden regularity" result: in fact, the first component of the solution of (1.7) enjoys more smoothness than expected.

Theorem 1.4. *Assume that*
$$(u_1, \partial_t u_1)|_{t=0} = (u_1^0, u_1^1) \in H^{s+3}(\Omega) \times H^{s+2}(\Omega).$$

Then, the first component of the solution (u_1, u_2) to System 1.7 satisfies the additional regularity $u_1 \in \bigcap_{k=0}^{3} \mathscr{C}^k(\mathbb{R}; H^{s+3-k}(\Omega))$. Moreover, for all $T > 0$ there exists a constant $C > 0$ such that

$$\sum_{k=0}^{3} \|\partial_t^k u_1\|_{L^\infty(0,T;H^{s+3-k}(\Omega))} + \|u_2\|_{L^\infty(0,T;H^{s+1}(\Omega))} + \|\partial_t u_2\|_{L^\infty(0,T;H^s(\Omega))}$$
$$\leqq C\Big(\|u_1^0\|_{H^{s+3}(\Omega)}+\|u_1^1\|_{H^{s+2}(\Omega)}+\|u_2^0\|_{H^{s+1}(\Omega)}+\|u_2^1\|_{H^s(\Omega)}+\|F\|_{L^1(0,T;H^s(\Omega))}\Big). \tag{1.8}$$

The proof is given in Section 6.

We now consider the associated control problem:

$$\begin{cases} Pu_1 + b(x)\, u_2 = 0 & \text{in } (0,T)\times\Omega, \\ P_\gamma u_2 = b_\omega(x)\, f & \text{in } (0,T)\times\Omega. \end{cases} \tag{1.9}$$

As a consequence of Theorem 1.4, for $f \in L^2((0,T)\times\Omega)$, starting from zero initial data, we have

$$u_1 \in \mathscr{C}^0(\mathbb{R}, H^3(\Omega)) \cap \mathscr{C}^1(\mathbb{R}, H^2(\Omega)), \quad u_2 \in \mathscr{C}^0(\mathbb{R}, H^1(\Omega)) \cap \mathscr{C}^1(\mathbb{R}, L^2(\Omega)),$$

leading to the following non controllability result.

Corollary 1.5. *For any $s < 2$ any $T > 0$, and any open sets ω and $\mathcal{O}$, if we start from zero initial data and if $f \in L^2((0,T)\times\Omega)$ we cannot reach any target $(u_1(T), u_2(T), \partial_t u_1(T), \partial_t u_2(T)) = (\tilde{u}_1^0, \tilde{u}_2^0, \tilde{u}_1^1, \tilde{u}_2^1)$ if*

$$(\tilde{u}_1^0, \tilde{u}_2^0, \tilde{u}_1^1, \tilde{u}_2^1) \in \Big(H^{s+1}(\Omega)\setminus H^3(\Omega)\Big) \times H^1(\Omega) \times \Big(H^s(\Omega)\setminus H^2(\Omega)\Big) \times L^2(\Omega).$$

In light of what precedes we define the following property.

Definition 1.6. Let $s \geqq 2$. We shall say that System (1.9) is controllable (from zero) in the space $H^{s+1}\times H^s$ in time $T > 0$ if for any target $(\tilde{u}_1^0, \tilde{u}_2^0, \tilde{u}_1^1, \tilde{u}_2^1) \in H^{s+1}(\Omega)\times H^1(\Omega)\times H^s(\Omega)\times L^2(\Omega)$, there exists a control function $f \in L^2((0,T)\times\Omega)$ such that the solution of (1.9) starting from rest $(u_1, u_2, \partial_t u_1, \partial_t u_2)|_{t=0} = (0,0,0,0)$ satisfies $(u_1, u_2, \partial_t u_1, \partial_t u_2)|_{t=T} = (\tilde{u}_1^0, \tilde{u}_2^0, \tilde{u}_1^1, \tilde{u}_2^1)$.

This notion is also equivalent to exact and null-controllability. Note that obviously, if $s' > s \geqq 2$, controllability in $H^{s+1}\times H^s$ implies controllability in $H^{s'+1}\times H^{s'}$.

Definition 1.7. For a subset $U \subset \Omega$ satisfying GCC and $\gamma > 0$, we define $T_U(\gamma)$ to be the infimum of times T such that every geodesic traveling at speed γ in Ω meets U in a time $t < T$.

In particular, with the notation above, we have

$$\gamma T_U(\gamma) = T_U(1) = T_U, \tag{1.10}$$

the usual GCC time of the subset U.

With these two definitions, we have the following result.

Theorem 1.8. 1. *Suppose that* $\overline{\omega} \cap \overline{\mathcal{O}}$ *does not satisfy GCC. Then for any* $s \geqq 2$ *and any* $T > 0$*, System* (1.9) *is not controllable in* $H^{s+1} \times H^s$ *in time* T.
2. *Suppose that* $\omega \cap \mathcal{O}$ *satisfies GCC. Then, System* (1.9) *is controllable in* $H^3 \times H^2$ *for* $T > \max\{T_{\omega\cap\mathcal{O}}, T_\omega(\gamma)\}$ *and is not controllable for* $T < \max\{T_{\overline{\omega}\cap\overline{\mathcal{O}}}, T_\omega(\gamma)\}$.

The proof is given in Section 6.

Remark 1.9. In fact, the following phenomena occur:

- In item 1, there exists an initial condition that generates a singularity for u_1 that propagates independently of the action of the control function.
- In item 2, in the case $T < \max\{T_{\overline{\omega}\cap\overline{\mathcal{O}}}(1), T_\omega(\gamma)\}$ there exists an initial condition that generates a singularity for u_1 that does not meet the region $\overline{\omega} \cap \overline{\mathcal{O}}$. The proof of Theorem 1.8 shows that this is the only region where this singularity can be affected by the control process.

Remark 1.10. Considering the case where $\omega \cap \mathcal{O}$ satisfies GCC of item 2, and the bounds obtained for the control time, one may wonder what occurs if we let $\gamma \to 1$. We have, with (1.10),

$$\max\{T_{\omega\cap\mathcal{O}}, T_\omega(\gamma)\} \to \max\{T_{\omega\cap\mathcal{O}}, T_\omega\} = T_{\omega\cap\mathcal{O}},$$
$$\max\{T_{\overline{\omega}\cap\overline{\mathcal{O}}}, T_\omega(\gamma)\} \to \max\{T_{\overline{\omega}\cap\overline{\mathcal{O}}}, T_\omega\} = T_{\overline{\omega}\cap\overline{\mathcal{O}}}.$$

In the case $\gamma = 1$, controllability is not considered in the same spaces. In fact it is natural to consider controllability in $H^2 \times H^1$, that is for a target $(\tilde{u}_1^0, \tilde{u}_2^0, \tilde{u}_1^1, \tilde{u}_2^1) \in H^2(\Omega) \times H^1(\Omega) \times H^1(\Omega) \times L^2(\Omega)$ as explained in the beginning of Section 1.1. Yet for system (1.1) one can restrict the control problem to smoother targets, that is, $(\tilde{u}_1^0, \tilde{u}_2^0, \tilde{u}_1^1, \tilde{u}_2^1) \in H^3(\Omega) \times H^1(\Omega) \times H^2(\Omega) \times L^2(\Omega)$. The minimal control time will then be less than the time $T_{\omega\to\mathcal{O}\to\omega}$ of Definition 1.2.

We have $T_{\omega\to\mathcal{O}\to\omega} \leqq T_{\overline{\omega}\cap\overline{\mathcal{O}}} \leqq T_{\omega\cap\mathcal{O}}$. Yet, in general $T_{\omega\to\mathcal{O}\to\omega} < T_{\overline{\omega}\cap\overline{\mathcal{O}}}$, as can be seen in the example of Fig. 2. We thus conclude that in the case where $\omega\cap\mathcal{O}$ satisfies GCC, even if we have controllability in $H^3 \times H^2$ for all values of $\gamma > 0$, the infimum of all times for which controllability holds may not be continuous as $\gamma \to 1$.

This non continuity result illustrates well the fact that the underlying geometrical dynamics changes radically as γ crosses 1. If $\gamma \neq 1$ the bicharacteristics of one wave operator, and hence the associated travelling singularities, never meet those of the second wave operator (which moreover explains the microlocal gain of regularity). However if $\gamma = 1$ the two systems of bicharacteristics coincide, allowing for "communication" in the coupling region that can only contribute to reducing the control time.

Remark 1.11. An extension of these results should be possible in the case of different Riemannian metrics yielding (partially or totally) non-intersecting characteristic sets of the two wave operators.

The previous results show that in the case where the two waves propagate with different speeds much stronger geometrical conditions on both ω and $\mathcal{O}$ are needed to achieve a positive controllability result if compared to the conditions of Theorem 1.3 in the case of two equal speeds.

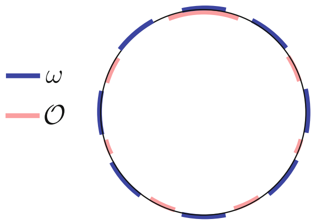

Fig. 2. A simple one-dimensional geometry for which $T_{\omega\to\mathcal{O}\to\omega} < T_{\overline{\omega}\cap\overline{\mathcal{O}}}$ in the case where $\omega \cap \mathcal{O}$ satisfies GCC

1.3. Comments and Outline

1.3.1. Regarding the Time $T_{\omega\to\mathcal{O}\to\omega}$ The time $T_{\omega\to\mathcal{O}\to\omega}$ might be surprising at first sight. It can be interpreted in the following way: to be able to detect the energy of both components of System (1.6) from the observation on ω of the second one only, the polarization of the state along each ray of geometric optics has to change its direction between two passages in the control region ω. This change of polarization arises only when this ray enters the coupling set $\mathcal{O}$.

A description of the notion of polarization, as well as an insight on this geometrical interpretation may be found in [13].

A comparable geometric condition already appears in the study of the decay rates for the thermoelasticity system, see [29] and [13].

1.3.2. Comparing the Different Methods of Proofs of Theorem 1.3 In the case of a scalar wave equation, there exist, to our knowledge, three different methods for proving the (high-frequency) observability on a compact manifold, with optimal conditions on the geometry and the control time. The first one, introduced by Rauch and Taylor [38], and further developed by Bardos, Lebeau and Rauch [8], deals with the propagation of wavefront sets and uses in a crucial way the Hörmander theorem on propagation of singularities.

The second method, introduced by Lebeau [27], further used by Burq [9] as well as Burq and Gérard [11], is based on microlocal defect measures and the propagation of their support.

The last method relies on the use of the Egorov theorem (that is, the theory of Fourier integral operators) and was recently proposed by Dehman and Lebeau [18]. Note on the one hand that the first two methods also apply (with considerable additional difficulties) in the case of a manifold with boundary. On the other hand, there is no analogue of the Egorov theorem in such a case, and the last method fails to apply. However, in [18], the authors show that the FIO Egorov approach provides additional insight on the control problem. In particular, they prove that the HUM operator (the optimal control operator) is (essentially) a pseudodifferential operator and they exhibit its principal symbol.

Here, we provide two different proofs of Theorem 1.3. The first one (using microlocal defect measures) has the advantage of working with limited smoothness (we basically only have to assume that $b_\omega \in \mathscr{C}^0(\Omega)$ and $b \in W^{1,\infty}(\Omega)$). Moreover, this method could be extended to boundary value problems.

The second proof, using the Egorov theorem, has the advantage of working as well with coupling functions b changing signs. Moreover, this method not only provides the observability inequality, but also several additional items of information on the microlocal nature of the HUM control operator.

Finally, note that a proof based on wavefront sets might be possible with the use of the polarization wavefront set of Dencker [17].

1.3.3. The Case of an Open Domain $\Omega \subset \mathbb{R}^n$ Naturally, the same problem can also be adressed on a bounded smooth open set $\Omega \subset \mathbb{R}^n$ (or a manifold with boundary), with (for instance) Dirichlet conditions on the boundary. The method of proof using microlocal defect measures may also work in this setting. However, one of its key points is a propagation result of the microlocal defect measures (analogous of Lemma 3.3 of the present article) up to the boundary (see [9,11,27] for scalar equations and [13] for systems). This technical point needs more care, and is the goal of an ongoing work.

1.3.4. Approximate Controllability Here, we do not touch the subject of approximate controllability. It is deeply related to the question of unique continuation for systems. This topic is widely open and we believe that the tools developed in the present article are not well suited for such a study.

1.3.5. Application to Parabolic Systems The "control transmutation method" allows one to transfer controllability results for certain types of equations to other types through appropriate transforms; we refer the reader to [33,35–37,40]. The name "control transmutation" was coined in [33]. For systems it was used in [3,4] and we can follow their approach here.

As a corollary of Theorem 1.3, we can obtain several null-controllability results for cascade parabolic (or Schrödinger) systems (for all positive time independently of the control time in the hyperbolic case), in cases where the control region ω and the coupling region $\mathcal{O}$ do not intersect. However, in such results, ω and $\mathcal{O}$ have to satisfy GCC, whereas for parabolic systems we expect a null-controllability result to hold without any geometric assumption on these two subsets.

Note that the controllability results for hyperbolic systems in the recent work [2] (obtained with different methods) yield similar results concerning parabolic systems via the "control transmutation method".

1.3.6. Application to Insensitizing Controls We first recall that the problem of insensitizing controls is equivalent (see [15] or [46]) to the fact that the observability inequality

$$E_{-1}(v_1(0)) \leqq C \int_0^T \int_\Omega |b_\omega v_2|^2 \, dx \, dt,$$

holds for every $(v_1, v_2) \in \mathscr{C}^0([0,T]; H^{-1}(\Omega) \times L^2(\Omega)) \cap \mathscr{C}^1([0,T]; H^{-2}(\Omega) \times H^{-1}(\Omega))$ solutions of

$$\begin{cases} Pv_1 = 0 & \text{in } (0,T) \times \Omega, \\ Pv_2 = b(x)\, v_1 & \text{in } (0,T) \times \Omega, \\ (v_2, \partial_t v_2)|_{t=T} = (0,0) & \text{in } \Omega. \end{cases}$$

Since Theorem 1.3 also holds for $b \leqq 0$, $\mathcal{O} = \{b < 0\}$ (changing v_1 in $-v_1$), we directly obtain the following result.

Corollary 1.12. *Suppose that both ω and $\mathcal{O}$ satisfy GCC, and that $T > T_{\omega \to \mathcal{O} \to \omega}$. Then for all $(u_0, u_1) \in H^1(\Omega) \times L^2(\Omega)$, there exists a control function $f \in L^2((0,T) \times \Omega)$ for System* (1.2) *that insensitizes the functional Φ defined in* (1.4).

Since GCC is necessary for both sets ω and $\mathcal{O}$, the geometric conditions obtained here are optimal. Note that the only known results to our knowledge are the one dimensional case, see [15], and the case where $\mathcal{O} \cap \omega$ satisfies the multiplier condition of Lions, see [46].

1.3.7. Outline The outline of this article is the following. In Section 2, we give some notation, define the tools used in the main part of the article and recall some basic well-posedness results.

In Section 3, we prove that the observability inequality holds if $T > T_{\omega \to \mathcal{O} \to \omega}$. Conversely, we prove in Section 4 that the observability inequality does not hold in the case $T < T_{\omega \to \mathcal{O} \to \omega}$.

In Section 5, we develop the Hilbert Uniqueness Method. We first prove the equivalence between controllability and observability in Section 5.1 and we define the HUM control operator in Section 5.2. Then, we give the explicit characterization of the HUM operator in Section 5.3.

Finally, in Section 6, we provide proofs for the positive and negative results concerning the case of coupled waves with different speeds.

2. Preliminary Remarks, Definitions and Notation

We define the manifold $M = \mathbb{R} \times \Omega$ and its restriction to $(0,T)$, $M_T = (0,T) \times \Omega = \{(t,x) \in M \text{ such that } t \in (0,T)\}$. We also write T^*M_T as the restriction of the cotangent bundle of M to $(0,T)$, that is $T^*M_T = \{(t,x,\tau,\eta) \in T^*M, t \in (0,T)\}$. Setting $|\eta|_x^2 = g_x(\eta,\eta)$ as the Riemannian norm in the cotangent space of Ω at x, we define

$$S^*M = \{(t,x,\tau,\eta) \in T^*M, |\tau|^2 + |\eta|_x^2 = 1\},$$

the cosphere bundle of M, and similarly S^*M_T its restriction to $(0,T)$. We denote by $\pi : S^*M \to M$ the natural projection, which also maps S^*M_T onto M_T. We shall also use the associated cosphere bundle in the spatial variables only,

$$S^*\Omega = \{(x,\eta) \in T^*\Omega, |\eta|_x^2 = 1/2\}.$$

2.1. Symbols, Operators and Measures on the Cosphere Bundle

Here, we follow [10, Section 1.1] for the notation. We denote by $H^k(X;\mathbb{C}^j)$ or $H^k_{\mathrm{loc}}(X;\mathbb{C}^j)$, with $j=1$ or 2 and $X=\Omega, M$, or M_T, the usual Sobolev space for functions valued in $\mathbb{C}^j$, endowed with the natural inner product and norm. In particular, the $L^2(X;\mathbb{C}^j)$ inner product is denoted by $(\cdot,\cdot)_{L^2(X;\mathbb{C}^j)}$.

We define $S^m_{\mathrm{phg}}(T^*M_T;\mathbb{C}^{j\times j})$, with $j=1$ or 2 as the set of matrix valued polyhomogeneous symbols of order m on M_T with *compact support* in M_T. We recall that symbols in the class $S^m_{\mathrm{phg}}(T^*\mathbb{R}^n;\mathbb{C}^{j\times j})$ behave well with respect to changes of variables, up to symbols in $S^{m-1}_{\mathrm{phg}}(T^*\mathbb{R}^n;\mathbb{C}^{j\times j})$ (see [23, Theorem 18.1.17 and Lemma 18.1.18]).

For any m, the restriction to the sphere

$$S^m_{\mathrm{phg}}(T^*M_T;\mathbb{C}^{j\times j})\to \mathscr{C}^\infty_c(S^*M_T;\mathbb{C}^{j\times j}),\quad a\to a|_{S^*M_T}, \tag{2.1}$$

is surjective. This will allow us to identify a *homogeneous* symbol with a smooth function on the sphere.

We denote by $\Psi^m_{\mathrm{phg}}(M_T;\mathbb{C}^{j\times j})$, with $j=1$ or 2 the space of polyhomogeneous pseudodifferential operators of order m on M_T, with a *compactly supported kernel* in $M_T\times M_T$: one says that $A\in\Psi^m_{\mathrm{phg}}(M_T;\mathbb{C})$ if

1. its kernel $K(x,y)\in\mathscr{D}'(M_T\times M_T)$ is such that $\mathrm{supp}(K)$ is compact in M_T;
2. $K(x,y)$ is smooth away from the diagonal $\Delta_{M_T}=\{(t,x;t,x);\ (t,x)\in M_T\}$;
3. for every coordinate patch $M_{T,\kappa}\subset M_T$ with coordinates $M_{T,\kappa}\ni(t,x)\mapsto\kappa(t,x)\in\tilde M_{T,\kappa}\subset\mathbb{R}^{n+1}$ and all $\phi_0,\phi_1\in\mathscr{C}^\infty_c(\tilde M_{T,\kappa})$ the map
$$u\mapsto\phi_1\left(\kappa^{-1}\right)^*A\kappa^*(\phi_0 u)$$
is in $\mathrm{Op}(S^m_{\mathrm{phg}}(\mathbb{R}^{n+1}\times\mathbb{R}^{n+1}))$.

For $A\in\Psi^m_{\mathrm{phg}}(M_T;\mathbb{C}^{j\times j})$, we denote by $\sigma_m(A)\in S^m_{\mathrm{phg}}(T^*M_T;\mathbb{C}^{j\times j})$ the principal symbol of A (see [23, Chapter 18.1]). Note that the principal symbol is uniquely defined in $S^m_{\mathrm{phg}}(T^*M_T;\mathbb{C}^{j\times j})$ because of the polyhomogeneous structure (see the remark following Definition 18.1.20 in [23]). The application σ_m enjoys the following properties

- $\sigma_m:\Psi^m_{\mathrm{phg}}(M_T;\mathbb{C}^{j\times j})\to S^m_{\mathrm{phg}}(T^*M_T;\mathbb{C}^{j\times j})$ is surjective.
- For all $A\in\Psi^m_{\mathrm{phg}}(M_T;\mathbb{C}^{j\times j})$, $\sigma_m(A)=0$ if and only if $A\in\Psi^{m-1}_{\mathrm{phg}}(M_T;\mathbb{C}^{j\times j})$.
- For all $A\in\Psi^m_{\mathrm{phg}}(M_T;\mathbb{C}^{j\times j})$, $\sigma_m(A^*)={}^t\overline{\sigma_m(A)}$.
- For all $A_1\in\Psi^{m_1}_{\mathrm{phg}}(M_T;\mathbb{C}^{j\times j})$ and $A_2\in\Psi^{m_2}_{\mathrm{phg}}(M_T;\mathbb{C}^{j\times j})$, we have $A_1A_2\in\Psi^{m_1+m_2}_{\mathrm{phg}}(M_T;\mathbb{C}^{j\times j})$ with
$$\sigma_{m_1+m_2}(A_1A_2)=\sigma_{m_1}(A_1)\sigma_{m_2}(A_2).$$
- For all $A_1\in\Psi^{m_1}_{\mathrm{phg}}(M_T;\mathbb{C})$ and $A_2\in\Psi^{m_2}_{\mathrm{phg}}(M_T;\mathbb{C})$, we have $[A_1,A_2]=A_1A_2-A_2A_1\in\Psi^{m_1+m_2-1}_{\mathrm{phg}}(M_T;\mathbb{C})$ with
$$\sigma_{m_1+m_2-1}([A_1,A_2])=\frac{1}{i}\{\sigma_{m_1}(A_1),\sigma_{m_2}(A_2)\}.$$

Here, $\{a_1, a_2\}$ denotes the Poisson bracket, given in local charts by

$$\{a_1, a_2\} = \partial_\tau a_1 \partial_t a_2 - \partial_t a_1 \partial_\tau a_2 + \sum_l (\partial_{\xi_l} a_1 \partial_{x_l} a_2 - \partial_{x_l} a_1 \partial_{\xi_l} a_2).$$

- If $A \in \Psi^m_{\mathrm{phg}}(M_T; \mathbb{C}^{j\times j})$, then A maps continuously $H^k(M_T; \mathbb{C}^j)$ into $H^{k-m}(M_T; \mathbb{C}^j)$ (resp. $H^k_{\mathrm{loc}}(M_T; \mathbb{C}^j)$ into $H^{k-m}_{\mathrm{loc}}(M_T; \mathbb{C}^j)$). In particular, for $m < 0$, A is compact on $L^2(M_T; \mathbb{C}^j)$.

Given an operator $A \in \Psi^m_{\mathrm{phg}}(M_T; \mathbb{C})$, we define $\mathrm{Char}(A) = \{\rho \in T^*M, \sigma_m(A)(\rho) = 0\}$.

At places we shall need to consider pseudodifferential operators acting on Ω yet depending upon the parameter $t \in (0, T)$ with some smoothness with respect to t. Let $k \in \mathbb{N}\cup\{\infty\}$, we say that $A_t \in \mathscr{C}^k\big((0,T), \mathrm{Op}(S^m_{\mathrm{phg}}(\mathbb{R}^n \times \mathbb{R}^n))\big)$ if $A_t = \mathrm{Op}(a_t)$ with $a_t \in \mathscr{C}^k((0,T), S^m_{\mathrm{phg}}(\mathbb{R}^n \times \mathbb{R}^n))$. Next we say that $A_t \in \mathscr{C}^k((0,T), \Psi^m_{\mathrm{phg}}(\Omega))$ if

1. its kernel $K_t(x, y)$ is in $\mathscr{C}^k((0,T), \mathscr{C}^\infty(\Omega \times \Omega \setminus \Delta_\Omega))$ where $\Delta_\Omega = \{(x,x);\ x \in \Omega\}$;
2. for every coordinate patch $\Omega_\kappa \subset \Omega$ with coordinates $\Omega_\kappa \ni x \mapsto \kappa(x) \in \tilde{\Omega}_\kappa \subset \mathbb{R}^n$ and all $\phi_0, \phi_1 \in \mathscr{C}^\infty_c(\tilde{\Omega}_\kappa)$ the map

$$u \mapsto \phi_1 \left(\kappa^{-1}\right)^* A_t \kappa^*(\phi_0 u)$$

is in $\mathscr{C}^k\Big((0,T), \mathrm{Op}(S^m_{\mathrm{phg}}(\mathbb{R}^n \times \mathbb{R}^n))\Big)$.

In particular we shall use the following form of the Egorov theorem.

Theorem 2.1. *Let $A_t \in \mathscr{C}^\infty\big((0,T), \Psi^1_{\mathrm{phg}}(\Omega)\big)$ with real principal symbol $a_{1,t}$ and $P \in \Psi^m_{\mathrm{phg}}(\Omega)$, $m \in \mathbb{R}$. Define $S(s', s)$ as the solution operator for the Cauchy problem*

$$\partial_t u + iA_t u = 0, \qquad u|_{t=s} = u_0,$$

that is, $u(s') = S(s', s)u_0$. Then there exists $Q_t \in \mathscr{C}^\infty\big((0,T), \Psi^m_{\mathrm{phg}}(\Omega)\big)$ such that, for all $\sigma, N \in \mathbb{R}$, we have

$$S(t,0)PS(0,t) - Q_t \in \mathscr{C}^\infty\Big((0,T), \mathcal{L}(H^\sigma(\Omega), H^{\sigma+N}(\Omega))\Big),$$

and the principal symbol of Q_t is given by $q_t \in \mathscr{C}^\infty((0,T), S^m_{\mathrm{phg}}(T^*\Omega))$ with $q_t = p \circ \chi_{0,t}$ where $\rho(s,t) = \chi_{s,t}(\rho_0)$ is given by the flow of the Hamiltonian vector field associated with $a_{1,t}$:

$$\frac{\mathrm{d}}{\mathrm{d}s}\rho(s,t) = H_{a_{1,t}}(\rho(s,t)), \qquad \rho(t,t) = \rho_0.$$

The proof can be adapted from that given in for instance [45, Theorem 0.9.A]. The notion of smoothing operators appearing in the statement of the above theorem is made precise in following definition.

Definition 2.2. (Smoothing operators) Let $A : \mathscr{D}'(\Omega) \to \mathscr{D}'(\Omega)$ be a linear operator and $k > 0$. We say that A is k-smoothing if $A \in \mathcal{L}(H^s(\Omega); H^{s+k}(\Omega))$ for all $s \in \mathbb{R}$. We say that A is infinitely smoothing if A is k-smoothing for all $k > 0$.

Moreover, we say that $A \in \mathcal{R}^k(\Omega)$ if $A \in \mathcal{L}(H^s(\Omega); H^{s+k}(\Omega))$ for all $s \geqq 0$. We set $\mathcal{R}^\infty(\Omega) = \bigcap_{k>0} \mathcal{R}^k(\Omega)$.

Note in particular that k-smoothing operators are in $\mathcal{R}^k(\Omega)$. Moreover, for $k > 0$, operators in $\Psi^{-k}_{\mathrm{phg}}(\Omega)$ are k-smoothing.

We recall that $-\Delta$ denotes the Laplace operator on Ω, and that we have

$$-\Delta \in \Psi^2_{\mathrm{phg}}(\Omega), \quad \text{with} \quad \sigma_2(-\Delta)(x, \eta) = |\eta|_x^2.$$

It will also be useful to define a function $\tilde{\lambda} \in \mathscr{C}^\infty(T^*M)$ such that

$$\begin{aligned} &\tilde{\lambda}(t, x, \tau, \eta) = (|\tau|^2 + |\eta|_x^2)^{\frac{1}{2}} \text{ for } (t, x, \tau, \eta) \in T^*M, \text{ with } (|\tau|^2 + |\eta|_x^2)^{\frac{1}{2}} \geqq \tfrac{1}{2} \\ &\tilde{\lambda}(t, x, \tau, \eta) \geqq C > 0 \text{ for } (t, x, \tau, \eta) \in T^*M, \text{ with } (|\tau|^2 + |\eta|_x^2)^{\frac{1}{2}} \leqq \tfrac{1}{2}. \end{aligned} \tag{2.2}$$

This gives $\chi \tilde{\lambda}^m \in S^m_{\mathrm{phg}}(T^*M; \mathbb{C})$ if $m \in \mathbb{Z}$ and $\chi \in \mathscr{C}^\infty_c(M)$.

Finally, we define $\mathcal{M}(S^*M_T; \mathbb{R})$ to be the set of real valued measures on S^*M_T, $\mathcal{M}_+(S^*M_T)$ the set of positive measures on S^*M_T, and $\mathcal{M}_+(S^*M_T; \mathbb{C}^{2\times 2})$ the set of measures with values in non-negative hermitian 2×2 matrices. For $\mu \in \mathcal{M}(S^*M_T; \mathbb{R})$ (resp. $\mu \in \mathcal{M}_+(S^*M_T; \mathbb{C}^{2\times 2})$) and $a \in \mathscr{C}^0_c(S^*M_T; \mathbb{R})$ (resp. $a \in \mathscr{C}^0_c(S^*M_T; \mathbb{C}^{2\times 2})$), we shall write

$$\langle \mu, a \rangle_{S^*M_T} = \int_{S^*M_T} a(\rho)\mu(\mathrm{d}\rho), \quad \left(\text{resp. } \langle \mu, a \rangle_{S^*M_T} = \int_{S^*M_T} \mathrm{tr}\{a(\rho)\mu(\mathrm{d}\rho)\}\right),$$

for the duality bracket. The same notation will also be used for $a \in S^0_{\mathrm{phg}}(T^*M_T; \mathbb{R})$ (resp. $a \in S^0_{\mathrm{phg}}(T^*M_T; \mathbb{C}^{2\times 2})$) according to the identification map (2.1).

Observe that the Laplace operator is not coercive since $-\Delta(1) = 0$. This can be cumbersome in places. As a remedy, we introduce more convenient spaces and a scalar product. Let $(e_j)_{j\in\mathbb{N}}$ be a Hilbert basis of eigenfunctions of $-\Delta$, associated to the eigenvalues $(\kappa_j)_{j\in\mathbb{N}}$. In particular, we have $\kappa_0 = 0$ and $e_0 = 1/\sqrt{|\Omega|}$. Following the notation of [18], we set

$$\begin{aligned} L^2_+(\Omega) &:= \left\{ \sum_{j\geqq 1} a_j e_j, (a_j) \in \ell^2 \right\} \\ &= \left\{ f \in L^2(\Omega), \int_\Omega f(x)\,\mathrm{d}x = 0 \right\} = \Pi_+ L^2(\Omega), \end{aligned}$$

with

$$\Pi_0 f = \left(\frac{1}{\sqrt{|\Omega|}} \int_\Omega f(x)\,\mathrm{d}x \right) e_0 = (f, e_0)_{L^2(\Omega)} e_0, \quad \text{and} \quad \Pi_+ f = f - \Pi_0 f.$$

Note that we have

$$\Pi_0 \in \Psi^{-\infty}_{\mathrm{phg}}(\Omega),$$

since Π_0 maps $\mathscr{D}'(\Omega)$ into $\mathscr{C}^\infty(\Omega)$, that is has a $\mathscr{C}^\infty$ kernel (this is true in fact in a more general setting of functional calculus, see [44, Chapter 12]). Hence

$$\Pi_+ = \operatorname{Id} - \Pi_0 \in \Psi^0_{\mathrm{phg}}(\Omega), \quad \text{with} \quad \sigma_0(\Pi_+) = 1.$$

We also define $H^s_+(\Omega) = \Pi_+ H^s(\Omega)$ for $s \in \mathbb{R}$, and in particular $H^s_+(\Omega) = H^s(\Omega) \cap L^2_+(\Omega)$ if $s \geqq 0$.

We shall often use the selfadjoint operator $\lambda = \sqrt{-\Delta}$, classically defined by

$$\lambda f = \sum_{j\in\mathbb{N}} \sqrt{\kappa_j}(f, e_j)_{L^2(\Omega)} e_j, \qquad D(\lambda) = H^1(\Omega).$$

In particular, we have $\lambda e_0 = 0$ and λ is an isomorphism from $H^{s+1}_+(\Omega)$ onto $H^s_+(\Omega)$. We shall denote by $\lambda^{-1} \in \mathcal{L}(H^s_+(\Omega); H^{s+1}_+(\Omega))$ its inverse. Moreover, according to [41] (or [42, Theorem 11.2]), we have

$$\lambda \in \Psi^1_{\mathrm{phg}}(\Omega), \quad \text{with} \quad \sigma_1(\lambda)(x,\eta) = |\eta|_x, \quad (x,\eta) \in T^*\Omega \setminus 0.$$

We denote by $(e^{it\lambda})_{t\in\mathbb{R}}$ the group on $H^s(\Omega)$ generated by $i\lambda$. Note that $e^{it\lambda}$ preserves the spaces $H^s_+(\Omega)$.

The decomposition (splitting) of the operator P into $P = -L_+L_-$, with

$$L_+ = \frac{1}{i}\partial_t - \lambda \quad \text{and} \quad L_- = \frac{1}{i}\partial_t + \lambda,$$

will also be useful in the following. Even though $L_\pm$ is not a pseudodifferential operator on M[1], we shall write

$$\ell_+ = \sigma_1(L_+) = \tau - |\eta|_x, \quad \ell_- = \sigma_1(L_-) = \tau + |\eta|_x, \tag{2.3}$$

and refer to these functions as “the principal symbol of L_+ and L_-”.

2.2. *Some Geometric Facts*

In local coordinates, we write g_{ij} for the metric g on the tangent bundle $T\Omega$. For $v \in T^*\Omega$ we set $|v|_x = (g_x(v,v))^{\frac{1}{2}}$, that is, $|v|_x = \left(\sum_{ij}(g_x)_{ij}v_iv_j\right)^{\frac{1}{2}}$ in local coordinates.

As a metric on the cotangent bundle $T^*\Omega$, g is given by g^{ij} in local coordinates, with $(g^{ij}) = (g_{ij})^{-1}$. Note that we keep the letter g for this metric by abuse of notation. For $\eta \in T^*_x\Omega$ we have $|\eta|_x = \left(g_x(\eta,\eta)\right)^{\frac{1}{2}}$, that is, $|\eta|_x = \left(\sum_{ij}(g_x)^{ij}\eta_i\eta_j\right)^{\frac{1}{2}}$ in local coordinates.

For all $v \in T_x\Omega$, we can define $v^* \in T^*_x\Omega$ uniquely by $g_x(v,w) = \langle v^*, w\rangle$ for any $w \in T_x\Omega$, which reads in local coordinates $v^*_i = \sum_j (g_x)_{ij}v_j$. Note that $|v|_x = |v^*|_x$.

[1] Observe that ℓ_+ and ℓ_- do not satisfy the proper estimate in the cone $|\tau| \geqq C|\eta|_x$.

2.2.1. Geodesics We start by defining geodesics on Ω associated with the metric g. With $k(x,\eta) = \frac{1}{2}|\eta|_x^2$, $(x,\eta) \in T^*\Omega \setminus 0$, we have the (maximal) integral curves $s \mapsto \zeta_s(x,\eta) \in T^*\Omega \setminus 0$:

$$\frac{\mathrm{d}}{\mathrm{d}s}\zeta_s(x,\eta) = H_k\left(\zeta_s(x,\eta)\right), \quad \zeta_0(x,\eta) = (x,\eta) \in T^*\Omega \setminus 0, \tag{2.4}$$

with the Hamilton vector field H_k given by $H_k = (\nabla_\eta k, -\nabla_x k)$ in local coordinates. In particular, if $\zeta_s(x,\eta) = (x(s),\eta(s))$, we have $\frac{\mathrm{d}}{\mathrm{d}s}x_i(s) = \sum_j (g_{x(s)})^{ij}\eta_j(s)$, that is $\eta(s) = \left(\frac{\mathrm{d}}{\mathrm{d}s}x(s)\right)^*$. Note also that the value of k is preserved along this integral curve as

$$\frac{\mathrm{d}}{\mathrm{d}s}k \circ \zeta_s|_{s=s_0} = H_k(k)(\zeta_{s_0}) = \{k,k\}(\zeta_{s_0}) = 0.$$

Let $S\Omega = \{(x,v) \in T\Omega, |v|_x = 1\}$. For $(x,v) \in S\Omega$, we consider the curve $(x(s),v(s))$ given by

$$(x(s),v(s)^*) = \zeta_s(x,v^*).$$

Note that we have $\frac{\mathrm{d}}{\mathrm{d}s}x(s) = v(s)$. In particular, $\frac{\mathrm{d}}{\mathrm{d}s}x(0) = v$ and moreover

$$|v(s)|_{x(s)}^2 = |v(s)^*|_{x(s)}^2 = 2k(x(s),v(s)^*) = 2k(x,v^*) = |v|_x^2 = 1.$$

We call the curve $s \mapsto x(s)$ on Ω the geodesic originating from $(x,v) \in S\Omega$ at time $s = 0$. We have $(x(t), \frac{\mathrm{d}x}{\mathrm{d}t}(t)) \in S\Omega$: the traveling speed of the geodesic is unitary.

2.2.2. Bicharacteristics of the d'Alembert Operator The principal symbol of the operator $P(x,\partial_t,\partial_x)$ is given by

$$\begin{aligned}\sigma_2(P)(t,x,\tau,\eta) = p(x,\tau,\eta) = -\tau^2 + |\eta|_x^2,\\ \text{for } (t,x,\tau,\eta) \in \mathbb{R}\times\Omega\times\mathbb{R}\times T_x^*\Omega \subset T^*M.\end{aligned} \tag{2.5}$$

We denote by H_p the associated Hamiltonian vector field. In local coordinates, we have

$$p = -|\tau|^2 + \sum_{i,j} g^{ij}\eta_i\eta_j \quad \text{and} \quad H_p = (\nabla_{\tau,\eta}p, -\nabla_{t,x}p).$$

Note that for $a \in S^m_{\mathrm{phg}}(T^*M;\mathbb{C})$, we have $H_p a = \{p,a\}$. We shall make use of the Hamiltonian flow map ϕ_s, that is the (maximal) solutions of

$$\frac{\mathrm{d}}{\mathrm{d}s}\phi_s(\rho) = H_p\left(\phi_s(\rho)\right), \quad \phi_0(\rho) = \rho = (t_0,x_0,\tau_0,\xi_0) \in T^*M \setminus 0. \tag{2.6}$$

Let Γ be an integral curve of (2.6). First notice that p is constant along Γ since $H_p p = 0$. In particular, the flow ϕ_s preserves $\mathrm{Char}(P)$. Moreover, as g is independent of time t, we have $\partial_t p = 0$. Writing $\phi_s(\rho) = (t(s),x(s),\tau(s),\eta(s))$, this implies that τ is constant along Γ. As $\mathrm{d}t/\mathrm{d}s = -2\tau_0$ we have $t(s) = -2\tau_0 s + t_0$.

Setting $\varphi_s(x_0, \eta_0) = \big(x(s), \eta(s)\big)$ we see that it is corresponds to the flow associated with the Hamiltonian $|\eta|_x^2$. Consequently, by Lemma B.1 we have $\varphi_s(x_0, \eta_0) = \xi_{2s}(x_0, \eta_0)$. We thus obtain

$$\phi_s(\rho) = (-2\tau_0 s + t_0, \tau_0, \zeta_{2s}(x_0, \eta_0)). \tag{2.7}$$

Let $\lambda \neq 0$ and M_λ be the map such that $M_\lambda(x, \eta) = (x, \lambda\eta)$, that is a multiplication by λ in the fiber. As k is homogeneous of degree 2 we have $M_\lambda \circ \zeta_{\lambda s} = \zeta_s \circ M_\lambda$ by Lemma B.2 for $\lambda > 0$. If $\tau_0 \neq 0$ with $t = -2\tau_0 s + t_0$, then, using Lemma B.3 as well, we find that

$$\begin{aligned}
\phi_s(\rho) &= (t, \tau_0, \zeta_{\frac{t_0 - t}{\tau_0}}(x_0, \eta_0)) \\
&= (t, \tau_0, M_{|\tau_0|} \circ \zeta_{(t_0 - t)\operatorname{sgn}(\tau_0)}(x_0, \eta_0/|\tau_0|)) \\
&= (t, \tau_0, M_{-\tau_0} \circ \zeta_{t - t_0}(x_0, -\eta_0/\tau_0)).
\end{aligned} \tag{2.8}$$

As is done classically, we call bicharacteristics the integral curves for which $p = 0$. Then $|\eta|_x^2 = |\tau|^2$ is also constant along bicharacteristics. Observe then that (2.6) defines a flow on the manifold

$$\operatorname{Char}(P) \cap S^*M = \{(t, x, \tau, \xi), |\tau|^2 = 1/2 \text{ and } |\eta|_x^2 = 1/2\}.$$

Let $(t(s), x(s), \tau, \eta(s))$ be a bicharacteristic curve of p with $(t(0), x(0), \tau, \eta(0)) = (t_0, x_0, \tau_0, \eta_0)$ and $|\tau_0| = \frac{1}{\sqrt{2}}$.

As we take $\tau_0 \neq 0$ here and $\frac{\mathrm{d}t}{\mathrm{d}s} = -2\tau_0 \neq 0$, we can use the variable t to parametrize the bicharacteristics. Setting $y(t) = x(s)$ and $\xi(t) = -\eta(s)/\tau_0$, with (2.8) we see that $(y(t), \xi(t)) = \zeta_{t-t_0}(x_0, -\eta_0/\tau_0)$. As $|\eta_0/\tau_0|_{x_0} = 1$, we see that $t \mapsto y(t)$ is the geodesic curve issued from (x_0, v_0) at time t_0, where $v_0^* = -\frac{1}{\tau_0}\eta_0$. Consequently, the projection of the bicharacteristic curves solution to (2.6), with $p = 0$, onto Ω, yields geodesics on Ω.

Conversely, if $t \mapsto y(t)$ is a geodesic issued from (x_0, v_0) at time t_0, it is the projection of such a bicharacteristic curve going through $(t_0, x_0, \tau_0, -\tau_0 v_0^*)$ with $\tau_0 = \pm\frac{1}{\sqrt{2}}$.

Now, we can rewrite the geometric condition given in Definition 1.2 in terms of bicharacteristics of the operator P.

Definition 2.3. The time $T_{\omega\to\mathcal{O}\to\omega}$ is the infimum of times $T > 0$ for which the following assertion is satisfied:

for any $\rho_* = (t_*, x_*, \tau_*, \eta_*) \in \operatorname{Char}(P) \cap S^*M_T$, there exists $s_0, s_1, s_2 \in \mathbb{R}$ with $0 < t(s_0) < t(s_1) < t(s_2) < T$, such that we have

$$\pi(\phi_{s_0}(\rho_*)) \in (0, T) \times \omega, \quad \pi(\phi_{s_1}(\rho_*)) \in (0, T) \times \mathcal{O}, \quad \text{and}$$
$$\pi(\phi_{s_2}(\rho_*)) \in (0, T) \times \omega.$$

2.2.3. Bicharacteristics of the Half-Wave Operators The set $\mathrm{Char}(P) \cap S^*M$ has two connected components,

$$\begin{aligned}\mathrm{Char}(P) \cap S^*M &= \left(\mathrm{Char}(L^+) \cap S^*M\right) \cup \left(\mathrm{Char}(L^-) \cap S^*M\right) \\ &= \{\tau = 1/\sqrt{2} \text{ and } |\eta|_x = 1/\sqrt{2}\} \cup \{\tau = -1/\sqrt{2} \text{ and } |\eta|_x = 1/\sqrt{2}\}.\end{aligned}$$

We now denote by $\phi_s^\pm$ the bicharacteristic flow associated with $\ell_\pm$ defined in (2.3), that is, the maximal solutions of

$$\frac{\mathrm{d}}{\mathrm{d}s}\phi_s^\pm(\rho) = H_{\ell_\pm}\left(\phi_s^\pm(\rho)\right), \quad \phi_0^\pm(\rho) = \rho \in T^*M \setminus 0, \text{ with } \rho \in \mathrm{Char}(\ell^\pm).$$

As we have $\partial_t \ell_\pm = 0$ and $\partial_\tau \ell_\pm = 1$, the flow $\phi_s^\pm$ can be written under the form

$$\phi_s^\pm(t, \tau, x, \eta) = (t + s, \tau, \varphi_s^\pm(x, \eta)),$$

where $\varphi_s^\pm$ is the flow associated to the Hamiltonian $\mp h(x, \eta)$ with $h(x, \eta) = |\eta|_x$, that is,

$$\frac{\mathrm{d}}{\mathrm{d}s}\varphi_s^\pm(x, \eta) = H_{\mp h}\left(\varphi_s^\pm(x, \eta)\right), \quad \varphi_0^\pm(x, \eta) = (x, \eta) \in T^*\Omega \setminus 0. \tag{2.9}$$

Note that $\varphi_s^\pm$ is the Hamiltonian flow associated with the operator $\mp\lambda$. In particular, by Lemma B.1 we have

$$\varphi_{-s}^-(x, \eta) = \varphi_s^+(x, \eta). \tag{2.10}$$

From Lemma B.1 (with $F(\alpha) = \mp\sqrt{2\alpha}$) we also deduce that

$$\varphi_s^\pm(x, \eta) = \zeta_{\mp s/|\eta|_x}(x, \eta), \quad (x, \eta) \in T^*\Omega \setminus 0. \tag{2.11}$$

We also have the following property.

Proposition 2.4. *Let $\rho_0 = (t_0, \tau_0, x_0, \eta_0) \in \mathrm{Char}(P) \cap S^*M$, with $\mathrm{sgn}(\tau_0) = \pm$, that is, $\rho_0 \in \mathrm{Char}(\ell^\pm)$. Then, for $t = t_0 - 2\tau_0 s$ we have*

$$\phi_s(\rho_0) = \phi_{t-t_0}^\pm(\rho_0) = (t, \tau_0, \varphi_{t-t_0}^\pm(x_0, \eta_0)).$$

Proof. By (2.11), we have

$$\begin{aligned}\phi_{t-t_0}^\pm(\rho_0) = \phi_{-2\tau_0 s}^\pm(\rho_0) &= (t_0 - 2\tau_0 s, \tau_0, \varphi_{-2\tau_0 s}^\pm(x_0, \eta_0)) \\ &= (t_0 - 2\tau_0 s, \tau_0, \zeta_{\pm 2\tau_0/|\eta|_x s}(x_0, \eta_0)).\end{aligned}$$

As $|\eta|_x = \pm\tau_0$ here, by (2.7), the result follows. □

With the flows $\varphi_s^\pm$, we now define the adapted minimal time for waves with positive/negative frequencies, $T^\pm_{\omega\to\mathcal{O}\to\omega}$, and we provide a characterization of the minimal control time $T_{\omega\to\mathcal{O}\to\omega}$.

Definition 2.5. The time $T^\pm_{\omega\to\mathcal{O}\to\omega}$ is the infimum of times $T > 0$ for which the following assertion is satisfied:

for any $(x, \eta) \in S^*\Omega$, there exists $0 < t_0 < t_1 < t_2 < T$ such that we have $b_\omega \circ \tilde{\pi} \circ \varphi^{\pm}_{t_0}(x, \eta) \neq 0, \quad b \circ \tilde{\pi} \circ \varphi^{\pm}_{t_1}(x, \eta) > 0, \quad b_\omega \circ \tilde{\pi} \circ \varphi^{\pm}_{t_2}(x, \eta) \neq 0,$

where $\tilde{\pi} : S^*\Omega \to \Omega$ is the natural projection.

Proposition 2.6. *We have* $T^+_{\omega\to\mathcal{O}\to\omega} = T^-_{\omega\to\mathcal{O}\to\omega} = T_{\omega\to\mathcal{O}\to\omega}$.

Proof. Let $T > T^+_{\omega\to\mathcal{O}\to\omega}$ and let $(x, \eta) \in T^*\Omega \setminus 0$. We set $(x_1, \eta_1) = \varphi^-_T(x, \eta)$. Then there exists $0 < t_0 < t_1 < t_2 < T$ such that

$$b_\omega \circ \tilde{\pi} \circ \varphi^+_{t_0}(x_1, \eta_1) \neq 0, \quad b \circ \tilde{\pi} \circ \varphi^+_{t_1}(x_1, \eta_1) > 0, \quad b_\omega \circ \tilde{\pi} \circ \varphi^+_{t_2}(x_1, \eta_1) \neq 0.$$

By (2.10) we have $\varphi^-_{T-t}(x, \eta) = \varphi^-_{-t} \circ \varphi^-_T(x, \eta) = \varphi^+_t(x_1, \eta_1)$, which yields

$$b_\omega \circ \tilde{\pi} \circ \varphi^-_{T-t_0}(x, \eta) \neq 0, \quad b \circ \tilde{\pi} \circ \varphi^-_{T-t_1}(x, \eta) > 0, \quad b_\omega \circ \tilde{\pi} \circ \varphi^-_{T-t_2}(x, \eta) \neq 0,$$

with $0 < T - t_2 < T - t_1 < T - t_0 < T$. Hence $T > T^+_{\omega\to\mathcal{O}\to\omega}$ implies $T \geqq T^-_{\omega\to\mathcal{O}\to\omega}$, which gives $T^-_{\omega\to\mathcal{O}\to\omega} \leqq T^+_{\omega\to\mathcal{O}\to\omega}$. The opposite inequality is proven similarly. We obtain $T^-_{\omega\to\mathcal{O}\to\omega} = T^+_{\omega\to\mathcal{O}\to\omega}$.

Also, with the same argument, using Proposition 2.4 we obtain $T^-_{\omega\to\mathcal{O}\to\omega} = T^+_{\omega\to\mathcal{O}\to\omega} = T_{\omega\to\mathcal{O}\to\omega}$. □

The previous proposition is important in what follows. In fact we shall base part of our analysis on the flows $\varphi^{\pm}_s$. With Definition 2.5 and Proposition 2.6 we thus have a precise characterization of the geometrical control condition of Definition 1.2 in this context.

In what follows, for concision, we shall omit the projection $\tilde{\pi}$ when composing functions on Ω with the flows $\varphi^{\pm}$, that is, we shall write $b \circ \varphi^{\pm}_{t_1}(x, \eta)$ in place of $b \circ \tilde{\pi} \circ \varphi^{\pm}_{t_1}(x, \eta)$.

2.3. Reformulation of the System in Symmetric Spaces

As one can see, work in asymmetric spaces can be awkward. We thus set $w_1 = (1 - \Delta)^{-\frac{1}{2}} v_1$, $w_2 = v_2$. Having (v_1, v_2) as the solution to (1.6) is then equivalent to having (w_1, w_2) as the solution of

$$\begin{cases} P w_1 = 0 & \text{in } (0, T) \times \Omega, \\ P w_2 = -b(x)\, (1 - \Delta)^{\frac{1}{2}} w_1 & \text{in } (0, T) \times \Omega, \end{cases} \tag{2.12}$$

as P and $(1 - \Delta)^{-\frac{1}{2}}$ commute. Hence, System (1.1) is exactly controllable in time T if and only if the inequality

$$E_0(w_1(0)) + E_0(w_2(0)) \leqq C \int_0^T \int_\Omega |b_\omega w_2|^2 \, dx \, dt \tag{2.13}$$

is satisfied for all $(w_1, w_2) \in \mathscr{C}^0([0,T]; L^2(\Omega;\mathbb{C}^2)) \cap \mathscr{C}^1([0,T]; H^{-1}(\Omega;\mathbb{C}^2))$ solutions of System (2.12). Note that the observability inequality (2.13) corresponds also to the exact controllability of the following system

$$\begin{cases} Pz_1 + (1-\Delta)^{\frac{1}{2}} b(x)\, z_2 = 0 & \text{in } (0,T)\times\Omega, \\ Pz_2 = b_\omega(x)\, f & \text{in } (0,T)\times\Omega, \end{cases} \tag{2.14}$$

with initial and final data in $H^1(\Omega;\mathbb{C}^2)\times L^2(\Omega;\mathbb{C}^2)$ (see Section 5.1 for details). Connexion with System (1.1) is obtained by setting $z_1 = (1-\Delta)^{\frac{1}{2}} u_1$ and $z_2 = u_2$.

To prove the well-posedness of System (2.12), we introduce the space

$$\mathscr{H} = L^2(\Omega;\mathbb{C}^2)\times H^{-1}(\Omega;\mathbb{C}^2),$$

endowed with the natural inner product.

Proposition 2.7. *For any* $(w_1^0, w_2^0, w_1^1, w_2^1) \in \mathscr{H}$ *and any* $T > 0$, *System* (2.12) *with*

$$(w_1, w_2, \partial_t w_1, \partial_t w_2)|_{t=0} = (w_1^0, w_2^0, w_1^1, w_2^1),$$

has a unique solution $(w_1, w_2) \in \mathscr{C}^0(-T,T; L^2(\Omega;\mathbb{C}^2)) \cap \mathscr{C}^1(-T,T; H^{-1}(\Omega;\mathbb{C}^2))$, *depending continuously on* $(w_1^0, w_2^0, w_1^1, w_2^1)$, *that is*

$$\sup_{t\in(-T,T)} \{E_0(w_1(t)) + E_0(w_2(t))\} \leqq C(T)\Big(\|w_1^0\|^2_{L^2(\Omega)} + \|w_1^1\|^2_{H^{-1}(\Omega)} + \|w_2^0\|^2_{L^2(\Omega)} + \|w_2^1\|^2_{H^{-1}(\Omega)}\Big). \tag{2.15}$$

System (2.12) can be written as the first-order system

$$\partial_t \mathscr{W} + \mathscr{A}\mathscr{W} = 0, \tag{2.16}$$

where $\mathscr{W} = {}^t(w_1, w_2, \partial_t w_1, \partial_t w_2)$ and the operator $\mathscr{A}$ is given by

$$\mathscr{A} = \begin{pmatrix} 0 & 0 & -\mathrm{Id} & 0 \\ 0 & 0 & 0 & -\mathrm{Id} \\ -\Delta & 0 & 0 & 0 \\ b(1-\Delta)^{\frac{1}{2}} & -\Delta & 0 & 0 \end{pmatrix}, \quad D(\mathscr{A}) = H^1(\Omega;\mathbb{C}^2)\times L^2(\Omega;\mathbb{C}^2). \tag{2.17}$$

The Lumer–Phillips Theorem [34] can be applied to (2.16) for positive and negative times t since the operators $\lambda_0 \,\mathrm{Id} \pm \mathscr{A}$ are maximal monotone for λ_0 sufficiently large (due to the cascade structure of the system). Hence $-\mathscr{A}$ generates a strongly continuous group that we shall denote by $(e^{-t\mathscr{A}})_{t\in\mathbb{R}}$.

In places, we shall also write System (2.12) in the form

$$\mathcal{P}W = 0, \quad W = (w_1, w_2)^T,$$

with

$$\mathcal{P} = \begin{pmatrix} P & 0 \\ B & P \end{pmatrix} \in \Psi^2_{\mathrm{phg}}(M;\mathbb{C}^{2\times 2}), \quad \text{and } B = b(x)(1-\Delta)^{\frac{1}{2}}.$$

According to [41] or [42, Theorem 11.2], we have $B \in \Psi^1_{\mathrm{phg}}(\Omega;\mathbb{C})$, with principal symbol $\sigma_1(B)(x,\eta) = b|\eta|_x$.

3. Observability for $T > T_{\omega\to\mathcal{O}\to\omega}$

In this section, we prove the following theorem.

Proposition 3.1. *Suppose that $T > T_{\omega\to\mathcal{O}\to\omega}$. Then, the observability inequality (2.13) holds for any $\mathscr{C}^0(0,T;L^2(\Omega;\mathbb{C}^2))\cap\mathscr{C}^1(0,T;H^{-1}(\Omega;\mathbb{C}^2))$-solution of (2.12).*

The positive controllability result of Theorem 1.3 is then a direct consequence of Theorem 3.1.

To prove Proposition 3.1, we follow the compactness–uniqueness method of [8, 9, 38], which consists of two steps. First we prove the observability inequality (2.13) in a weaker form, with additional compact terms on the right hand-side. This allows one to handle high frequencies. Second, we use a uniqueness argument to handle low frequencies and conclude the proof of the observability inequality (2.13).

3.1. A Relaxed Observability Inequality

We shall prove the following result.

Proposition 3.2. *Suppose that $T > T_{\omega\to\mathcal{O}\to\omega}$. Then, the observability inequality*

$$\begin{aligned} &E_0(w_1(0)) + E_0(w_2(0)) \\ &\leqq C\left(\int_0^T\int_\Omega |b_\omega w_2|^2\,\mathrm{d}x\,\mathrm{d}t + E_{-1}(w_1(0)) + E_{-1}(w_2(0))\right) \end{aligned} \tag{3.1}$$

holds for any $\mathscr{C}^0(0,T;L^2(\Omega;\mathbb{C}^2))\cap\mathscr{C}^1(0,T;H^{-1}(\Omega;\mathbb{C}^2))$-solution of (2.12).

Proof. We proceed by contradiction and suppose that the observability inequality (3.1) is not satisfied. Thus, there exists a sequence $(w_1^k, w_2^k)_{k\in\mathbb{N}}$ of $\mathscr{C}^0(0,T;L^2(\Omega))\cap\mathscr{C}^1(0,T;H^{-1}(\Omega))$-solutions of

$$\begin{cases} Pw_1^k = 0 & \text{in } (0,T)\times\Omega, \\ Pw_2^k + Bw_1^k = 0 & \text{in } (0,T)\times\Omega, \end{cases} \tag{3.2}$$

such that

$$E_0(w_1^k(0)) + E_0(w_2^k(0)) = 1, \tag{3.3}$$

$$\int_0^T\int_\Omega |b_\omega w_2^k|^2\,\mathrm{d}x\,\mathrm{d}t \to 0, \quad k\to\infty, \tag{3.4}$$

$$E_{-1}(w_1^k(0)) + E_{-1}(w_2^k(0)) \to 0, \quad k\to\infty. \tag{3.5}$$

According to (3.3) and the continuity of the solution with respect to the initial data, the sequence (w_1^k, w_2^k) is bounded in $L^2(M_T;\mathbb{C}^2)$. According to (3.5), we have $(w_1^k(0), w_2^k(0), \partial_t w_1^k(0), \partial_t w_2^k(0)) \rightharpoonup (0,0,0,0)$ in $L^2(\Omega;\mathbb{C}^2)\times H^{-1}(\Omega;\mathbb{C}^2)$. The continuity of the solution with respect to the initial data yields

$$(w_1^k, w_2^k) \rightharpoonup (0,0) \quad \text{in } L^2(M_T;\mathbb{C}^2).$$

As a consequence of [21, Theorem 1], there exists a subsequence of $(W^k)_{k\in\mathbb{N}} = (w_1^k, w_2^k)_{k\in\mathbb{N}}$ (still denoted $(W^k)_{k\in\mathbb{N}} = (w_1^k, w_2^k)_{k\in\mathbb{N}}$ in what follows) and a microlocal defect measure

$$\mu = \begin{pmatrix} \mu_1 & \mu_{12} \\ \overline{\mu_{12}} & \mu_2 \end{pmatrix} \in \mathcal{M}_+(S^*M_T; \mathbb{C}^{2\times 2}),$$

(according to [21,43], see also [10, Proposition 5], this measure is intrinsically defined on S^*M_T) such that for any $\mathcal{A} \in \Psi^0_{\mathrm{phg}}(M_T; \mathbb{C}^{2\times 2})$ (recall that symbols are compactly supported in time t here, see Section 2.1),

$$\lim_{k\to\infty} (\mathcal{A}W^k, W^k)_{L^2(M_T;\mathbb{C}^2)} = \int_{S^*M_T} \mathrm{tr}\{\sigma_0(\mathcal{A})(\rho)\mu(\mathrm{d}\rho)\}. \tag{3.6}$$

Testing the measure μ on different operators $\mathcal{A}$, the limit equation (3.6) can be equivalently written as

$$\begin{cases} \lim_{k\to\infty} (Aw_1^k, w_1^k)_{L^2(M_T;\mathbb{C})} = \langle \mu_1, \sigma_0(A)\rangle_{S^*M_T}, \\ \lim_{k\to\infty} (Aw_2^k, w_2^k)_{L^2(M_T;\mathbb{C})} = \langle \mu_2, \sigma_0(A)\rangle_{S^*M_T}, \\ \lim_{k\to\infty} (Aw_1^k, w_2^k)_{L^2(M_T;\mathbb{C})} = \langle \mu_{12}, \sigma_0(A)\rangle_{S^*M_T}, \end{cases} \tag{3.7}$$

for any $A \in \Psi^0_{\mathrm{phg}}(M_T; \mathbb{C})$.

The following lemma gives the properties of the three measures μ_1, μ_2 and μ_{12}, and is a key point in the proof of Proposition 3.2.

Lemma 3.3. (Properties of the measure μ)

1. *Suppose that the sequence* $(W^k)_{k\in\mathbb{N}}$ *satisfies* (3.2) *and* (3.5). *Then, we have* $\mu_1, \mu_2 \in \mathcal{M}_+(S^*M_T)$, $\mathrm{supp}(\mu_1) \subset \mathrm{Char}(P)$, $\mathrm{supp}(\mu_2) \subset \mathrm{Char}(P)$. *Moreover,* $\mathrm{supp}(\mu_{12}) \subset \mathrm{supp}(\mu_1) \cap \mathrm{supp}(\mu_2) \subset \mathrm{Char}(P)$. *Finally, these three measures satisfy the equations*

$$\begin{cases} \langle \mu_1, H_p a\rangle_{S^*M_T} = 0, \\ \langle \mu_2, H_p a\rangle_{S^*M_T} = -\langle \mathrm{Im}(\mu_{12}), 2b|\eta|_x a\rangle_{S^*M_T}, \\ \langle \mathrm{Im}(\mu_{12}), H_p a\rangle_{S^*M_T} = -\langle \mu_1, b|\eta|_x a\rangle_{S^*M_T}, \\ \langle \mathrm{Re}(\mu_{12}), H_p a\rangle_{S^*M_T} = 0, \end{cases} \tag{3.8}$$

   for any $a \in S^{-1}_{\mathrm{phg}}(T^*M_T; \mathbb{C})$.
2. *Moreover if the sequence* $(W^k)_{k\in\mathbb{N}}$ *satisfies* (3.4), *then we also have* $\mathrm{supp}(\mu_2) \cap \pi^{-1}((0,T)\times\omega) = \emptyset$ *and* $\mathrm{supp}(\mu_{12}) \cap \pi^{-1}((0,T)\times\omega) = \emptyset$.

The proof of this lemma is given in Appendix B.2.

We shall prove below that the first and the last equations in (3.8) yield a "free" propagation result for the measures μ_1 and $\mathrm{Re}(\mu_{12})$. In particular, since $\mu_{12} = 0$ on $\pi^{-1}((0,T)\times\omega)$, and $T > T_{\omega\to\mathcal{O}\to\omega} \geqq T_\omega$, this will give $\mathrm{Re}(\mu_{12}) = 0$.

The most important equation in (3.8) is the third one that is to be viewed as a transport equation for the measure $\mathrm{Im}(\mu_{12})$.

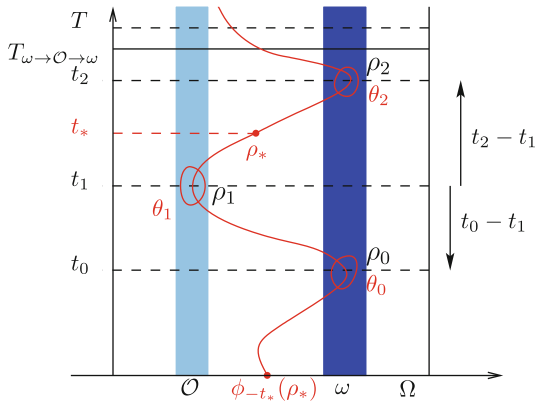

Fig. 3. Geometrical situation in the case $T > T_{\omega\to\mathcal{O}\to\omega}$

We already know that the supports of the three measures are subsets of $\mathrm{Char}(P)\cap S^*M_T$. Let us pick $\rho_* = (t_*, x_*, \tau_*, \eta_*) \in \mathrm{Char}(P)\cap S^*M_T$. According to Definition 2.3, there exist $0 < t_0 < t_1 < t_2 < T$ such that

$$\begin{aligned}&\phi_{t_0-t_*}(\rho_*) \in \pi^{-1}((0,T)\times\omega), \quad \phi_{t_1-t_*}(\rho_*) \in \pi^{-1}((0,T)\times\mathcal{O}),\\ &\phi_{t_2-t_*}(\rho_*) \in \pi^{-1}((0,T)\times\omega).\end{aligned}$$

We take three open subsets $\theta_0, \theta_1, \theta_2 \subset S^*M_T$, satisfying

$$\begin{gathered}\rho_j = \phi_{t_j-t_*}(\rho_*) \in \theta_j \text{ for } j = 0,1,2,\\ \pi(\theta_0)\subset(0,T)\times\omega, \quad \pi(\theta_1)\subset(0,T)\times\mathcal{O}, \quad \pi(\theta_2)\subset(0,T)\times\omega.\end{gathered} \tag{3.9}$$

This geometrical situation is illustrated in Fig. 3.

We now choose a function $e \in \mathscr{C}_c^\infty(S^*M_T)$ such that

$$\mathrm{supp}(e) \subset \phi_{t_1-t_0}(\theta_0)\cap\theta_1\cap\phi_{t_1-t_2}(\theta_2), \quad e \geqq 0, \quad \text{and} \quad e(\rho_1) = 1. \tag{3.10}$$

Note that the set $\phi_{t_1-t_0}(\theta_0)\cap\theta_1\cap\phi_{t_1-t_2}(\theta_2)$ is open since $\phi_t : S^*M_T \to S^*M_T$ is bicontinuous and is nonempty since it contains ρ_1, according to (3.9).

Now, we apply the third identity of (3.8) to $a = e\circ\phi_s \in \mathscr{C}_c^\infty(S^*M_T)$ (which can be extended as a symbol, namely $\tilde{\lambda}^{-1}a(t,x,\tau/\tilde{\lambda},\eta/\tilde{\lambda}) \in S^{-1}_{\mathrm{phg}}(T^*M_T;\mathbb{R})$, where $\tilde{\lambda}$ is defined in (2.2)) for $s \in (t_1-t_2, t_1-t_0)$, as a consequence of (3.9)–(3.10). From the third equation in (3.8) we obtain

$$\begin{aligned}-\langle\mu_1, b|\eta|_x\, e\circ\phi_s\rangle_{S^*M_T} &= \langle \mathrm{Im}(\mu_{12}), H_p(e\circ\phi_s)\rangle_{S^*M_T}\\ &= \left\langle \mathrm{Im}(\mu_{12}), \frac{\mathrm{d}}{\mathrm{d}s} e\circ\phi_s\right\rangle_{S^*M_T}\\ &= \frac{\mathrm{d}}{\mathrm{d}s}\langle \mathrm{Im}(\mu_{12}), e\circ\phi_s\rangle_{S^*M_T}.\end{aligned}$$

Integrating this equation on the interval $(t_1 - t_2, t_1 - t_0)$ gives

$$- \int_{t_1-t_2}^{t_1-t_0} \langle \mu_1, b|\eta|_x \, e \circ \phi_s \rangle_{S^*M_T} \, \mathrm{d}s = \left\langle \mathrm{Im}(\mu_{12}), e \circ \phi_{t_1-t_0} \right\rangle_{S^*M_T} - \left\langle \mathrm{Im}(\mu_{12}), e \circ \phi_{t_1-t_2} \right\rangle_{S^*M_T}. \quad (3.11)$$

Moreover, from (3.10), we have $\mathrm{supp}(e \circ \phi_{t_1-t_2}) \subset \theta_2$ and $\mathrm{supp}(e \circ \phi_{t_1-t_0}) \subset \theta_0$. Since $\pi(\theta_0), \pi(\theta_2) \subset (0,T) \times \omega$ and $\pi(\mathrm{supp}(\mu_{12})) \cap (0,T) \times \omega = \emptyset$ from Lemma 3.3, the right hand-side of (3.11) vanishes, and we obtain

$$\int_{t_1-t_2}^{t_1-t_0} \langle \mu_1, b|\eta|_x \, e \circ \phi_s \rangle_{S^*M_T} \, \mathrm{d}s = 0.$$

In this expression, the measure μ_1 and the functions $|\eta|_x$, b and $e \circ \phi_s$ are non-negative on the interval $(t_1 - t_2, t_1 - t_0)$. We thus obtain, for any $s \in (t_1 - t_2, t_1 - t_0)$,

$$\langle \mu_1, b|\eta|_x \, e \circ \phi_s \rangle_{S^*M_T} = 0,$$

and in particular for $s = 0 \in (t_1 - t_2, t_1 - t_0)$ we find

$$\langle \mu_1, b|\eta|_x \, e \rangle_{S^*M_T} = 0.$$

We have $b\big(\pi(\rho_1)\big) > 0$ since $\pi(\rho_1) \in (0,T) \times \mathcal{O}$, and $e(\rho_1) = 1$. Hence we have that

$$\mu_1 \text{ vanishes in a neighborhood of } \rho_1. \quad (3.12)$$

We can now prove that μ_1 is identically zero on this bicharacteristic. This is a direct consequence of the first equation of (3.8): applied to $a \circ \phi_s$ in place of a as long as $\mathrm{supp}(a \circ \phi_s) \subset S^*M_T$, it yields

$$0 = \left\langle \mu_1, H_p a \circ \phi_s \right\rangle_{S^*M_T} = \left\langle \mu_1, \frac{\mathrm{d}}{\mathrm{d}s} a \circ \phi_s \right\rangle_{S^*M_T} = \frac{\mathrm{d}}{\mathrm{d}s} \langle \mu_1, a \circ \phi_s \rangle_{S^*M_T}.$$

This directly gives

$$\langle \mu_1, a \circ \phi_s \rangle_{S^*M_T} = \langle \mu_1, a \rangle_{S^*M_T}, \quad \text{for all } a \in S^{-1}_{\mathrm{phg}}(T^*M_T; \mathbb{C}), \\ \text{for all } s \in \mathbb{R} \text{ such that } \mathrm{supp}(a \circ \phi_s) \subset S^*M_T. \quad (3.13)$$

From the argument above, there exists $a \in S^{-1}_{\mathrm{phg}}(T^*M_T; \mathbb{C})$ with a small support in a neighborhood of ρ_1 and $a(\rho_1) = 1$, such that $\langle \mu_1, a \rangle_{S^*M_T} = 0$. Taking $s = t_1 - t_*$ in (3.13) then yields

$$\left\langle \mu_1, a \circ \phi_{t_1-t_*} \right\rangle_{S^*M_T},$$

with $a \circ \phi_{t_1-t_*}(\rho_*) = 1$. This implies that μ_1 vanishes in a neighborhood of ρ_*. Here ρ_* was chosen arbitrarily in S^*M_T. We thus have $\mu_1 = 0$ on S^*M_T.

The third equation of (3.8) then becomes $\left\langle \mathrm{Im}(\mu_{12}), H_p a \right\rangle_{S^*M_T} = 0$ for all $a \in S^{-1}_{\mathrm{phg}}(T^*M_T; \mathbb{C})$. The same analysis directly yields the propagation of the measure $\mathrm{Im}(\mu_{12})$. This, together with $\mathrm{supp}(\mathrm{Im}(\mu_{12})) \cap \pi^{-1}((0,T) \times \omega) = \emptyset$

gives $\operatorname{Im}(\mu_{12}) = 0$ on S^*M_T, as $T > T_{\omega\to\mathcal{O}\to\omega} \geqq T_\omega$. Similarly, we obtain $\operatorname{Re}(\mu_{12}) = 0$ on S^*M_T with the last equation of (3.8). Finally, the second equation of (3.8) now reads $\langle \mu_2, H_p a\rangle_{S^*M_T} = 0$ for all $a \in S^{-1}_{\mathrm{phg}}(T^*M_T;\mathbb{C})$. This implies $\mu_2 = 0$ on S^*M_T as we already know that $\operatorname{supp}(\mu_2) \cap \pi^{-1}((0,T)\times\omega) = \emptyset$ from Lemma 3.3.

Since $\mu = 0$, we have

$$(w_1^k, w_2^k) \to (0,0) \quad \text{strongly in } L^2_{\mathrm{loc}}(M_T;\mathbb{C}^2). \tag{3.14}$$

Let us take $\chi \in \mathscr{C}_c^\infty(0,T;\mathbb{C})$, multiply the second equation in (3.2) by $\chi(1-\Delta)^{-1}w_2^k$ and integrate on $(0,T)\times\Omega$. After an integration by parts in time, this gives

$$\begin{aligned}\int_0^T\int_\Omega \chi|(1-\Delta)^{-\frac12}\partial_t w_2^k|^2\,\mathrm{d}x\,\mathrm{d}t + \int_0^T\int_\Omega \partial_t\chi\,\partial_t w_2^k(1-\Delta)^{-1}w_2^k\,\mathrm{d}x\,\mathrm{d}t \\ + \int_0^T\int_\Omega \chi\Delta w_2^k(1-\Delta)^{-1}w_2^k\,\mathrm{d}x\,\mathrm{d}t = \int_0^T\int_\Omega \chi B w_1^k(1-\Delta)^{-1}w_2^k\,\mathrm{d}x\,\mathrm{d}t.\end{aligned}$$

In this expression, we have

$$\begin{aligned}\int_0^T\int_\Omega \partial_t\chi\,\partial_t w_2^k(1-\Delta)^{-1}w_2^k \to 0 \quad &\text{since } \partial_t\chi\,\partial_t(1-\Delta)^{-1} \in \Psi^{-1}_{\mathrm{phg}}(M_T;\mathbb{C}),\\ \int_0^T\int_\Omega \chi B w_1^k(1-\Delta)^{-1}w_2^k \to 0 \quad &\text{since } \chi(1-\Delta)^{-1}B \in \Psi^{-1}_{\mathrm{phg}}(M_T;\mathbb{C}),\end{aligned}$$

and

$$\int_0^T\int_\Omega \chi\Delta w_2^k(1-\Delta)^{-1}w_2^k \to \langle\mu_2,\chi\rangle_{S^*M_T} = 0.$$

As a consequence, we obtain, for all $\chi \in \mathscr{C}_c^\infty(0,T;\mathbb{C})$,

$$\int_0^T\int_\Omega \chi|(1-\Delta)^{-\frac12}\partial_t w_2^k|^2 \to 0.$$

This, together with (3.14), yields, for all $0 < \varepsilon_1 < \varepsilon_2 < T$,

$$\int_{\varepsilon_1}^{\varepsilon_2} E_0(w_2^k(t))\,\mathrm{d}t \to 0.$$

The same method with the first equation of (3.2) also gives $\int_{\varepsilon_1}^{\varepsilon_2} E_0(w_1^k(t))\,\mathrm{d}t \to 0$. Hence, $E_0(w_2^k(t)) + E_0(w_1^k(t)) \to 0$ for almost every $t \in (\varepsilon_1,\varepsilon_2)$. Picking such a time, say t_3, the (backward) well-posedness result of Proposition 2.7 for the Cauchy problem (3.2) with data given at t_3 gives

$$E_0(w_1^k(0)) + E_0(w_2^k(0)) \leqq C\left(E_0(w_1^k(t_3)) + E_0(w_2^k(t_3))\right).$$

This yields $E_0(w_1^k(0)) + E_0(w_2^k(0)) \to 0$, gives a contradiction with (3.3), and concludes the proof of the proposition. □

3.2. *End of the Proof of Proposition 3.1*

With the relaxed observability inequality of Proposition 3.2, we are now able to handle the low-frequencies and conclude the proof Theorem 3.1. The main point here is a unique continuation result for solutions of the elliptic problem associated with System (2.12). The idea of reducing the observability for the low frequencies to an elliptic unique continuation result and associated technology are due to [8]. Here we follow the expository lectures [12].

We first define for any $T > 0$ the set of invisible solutions (see [8]) from $(0, T) \times \omega$:

$$\mathcal{N}(T) = \{\mathscr{W} = (w_1^0, w_2^0, w_1^1, w_2^1) \in \mathscr{H} \text{ such that the associated solution of (2.12)} \\ \text{satisfies } w_2(t, x) = 0 \text{ for all } (t, x) \in (0, T) \times \omega\}.$$

We have the following key lemma, which is proved at the end of this section.

Lemma 3.4. *For $T > T_{\omega\to\mathcal{O}\to\omega}$, we have $\mathcal{N}(T) = \{0\}$.*

As for the proof of the relaxed observability inequality of Proposition 3.2 we proceed by contradiction. We suppose that the result of Theorem 3.1 is false. Thus, there exists a sequence $(w_1^k, w_2^k)_{k\in\mathbb{N}}$ of $\mathscr{C}^0(0, T; L^2(\Omega)) \cap \mathscr{C}^1(0, T; H^{-1}(\Omega))$-solutions of (2.12) such that

$$E_0(w_1^k(0)) + E_0(w_2^k(0)) = 1, \tag{3.15}$$

$$\int_0^T \int_\Omega |b_\omega w_2^k|^2 \, \mathrm{d}x \, \mathrm{d}t \to 0. \tag{3.16}$$

Equation (3.15) and Proposition 2.7 imply that the sequence $(w_1^k, w_2^k)_{k\in\mathbb{N}}$ is bounded in $L^2(M_T; \mathbb{C}^2)$. Hence, there exists a subsequence (also denoted $(w_1^k, w_2^k)_{k\in\mathbb{N}}$ in what follows) weakly converging in $L^2(M_T; \mathbb{C}^2)$, towards $(w_1, w_2) \in L^2(M_T; \mathbb{C}^2)$. From (3.16), this limit satisfies, for all $t \in (0, T)$, $w_2|_\omega = 0$, and is moreover a solution of (2.12). Hence, we have $(w_1(0), w_2(0), \partial_t w_1(0), \partial_t w_2(0)) \in \mathcal{N}(T)$. According to Lemma 3.4, this yields $(w_1, w_2) = (0, 0)$. Also, the imbedding $\mathscr{H} \hookrightarrow H^{-1}(\Omega)^2 \times H^{-2}(\Omega)^2$ is compact. This yields

$$E_{-1}(w_1^k(0)) + E_{-1}(w_2^k(0)) \to E_{-1}(w_1(0)) + E_{-1}(w_2(0)).$$

The relaxed observability inequality (3.1) hence yields

$$1 \leqq C\left(E_{-1}(w_1(0)) + E_{-1}(w_2(0))\right),$$

which contradicts the fact that $(w_1, w_2) = (0, 0)$, and concludes the proof of the Proposition 3.1. □

It only remains to prove Lemma 3.4.

Proof of Lemma 3.4. First, Proposition 2.7 implies that $\mathcal{N}(T)$ is a closed subspace of $\mathscr{H}$. Second, applying the relaxed observability inequality (3.1) to an element of $\mathcal{N}(T)$ gives

$$\|\mathscr{W}\|_{\mathscr{H}}^2 = E_0(w_1(0)) + E_0(w_2(0)) \leqq C\left(E_{-1}(w_1(0)) + E_{-1}(w_2(0))\right). \tag{3.17}$$

Using the compact imbedding $\mathscr{H} \hookrightarrow H^{-1}(\Omega)^2 \times H^{-2}(\Omega)^2$, this implies that $\mathcal{N}(T)$ has a finite dimension, and is thus complete for any norm. Moreover, setting $\delta = \frac{1}{2}(T - T_{\omega\to\mathcal{O}\to\omega}) > 0$, we remark that (3.17) is also satisfied by all $\mathscr{W} \in \mathcal{N}(T-\delta)$. Taking $\mathscr{W} \in \mathcal{N}(T)$ implies that, for all $\varepsilon \in (0,\delta)$, we have $e^{-\varepsilon\mathscr{A}}\mathscr{W} \in \mathcal{N}(T-\delta)$. We also have, for λ_0 sufficiently large, according to Section 2.3,

$$\begin{aligned}(\lambda_0 + \mathscr{A})^{-1}\frac{1}{\varepsilon}(\mathrm{Id} - e^{-\varepsilon\mathscr{A}})\mathscr{W} &= \frac{1}{\varepsilon}(\mathrm{Id} - e^{-\varepsilon\mathscr{A}})(\lambda_0 + \mathscr{A})^{-1}\mathscr{W} \\ &\underset{\varepsilon\to 0^+}{\to} \mathscr{A}(\lambda_0 + \mathscr{A})^{-1}\mathscr{W} \quad \text{in } \mathscr{H},\end{aligned}$$

as $(\lambda_0 + \mathscr{A})^{-1}\mathscr{W} \in D(\mathscr{A})$. As a consequence, the sequence $\left(\frac{1}{\varepsilon}(\mathrm{Id} - e^{-\varepsilon\mathscr{A}})\mathscr{W}\right)_{\varepsilon>0}$ is a Cauchy sequence in $\mathcal{N}(T-\delta)$, endowed with the norm $\|(\lambda_0 + \mathscr{A})^{-1}\cdot\|_{\mathscr{H}}$. As all norms are equivalent, the sequence $\left(\frac{1}{\varepsilon}(\mathrm{Id} - e^{-\varepsilon\mathscr{A}})\mathscr{W}\right)_{\varepsilon>0}$ is thus also a Cauchy sequence in $\mathcal{N}(T-\delta)$, endowed with the norm $\|\cdot\|_{\mathscr{H}}$, which yields $\mathscr{A}\mathscr{W} \in \mathscr{H}$. Hence, we have $\mathcal{N}(T) \subset D(\mathscr{A})$. Denoting by $\tilde{\mathscr{W}}(t)$ the solution of

$$\begin{aligned}&\partial_t\tilde{\mathscr{W}} + \mathscr{A}\tilde{\mathscr{W}} = 0, \quad \tilde{\mathscr{W}}|_{t=0} = \mathscr{W}, \\ &\quad \text{with } \tilde{\mathscr{W}} \in \mathscr{C}^1([0,T];\mathscr{H}) \cap \mathscr{C}^0([0,T];D(\mathscr{A})),\end{aligned}$$

from semigroup theory, we remark that we have

$$-\mathscr{A}\mathscr{W} = \partial_t\tilde{\mathscr{W}}|_{t=0} \in \mathcal{N}(T) \quad \text{if} \quad \mathscr{W} \in \mathcal{N}(T).$$

In fact, as $w_2(t,x) = 0$ for all $(t,x) \in (0,T)\times\omega$, the same holds for $\partial_t w_2$. Consequently $\mathscr{A}\mathcal{N}(T) \subset \mathcal{N}(T)$.

Since $\mathcal{N}(T)$ is a finite dimensional subspace of $D(\mathscr{A})$, stable by the action of the operator $\mathscr{A}$, it contains an eigenfunction of $\mathscr{A}$. There exist $\mu \in \mathbb{C}$ and $\mathscr{W}_\mu \in \mathcal{N}(T)$ such that $\mathscr{A}\mathscr{W}_\mu = \mu\mathscr{W}_\mu$. Writing $\mathscr{W}_\mu = {}^t(w_1^0, w_2^0, w_1^1, w_2^1)$, this is equivalent to having

$$\begin{cases} -w_1^1 = \mu w_1^0, \\ -w_2^1 = \mu w_2^0, \\ -\Delta w_1^0 = \mu w_1^1, \\ -\Delta w_2^0 + b(x)(1-\Delta)^{\frac{1}{2}} w_1^0 = \mu w_2^1. \end{cases}$$

This system implies

$$\begin{cases} -\Delta w_1^0 = -\mu^2 w_1^0, \\ -\Delta w_2^0 + b(x)(1-\Delta)^{\frac{1}{2}} w_1^0 = -\mu^2 w_2^0. \end{cases} \tag{3.18}$$

We first prove that $w_1^0 = 0$ on Ω. If not, the first equation gives $\mu = i\sqrt{\kappa}$ with $\kappa \in \mathrm{Sp}(-\Delta) \subset \mathbb{R}_+$ and w_1^0 is an eigenfunction of $-\Delta$ associated to κ. Hence, taking the $L^2(\Omega)$-inner product of the first line of (3.18) with $(1-\Delta)^{\frac{1}{2}} w_2^0$, and that of the second line of (3.18) with $(1-\Delta)^{\frac{1}{2}} w_1^0$, we obtain

$$\int_\Omega b(x)|(1-\Delta)^{\frac{1}{2}} w_1^0|^2 \, \mathrm{d}x = 0.$$

Since $b \geqq 0$ and b does not vanish identically, this proves that $(1-\Delta)^{\frac{1}{2}} w_1^0 = 0$ on $\mathcal{O}$. As $(1-\Delta)^{\frac{1}{2}} w_1^0$ is an eigenfunction of the Laplace operator vanishing on $\mathcal{O}$, a unique continuation result (see for instance the classical reference [6,7], the book [47] or the exposition article [25]) yields $(1-\Delta)^{\frac{1}{2}} w_1^0 = 0$ on Ω. Hence $w_1^0 = 0$ on Ω.

Moreover, $\mathscr{W} \in \mathcal{N}(T)$ yields $w_2^0 = 0$ on ω. This proves that $w_2^0 = 0$ on Ω, as w_2^0 is an eigenfunction of the Laplace operator as a consequence of (3.18). This concludes the proof of the Lemma 3.4. □

Remark 3.5. Note that elliptic unique continuation properties such as those used here are not known in general for 2×2 elliptic systems. For these types of general systems, such a result holds if $\omega \cap \mathcal{O} \neq \emptyset$ [26, Proposition 5.1]. However the case $\omega \cap \mathcal{O} = \emptyset$ remains open in general.

Here, the cascade structure of System (2.12) allows us to bypass a more involved unique continuation theorem. We use that the eigenvalues and eigenfunctions of the operator

$$\begin{pmatrix} -\Delta & 0 \\ b(1-\Delta)^{\frac{1}{2}} & -\Delta \end{pmatrix},$$

are $(\kappa_j, {}^t(0, \varphi_j))_{j \in \mathbb{N}}$, where $(\kappa_j, \varphi_j)_{j \in \mathbb{N}}$ are the eigenvalues and eigenfunctions of $-\Delta$. This is a very particular feature of cascade systems with twice the same elliptic operator on the diagonal.

4. Lack of Observability for $T < T_{\omega \to \mathcal{O} \to \omega}$

In this section, we prove the following theorem.

Theorem 4.1. *Suppose that $T < T_{\omega \to \mathcal{O} \to \omega}$. Then, there exists a bounded sequence $(w_1^k, w_2^k)_{k \in \mathbb{N}}$ of $\mathscr{C}^0(0, T; L^2(\Omega)^2) \cap \mathscr{C}^1(0, T; H^{-1}(\Omega)^2)$-solutions of (2.12) satisfying*

$$\begin{cases} \liminf_{k \to \infty} \left(E_0(w_1^k(0)) + E_0(w_2^k(0)) \right) \geqq 1, \\ (w_1^k, w_2^k) \rightharpoonup 0 \text{ in } L^2(M_T; \mathbb{C}^2), \end{cases} \tag{4.1}$$

such that the sequence $(w_2^k)_{k \in \mathbb{N}}$ is pure and its microlocal defect measure μ_2 satisfies

$$\left\langle \mu_2, \chi^2 |b_\omega|^2 \right\rangle_{S^* M_T} = 0, \tag{4.2}$$

for any $\chi \in \mathscr{C}_c^\infty(0, T)$.

We refer to [21, Definition 1.3] for the definition of a pure sequence. As a direct consequence of this theorem, we have the following non-observability result.

Corollary 4.2. *If* $T < T_{\omega\to\mathcal{O}\to\omega}$*, the observability inequality* (2.13) *does not hold.*

The negative controllability result of Theorem 1.3 is then a direct consequence of Corollary 4.2 and HUM.

Proof of Corollary 4.2. Suppose that the observability inequality (2.13) holds for some constant $C > 0$, for all solutions of (2.12). Then, for the sequence $(w_1^k, w_2^k)_{k\in\mathbb{N}}$ given by Theorem 4.1, we have, for k sufficiently large,

$$\frac{1}{2} \leqq E_0(w_1^k(0)) + E_0(w_2^k(0)) \leqq K \int_0^T \int_\Omega |b_\omega w_2^k|^2 \,\mathrm{d}x\,\mathrm{d}t, \quad K > 0. \tag{4.3}$$

With the sequence $(w_1^k, w_2^k)_{k\in\mathbb{N}}$ being bounded in the energy space, we have, in particular

$$\int_\Omega |b_\omega w_2^k(t,x)|^2 \,\mathrm{d}x \leqq C_0, \quad \text{for some } C_0 > 0, \text{ and all } k \in \mathbb{N}, t \in [0,T].$$

This yields, for any $\varepsilon \in (0, \frac{T}{2})$,

$$\begin{aligned}\int_0^T \int_\Omega |b_\omega w_2^k|^2 \,\mathrm{d}x\,\mathrm{d}t &= \int_0^\varepsilon \int_\Omega |b_\omega w_2^k|^2 \,\mathrm{d}x\,\mathrm{d}t + \int_\varepsilon^{T-\varepsilon} \int_\Omega |b_\omega w_2^k|^2 \,\mathrm{d}x\,\mathrm{d}t \\ &\quad + \int_{T-\varepsilon}^T \int_\Omega |b_\omega w_2^k|^2 \,\mathrm{d}x\,\mathrm{d}t \\ &\leqq 2\varepsilon C_0 + \int_\varepsilon^{T-\varepsilon} \int_\Omega |b_\omega w_2^k|^2 \,\mathrm{d}x\,\mathrm{d}t \\ &\leqq 2\varepsilon C_0 + \int_0^T \int_\Omega |\chi b_\omega w_2^k|^2 \,\mathrm{d}x\,\mathrm{d}t,\end{aligned}$$

with $\chi \in \mathscr{C}_c^\infty(0,T)$ such that $\chi = 1$ on $(\varepsilon, T-\varepsilon)$. We fix ε such that $2\varepsilon C_0 = \frac{1}{8K}$ and obtain

$$\int_0^T \int_\Omega |b_\omega w_2^k|^2 \,\mathrm{d}x\,\mathrm{d}t \leqq \frac{1}{8K} + \int_0^T \int_\Omega |\chi b_\omega w_2^k|^2 \,\mathrm{d}x\,\mathrm{d}t.$$

According to (4.2), this gives

$$\int_0^T \int_\Omega |b_\omega w_2^k|^2 \,\mathrm{d}x\,\mathrm{d}t \leqq \frac{1}{4K},$$

for k sufficiently large. This yields a contradiction with (4.3), and concludes the proof of the corollary. □

We shall use the following lemma in the proof of Theorem 4.1. A proof is given in Appendix B.3.

Lemma 4.3. *For any bicharacteristic curve* Γ *of the d'Alembert operator* P*, there exists a pure sequence* $(w^k)_{k\in\mathbb{N}}$ *of* $\mathscr{C}^0(0,T; L^2(\Omega)) \cap \mathscr{C}^1(0,T; H^{-1}(\Omega))$*-solutions of* $Pw^k = 0$ *such that*

$$\begin{cases} \liminf_{k\to\infty} E_0(w^k(0)) \geqq 1, \\ w^k \rightharpoonup 0 \text{ in } L^2(M_T;\mathbb{C}), \end{cases} \tag{4.4}$$

and the microlocal defect measure of $(w^k)_{k\in\mathbb{N}}$ *is supported in* Γ.

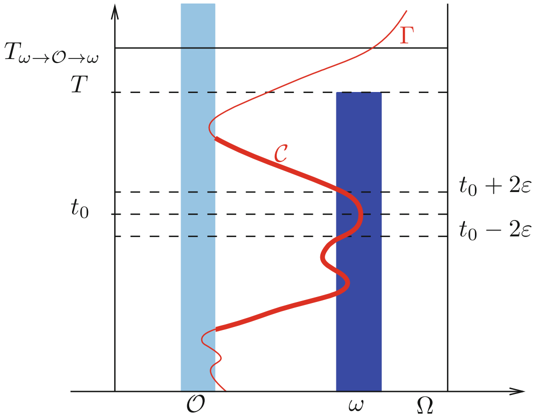

Fig. 4. Geometrical situation in the case $T < T_{\omega\to\mathcal{O}\to\omega}$ with $\pi(\Gamma) \cap \big((0, T) \times \omega\big) \neq \emptyset$

We now prove Theorem 4.1.

Proof of Theorem 4.1. Since $T < T_{\omega\to\mathcal{O}\to\omega}$, Definition 2.3 gives the existence of $\rho_* = (0, x_*, \tau_*, \eta_*) \in \mathrm{Char}(P) \cap S^*M$ such that the following condition holds:

$$\begin{array}{l} \text{For any } 0 < t_0 < t_1 < t_2 < T, \text{ we have } \ \pi(\phi_{t_0}(\rho_*)) \notin (0, T) \times \omega, \\ \text{or } \ \pi(\phi_{t_1}(\rho_*)) \notin (0, T) \times \mathcal{O}, \quad \text{or } \ \pi(\phi_{t_2}(\rho_*)) \notin (0, T) \times \omega. \end{array} \tag{4.5}$$

We set

$$\Gamma = \{\phi_s(\rho_*), s \in [0, T]\}.$$

We now construct the sequence $(w_1^k, w_2^k)_{k\in\mathbb{N}}$.

First, if $\pi(\Gamma) \cap \big((0, T) \times \omega\big) = \emptyset$ (which only occurs if $T < T_\omega$), we take $w_1^k = 0$ for all $k \in \mathbb{N}$ and $(w_2^k)_{k\in\mathbb{N}}$ as given by Lemma 4.3. In this case, (4.2) is clear as $\mathrm{supp}(\mu_2) \subset \Gamma$ and $\pi(\Gamma) \cap \big((0, T) \times \omega\big) = \emptyset$ (this is a classical non-observability result for a single wave equation [8,9,11]).

Second, if $\pi(\Gamma) \cap \big((0, T) \times \omega\big) \neq \emptyset$, we choose $(w_1^k)_{k\in\mathbb{N}}$ as given by Lemma 4.3. In particular, according to Lemma 4.3, this gives $\liminf_{k\to\infty} \big(E_0(w_1^k(0)) + E_0(w_2^k(0))\big) \geqq 1$ for all $\mathscr{C}^0(0, T; L^2(\Omega)) \cap \mathscr{C}^1(0, T; H^{-1}(\Omega))$-sequence $(w_2^k)_{k\in\mathbb{N}}$. We also pick some $t_0 \in (0, T)$ such that $\pi(\phi_{t_0}(\rho_*)) \in (0, T) \times \omega$. The geometrical situation is sketched out in Fig. 4.

We choose w_2^k to be the unique (forward and backward) solution of

$$\begin{cases} P w_2^k = -B w_1^k & \text{in } (0, T) \times \Omega, \\ (w_2^k, \partial_t w_2^k)|_{t=t_0} = (0, 0) & \text{on } \Omega. \end{cases} \tag{4.6}$$

In particular, the well-posedness of the wave equation yields $w_2^k \in \mathscr{C}^0(0, T; L^2(\Omega)) \cap \mathscr{C}^1(0, T; H^{-1}(\Omega))$ and

$$\|w_2^k\|_{L^2(M_T)} \leqq C \|B w_1^k\|_{L^2(0,T;H^{-1}(\Omega))} \leqq C \|w_1^k\|_{L^2(M_T)},$$

as $B = b(1-\Delta)^{\frac{1}{2}}$. Since $(w_1^k)_{k\in\mathbb{N}}$ is bounded in $L^2(M_T;\mathbb{C})$ from Lemma 4.3, the sequence $(w_1^k, w_2^k)_{k\in\mathbb{N}}$ is bounded in $L^2(M_T;\mathbb{C}^2)$. Moreover, the well-posedness of the wave equation in $\mathscr{C}^0(0,T;H^{-1}(\Omega))\cap\mathscr{C}^1(0,T;H^{-2}(\Omega))$ also gives

$$\|w_2^k\|_{H^{-1}(M_T)} \leqq C\|Bw_1^k\|_{L^2(0,T;H^{-2}(\Omega))} \leqq C\|w_1^k\|_{H^{-1}(M_T)} \to 0,$$

since $w_1^k \rightharpoonup 0$ in $L^2(M_T;\mathbb{C})$. Finally, the sequence $(w_1^k, w_2^k)_{k\in\mathbb{N}}$ satisfies all the conditions (4.1).

Since $(w_1^k, w_2^k) \rightharpoonup 0$ in $L^2(M_T;\mathbb{C}^2)$, and because of [21, Theorem 1], there exists a subsequence (still denoted $(W^k)_{k\in\mathbb{N}} = (w_1^k, w_2^k)_{k\in\mathbb{N}}$) and a microlocal defect measure

$$\mu = \begin{pmatrix} \mu_1 & \mu_{12} \\ \bar{\mu}_{12} & \mu_2 \end{pmatrix} \in \mathcal{M}_+(S^*M_T;\mathbb{C}^{2\times 2}),$$

such that for any $\mathcal{A} \in \Psi^0_{\text{phg}}(M_T;\mathbb{C}^{2\times 2})$,

$$\lim_{k\to\infty} (\mathcal{A}W^k, W^k)_{L^2(M_T;\mathbb{C}^2)} = \langle \mu, \sigma_0(\mathcal{A})\rangle_{S^*M_T}.$$

Moreover, μ_1 is the microlocal defect measure associated with the sequence $(w_1^k)_{k\in\mathbb{N}}$. As this sequence is (a subsequence of a sequence) chosen by means of Lemma 4.3, we have $\operatorname{supp}(\mu_1) \subset \Gamma$. Observe now that the sequence $(W^k)_{k\in\mathbb{N}}$ satisfies all the assumptions of the first part of Lemma 3.3, which gives $\mu_2 \in \mathcal{M}_+(S^*M_T)$, $\operatorname{supp}(\mu_2) \subset \operatorname{Char}(P)$, and $\operatorname{supp}(\mu_{12}) \subset \operatorname{supp}(\mu_1) \subset \Gamma$. Finally these three measures satisfy the equations

$$\begin{cases} \langle \mu_1, H_p a\rangle_{S^*M_T} = 0, \\ \langle \mu_2, H_p a\rangle_{S^*M_T} = -\langle \operatorname{Im}(\mu_{12}), 2b|\eta|_x a\rangle_{S^*M_T}, \\ \langle \operatorname{Im}(\mu_{12}), H_p a\rangle_{S^*M_T} = -\langle \mu_1, b|\eta|_x a\rangle_{S^*M_T}, \end{cases} \tag{4.7}$$

for any $a \in S^{-1}_{\text{phg}}(T^*M_T;\mathbb{C})$.

We denote by $\mathcal{C}$ the connected component of $\Gamma\setminus\pi^{-1}((0,T)\times\overline{\mathcal{O}})$ that contains $\Gamma\cap\pi^{-1}((0,T)\times\omega)$ (indicated by a thick line in Fig. 4). We have $\phi_{t_0}(\rho_*) \in \mathcal{C}$. In the present case, Condition (4.5) gives

$$\pi(\Gamma)\cap((0,T)\times\omega)\cap((0,T)\times\mathcal{O}) = \emptyset. \tag{4.8}$$

Moreover, condition (4.5) yields the existence and uniqueness of such a connected component.

The end of the proof consists in showing that $\operatorname{supp}(\mu_2) \subset \Gamma$ and that $\mathcal{C}\cap\operatorname{supp}(\mu_2) = \emptyset$. For this, we first prove that μ_2 vanishes identically on $\pi^{-1}\big((t_0 - 2\varepsilon, t_0+2\varepsilon)\times\Omega\big)$ for $\varepsilon > 0$ sufficiently small. Then, a propagation argument with (4.7) gives $\operatorname{supp}(\mu_2)\subset\Gamma$ and that μ_2 vanishes on $\mathcal{C}$.

Let us take $\varepsilon > 0$ with $2\varepsilon < \min(t_0, T-t_0)$, to be chosen below. The well-posedness for the wave equation (4.6) gives

$$\begin{aligned} \|w_2^k\|_{L^2((t_0-\varepsilon,t_0+\varepsilon)\times\Omega)} &\leqq C\|Bw_1^k\|_{L^2(t_0-\varepsilon,t_0+\varepsilon;H^{-1}(\Omega))} \\ &\leqq C\|(1-\Delta)^{-\frac{1}{2}}b(1-\Delta)^{\frac{1}{2}}w_1^k\|_{L^2((t_0-\varepsilon,t_0+\varepsilon)\times\Omega)} \\ &\leqq C\|\chi(1-\Delta)^{-\frac{1}{2}}b(1-\Delta)^{\frac{1}{2}}w_1^k\|_{L^2((t_0-2\varepsilon,t_0+2\varepsilon)\times\Omega)} \end{aligned} \tag{4.9}$$

for some $\chi \in \mathscr{C}_c^\infty(t_0 - 2\varepsilon, t_0 + 2\varepsilon)$ satisfying $0 \leqq \chi \leqq 1$ and $\chi = 1$ on $(t_0 - \varepsilon, t_0 + \varepsilon)$. In this last expression, we have $\chi(1-\Delta)^{-\frac{1}{2}} b(1-\Delta)^{\frac{1}{2}} \in \Psi^0_{\mathrm{phg}}(M_T; \mathbb{C})$, with principal symbol $\sigma_0\big(\chi(1-\Delta)^{-\frac{1}{2}} b(1-\Delta)^{\frac{1}{2}}\big) = \chi b$. As a consequence, we have

$$\|\chi(1-\Delta)^{-\frac{1}{2}} b(1-\Delta)^{\frac{1}{2}} w_1^k\|^2_{L^2((t_0-2\varepsilon,t_0+2\varepsilon)\times\Omega)} \to \left\langle \mu_1, \chi^2 b^2 \right\rangle_{S^*M_T}. \tag{4.10}$$

Since $\pi^{-1}((0,T)\times\omega)$ is an open set in S^*M_T, containing $\phi_{t_0}(\rho_*)$, there exists $\varepsilon > 0$ such that

$$\phi_s(\rho_*) \in \pi^{-1}((0,T)\times\omega), \quad \text{for all } s \in (t_0 - 2\varepsilon, t_0 + 2\varepsilon). \tag{4.11}$$

For this choice of ε, we have

$$\begin{aligned} &\mathrm{supp}(\mu_1) \cap \pi^{-1}((t_0 - 2\varepsilon, t_0 + 2\varepsilon)\times\Omega) \subset \Gamma \cap \pi^{-1}((t_0 - 2\varepsilon, t_0 + 2\varepsilon)\times\Omega) \\ &\quad = \{\phi_s(\rho_*), s \in (t_0 - 2\varepsilon, t_0 + 2\varepsilon)\}. \end{aligned}$$

According to (4.11), this last set is contained in $\Gamma \cap \pi^{-1}\big((0,T)\times\omega\big)$, i.e:

$$\mathrm{supp}(\mu_1) \cap \pi^{-1}((t_0 - 2\varepsilon, t_0 + 2\varepsilon)\times\Omega) \subset \Gamma \cap \pi^{-1}((0,T)\times\omega).$$

From (4.8), we then obtain $\mathrm{supp}(\mu_1) \cap \pi^{-1}\big((t_0 - 2\varepsilon, t_0 + 2\varepsilon)\times\Omega\big) \cap \pi^{-1}\big((0,T)\times \mathcal{O}\big) = \emptyset$, which gives

$$\mathrm{supp}(\mu_1) \cap \pi^{-1}(\mathrm{supp}(\chi)) \cap \pi^{-1}(\mathrm{supp}(b)) = \emptyset.$$

This, together with (4.10), gives

$$\|\chi(1-\Delta)^{-\frac{1}{2}} b(1-\Delta)^{\frac{1}{2}} w_1^k\|_{L^2((t_0-2\varepsilon,t_0+2\varepsilon)\times\Omega)} \to 0.$$

Using (4.9), we now obtain $\|w_2^k\|_{L^2((t_0-\varepsilon,t_0+\varepsilon)\times\Omega)} \to 0$, and thus

$$\mathrm{supp}(\mu_2) \cap \pi^{-1}((t_0 - \varepsilon, t_0 + \varepsilon)\times\Omega)) = \emptyset. \tag{4.12}$$

As $\mathrm{supp}(\mu_{12}) \subset \Gamma$, the second equation of (4.7) yields

$$\left\langle \mu_2, H_p a \right\rangle_{S^*M_T} = 0 \tag{4.13}$$

for any $a \in S^{-1}_{\mathrm{phg}}(T^*M_T; \mathbb{C})$ such that $a = 0$ in a neighborhood of Γ. As in the proof of Proposition 3.2, this gives the invariance of the measure μ_2 along the flow ϕ_s away from Γ. Together with (4.12), this also yields

$$\mathrm{supp}(\mu_2) \subset \Gamma. \tag{4.14}$$

The second equation of (4.7) also gives (4.13) for any $a \in S^{-1}_{\mathrm{phg}}(T^*M_T; \mathbb{C})$ such that $a = 0$ in a neighborhood of $\pi^{-1}\big((0,T)\times\mathcal{O}\big)$, as b is supported in $\overline{\mathcal{O}}$. Once again, this gives the invariance of the measure μ_2 along the flow ϕ_s

on $S^*M_T \setminus \pi^{-1}\big((0,T)\times\overline{\mathcal{O}}\big)$. Together with Equation (4.12), this gives, with a propagation argument,

$$\operatorname{supp}(\mu_2)\cap\mathcal{C}=\emptyset. \tag{4.15}$$

Finally, Equations (4.14) and (4.15) together with the definition of $\mathcal{C}$ give

$$\operatorname{supp}(\mu_2)\cap\pi^{-1}\left((0,T)\times\omega\right)=\emptyset.$$

This implies (4.2) as b_ω is supported in $\overline{\omega}$, and concludes the proof of Theorem 4.1. □

5. The Hilbert Uniqueness Method and the HUM Operator

5.1. Controllability and Observability for Cascade Systems

In this section, we prove the equivalence between controllability and observability for the systems under view. This result is classical for waves [19] and was further developed by Lions [31], who coined the name HUM. For completeness, we provide the details of this result in the case of our cascade wave system. To also fit the setting of Section 6 below, where waves with different speeds are studied, we give this equivalence for the following control system with $\gamma > 0$ and $\sigma \geqq 0$ (see also Section 1.2.3):

$$\begin{cases} Pu_1+(1-\Delta)^{\frac{\sigma}{2}}b(x)\,u_2=0 & \text{in } (0,T)\times\Omega,\\ P_\gamma u_2=b_\omega(x)\,f & \text{in } (0,T)\times\Omega. \end{cases} \tag{5.1}$$

Proposition 5.1. *Let $s\in\mathbb{R}$. Assume that System (5.1) is well-posed in the space $H^{s-\sigma+1}(\Omega)\times H^1(\Omega)\times H^{s-\sigma}(\Omega)\times L^2(\Omega)$, that is*

$$(u_1,u_2,\partial_t u_1,\partial_t u_2)\in\mathscr{C}^0([0,T];H^{s-\sigma+1}(\Omega)\times H^1(\Omega)\times H^{s-\sigma}(\Omega)\times L^2(\Omega)).$$

Then, it is controllable in time $T>0$ in this space if and only if the observability inequality

$$E_{\sigma-s}(v_1(0))+E_0(v_2(0))\leqq C\int_0^T \|b_\omega v_2\|^2_{L^2(\Omega)}\,\mathrm{d}t \tag{5.2}$$

holds for all solutions (v_1,v_2) to

$$\begin{cases} Pv_1=0 & \textit{in } (0,T)\times\Omega,\\ P_\gamma v_2+b(x)\,(1-\Delta)^{\frac{\sigma}{2}}v_1=0 & \textit{in } (0,T)\times\Omega, \end{cases} \tag{5.3}$$

assumed to be well-posed in $H^{\sigma-s}(\Omega)\times L^2(\Omega)\times H^{\sigma-s-1}(\Omega)\times H^{-1}(\Omega)$, that is

$$(v_1,v_2,\partial_t v_1,\partial_t v_2)\in\mathscr{C}^0([0,T];H^{\sigma-s}(\Omega)\times L^2(\Omega)\times H^{\sigma-s-1}(\Omega)\times H^{-1}(\Omega)).$$

Remark 5.2. Note that we have $s=1$ and $\sigma=0$ in the duality between Systems (1.1) and (1.6). We have $s=1$ and $\sigma=1$ in the duality between (2.14) and (2.12).

As a preliminary of the proof we first present some duality framework. According to Definition 1.1, we shall only consider zero initial data: $(u_1(0), u_2(0), \partial_t u_1(0), \partial_t u_2(0)) = (0, 0, 0, 0)$. Firstly, we suppose that the data $(v_1(0), v_2(0), \partial_t v_1(0), \partial_t v_2(0))$ for System (5.3) and the control function f in (5.1) are smooth. Taking the inner product of the first line of (5.1) with v_1, we obtain

$$\begin{aligned} -((1-\Delta)^{\frac{\sigma}{2}} b u_2, v_1)_{L^2(M_T)} &= (P u_1, v_1)_{L^2(M_T)} \\ &= (u_1, P v_1)_{L^2(M_T)} + (\partial_t u_1(T), v_1(T))_{L^2(\Omega)} \\ &\quad -(u_1(T), \partial_t v_1(T))_{L^2(\Omega)}, \end{aligned}$$

after an integration by parts. Similarly, taking the inner product of the second line of (1.9) with v_2, we find

$$\begin{aligned} (b_\omega f, v_2)_{L^2(M_T)} &= (P_\gamma u_2, v_2)_{L^2(M_T)} \\ &= (u_2, P_\gamma v_2)_{L^2(M_T)} + (\partial_t u_2(T), v_2(T))_{L^2(\Omega)} \\ &\quad -(u_2(T), \partial_t v_2(T))_{L^2(\Omega)}. \end{aligned}$$

Summing the last two identities and recalling that (v_1, v_2) satisfies (5.3), we obtain the duality identity

$$\begin{aligned} (f, b_\omega v_2)_{L^2(M_T)} &= (\partial_t u_1(T), v_1(T))_{L^2(\Omega)} - (u_1(T), \partial_t v_1(T))_{L^2(\Omega)} \\ &\quad +(\partial_t u_2(T), v_2(T))_{L^2(\Omega)} - (u_2(T), \partial_t v_2(T))_{L^2(\Omega)}. \end{aligned}$$

Secondly, using a density argument, together with the well-posedness assumptions, we see that we have

$$\begin{aligned} (f, b_\omega v_2)_{L^2(M_T)} &= \langle \partial_t u_1(T), v_1(T)\rangle_{H^{s-\sigma}(\Omega), H^{\sigma-s}(\Omega)} \\ &\quad -\langle u_1(T), \partial_t v_1(T)\rangle_{H^{s-\sigma+1}(\Omega), H^{\sigma-s-1}(\Omega)} \\ &\quad +(\partial_t u_2(T), v_2(T))_{L^2(\Omega)} - \langle u_2(T), \partial_t v_2(T)\rangle_{H^1(\Omega), H^{-1}(\Omega)}. \end{aligned} \tag{5.4}$$

For the proof below we also introduce the following continuous map

$$S : L^2(M_T) \to H^{s-\sigma+1}(\Omega) \times H^1(\Omega) \times H^{s-\sigma}(\Omega) \times L^2(\Omega), \tag{5.5}$$

$$f \mapsto (u_1, u_2, \partial_t u_1, \partial_t u_2)|_{t=T}, \tag{5.6}$$

where $u = (u_1, u_2)$ is the solution to (5.1) with zero initial conditions.

Proof of Proposition 5.1. Controllability ⇒ observability.

We start by assuming controllability, that is that S is surjective. The open mapping theorem then yields that $S(B_{L^2(M_T)}(0, 1))$ is a neighborhood of $(0, 0, 0, 0)$. Then, for some $\eta > 0$ we have

$$B_{H^{s-\sigma+1}\times H^1\times H^{s-\sigma}\times L^2}(0, \eta) \subset S(B_{L^2(M_T)}(0, 1)),$$

where $B_H(0, r)$ denotes the open ball of radius r centered at 0 in the space H. By linearity, this yields

$$B_{H^{s-\sigma+1}\times H^1\times H^{s-\sigma}\times L^2}(0, 2) \subset S(B_{L^2(M_T)}(0, 2\eta^{-1})).$$

Now, take $V = (v_1^0, v_2^0, v_1^1, v_2^1) \in H^{\sigma-s}(\Omega)\times H^1(\Omega)\times H^{\sigma-s-1}(\Omega)\times L^2(\Omega)$. With the Riesz representation theorem, we choose $U = (u_1^0, u_2^0, u_1^1, u_2^1) \in H^{s-\sigma+1}(\Omega)\times H^1(\Omega)\times H^{s-\sigma}(\Omega)\times L^2(\Omega)$, such that $\|U\|_{H^{s-\sigma+1}(\Omega)\times H^1(\Omega)\times H^{s-\sigma}(\Omega)\times L^2(\Omega)} = 1$ and

$$\begin{aligned}\langle u_1^1, v_1^0\rangle_{H^{s-\sigma}(\Omega),H^{\sigma-s}(\Omega)} &- \langle u_1^0, v_1^1\rangle_{H^{s-\sigma+1}(\Omega),H^{\sigma-s-1}(\Omega)}\\ &+(u_2^1, v_2^0)_{L^2(\Omega)} - \langle u_2^0, v_2^1\rangle_{H^1(\Omega),H^{-1}(\Omega)}\\ &= \|V\|_{H^{\sigma-s}(\Omega)\times H^1(\Omega)\times H^{\sigma-s-1}(\Omega)\times L^2(\Omega)}.\end{aligned}$$

Then, take f such that $S(f) = U$ and $\|f\|_{L^2(M_T)} \leqq 2\eta^{-1}$. By (5.4), have

$$\begin{aligned}(E_{\sigma-s}(v_1(T)) + E_0(v_2(T)))^{\frac{1}{2}} &= \|V\|_{H^{\sigma-s}(\Omega)\times H^1(\Omega)\times H^{\sigma-s-1}(\Omega)\times L^2(\Omega)}\\ &= (f, b_\omega v_2)_{L^2(M_T)}\\ &\leqq \|f\|_{L^2(M_T)}\|b_\omega v_2\|_{L^2(M_T)} \leqq 2\eta^{-1}\|b_\omega v_2\|_{L^2(M_T)},\end{aligned}$$

where (v_1, v_2) is the backward solution of System (5.3), associated with the final data $(v_1, v_2, \partial_t v_1, \partial_t v_2)|_{t=T} = V$. This yields the observability inequality

$$E_{\sigma-s}(v_1(T)) + E_0(v_2(T)) \leqq C\|b_\omega v_2\|^2_{L^2(M_T)},$$

for all backward solutions of (5.3). Changing t in $T - t$ in System (5.3) yields Inequality (5.2).

Observability ⇒ controllability.

Given $U = (u_1^0, u_2^0, u_1^1, u_2^1) \in H^{s-\sigma+1}(\Omega) \times H^1(\Omega) \times H^{s-\sigma}(\Omega) \times L^2(\Omega)$, we define the following functional

$$\begin{aligned}J(V) = \frac{1}{2}\|b_\omega v_2\|^2_{L^2(M_T)} - \Big(&\langle u_1^1, v_1^0\rangle_{H^{s-\sigma}(\Omega),H^{\sigma-s}(\Omega)} - \langle u_1^0, v_1^1\rangle_{H^{s-\sigma+1}(\Omega),H^{\sigma-s-1}(\Omega)}\\ &+(u_2^1, v_2^0)_{L^2(\Omega)} - \langle u_2^0, v_2^1\rangle_{H^1(\Omega),H^{-1}(\Omega)}\Big),\end{aligned}$$

where $(v_1, v_2)(t, x)$ is the backward solution of (5.3) with the data $V = (v_1^0, v_2^0, v_1^1, v_2^1)$ at time $t = T$. This quadratic functional is continuous, strictly convex and the observability inequality (5.2) (after having changed t in $T - t$ in System (5.3)) implies that it is coercive. Hence, J admits a unique minimizer $\underline{V} = (\underline{v}_1^0, \underline{v}_2^0, \underline{v}_1^1, \underline{v}_2^1)$, satisfying the Euler equation

$$\begin{aligned}0 = (b_\omega \underline{v}_2, b_\omega v_2)_{L^2(M_T)} - \Big(&\langle u_1^1, v_1^0\rangle_{H^{s-\sigma}(\Omega),H^{\sigma-s}(\Omega)} - \langle u_1^0, v_1^1\rangle_{H^{s-\sigma+1}(\Omega),H^{\sigma-s-1}(\Omega)}\\ &+(u_2^1, v_2^0)_{L^2(\Omega)} - \langle u_2^0, v_2^1\rangle_{H^1(\Omega),H^{-1}(\Omega)}\Big),\end{aligned}$$

where $(\underline{v}_1, \underline{v}_2)(t, x)$ is the backward solution of (5.3) with the data $\underline{V}$ at time $t = T$. In view of (5.4), this means exactly that $f(t) := b_\omega \underline{v}_2(t)$ realizes a control for System (5.1). □

5.2. The HUM Operator

Here we use some of the notation and results of the proof of Proposition 5.1 where a control is defined. It is in fact of minimal L^2 norm. We shall associate an operator to this control.

We set the map

$$Q : H^{\sigma-s}(\Omega) \times L^2(\Omega) \times H^{\sigma-s-1}(\Omega) \times H^{-1}(\Omega) \to L^2(M_T),$$
$$(v_1(T), v_2(T), \partial_t v_1(T), \partial_t v_2(T)) \mapsto b_\omega v_2,$$

where $v = (v_1, v_2)$ is the solution to system (5.3) with final condition $(v_1(T), v_2(T), \partial_t v_1(T), \partial_t v_2(T)) \in H^{\sigma-s}(\Omega) \times L^2(\Omega) \times H^{\sigma-s-1}(\Omega) \times H^{-1}(\Omega)$. We define the natural duality bracket

$$\left\langle (u_1^0, u_2^0, u_1^1, u_2^1), (v_1^0, v_2^0, v_1^1, v_2^1) \right\rangle_* = \left\langle u_1^1, v_1^0 \right\rangle_{H^{s-\sigma}(\Omega), H^{\sigma-s}(\Omega)} - \left\langle u_1^0, v_1^1 \right\rangle_{H^{s-\sigma+1}(\Omega), H^{\sigma-s-1}(\Omega)} + (u_2^1, v_2^0)_{L^2(\Omega)} - \left\langle u_2^0, v_2^1 \right\rangle_{H^1(\Omega), H^{-1}(\Omega)},$$

between the spaces $H^{s-\sigma+1}(\Omega) \times H^1(\Omega) \times H^{s-\sigma}(\Omega) \times L^2(\Omega)$ and $H^{\sigma-s}(\Omega) \times L^2(\Omega) \times H^{\sigma-s-1}(\Omega) \times H^{-1}(\Omega)$.

From (5.4) we have

$$\|Q(V)\|^2_{L^2(M_T)} = \langle S \circ Q(V), V \rangle_*, \qquad V = (v_1(T), v_2(T), \partial_t v_1(T), \partial_t v_2(T)).$$

We set $\mathcal{L}_T = S \circ Q$ (the Gramian operator). If the observability inequality holds, it reads (change t in $T - t$)

$$\|Q(V)\|_{L^2(M_T)} \geqq C \|V\|_{H^{\sigma-s}(\Omega) \times H^1(\Omega) \times H^{\sigma-s-1}(\Omega) \times L^2(\Omega)}.$$

Then this yields the invertibility of $\mathcal{L}_T$ by the Lax-Milgram theorem, which allows one to define the HUM operator:

$$\mathcal{H}_T = \mathcal{L}_T^{-1}, \qquad \text{(HUM)}$$

that maps continuously $H^{s-\sigma+1}(\Omega) \times H^1(\Omega) \times H^{s-\sigma}(\Omega) \times L^2(\Omega)$ onto $H^{\sigma-s}(\Omega) \times L^2(\Omega) \times H^{\sigma-s-1}(\Omega) \times H^{-1}(\Omega)$.

For a final data $U = (u_1^0, u_2^0, u_1^1, u_2^1) \in H^{s-\sigma+1}(\Omega) \times H^1(\Omega) \times H^{s-\sigma}(\Omega) \times L^2(\Omega)$ we can define $f_{HUM} = Q \circ \mathcal{H}_T(U)$ which is a control reaching the target U at final time T for system (5.1):

$$S f_{HUM} = S \circ Q \circ \mathcal{H}_T(U) = S \circ Q \circ (S \circ Q)^{-1}(U) = U.$$

The identity (5.4) yields that f_{HUM} is precisely the control built in the second part of the proof of Proposition 5.1. The Fenchel-Rockafellar convex-optimization theory [20] implies that, actually, f_{HUM} is the unique minimizer of the cost functional $\|f\|^2_{L^2(M_T)}$ among all controls $f \in L^2(M_T)$ for System (5.1).

5.3. Microlocal Characterization of the HUM Operator

In this section, we develop a precise analysis of the Gramian operator associated with the observation system (2.12). More precisely, we prove that this operator is a matrix of pseudodifferential operators of order zero. We also analyze its ellipticity properties, providing a second proof of Theorem 1.3, together with additional microlocal properties.

Here we actually follow the program developed by Dehman-Lebeau [18], of which the reader can find very nice illustrations in [28].

We shall make an intensive use of the Egorov theorem as given in Theorem 2.1. We shall also use smoothing properties of some very particular Fourier integral operators; these results are collected in Appendix A.

5.3.1. A Simplified Model: A System of Coupled Half-Wave Equations In this section, we consider the control problem for two coupled half-wave equations

$$\begin{cases} (\partial_t - i\lambda)u_1 - \frac{1}{2i}bu_2 = 0 & \text{in } (0,T)\times\Omega, \\ (\partial_t - i\lambda)u_2 = b_\omega f & \text{in } (0,T)\times\Omega. \end{cases} \tag{5.7}$$

This system is much simpler than System (1.1). This section will, however, help the reader to understand the key aspects of the microlocal characterization of the HUM operator without the additional technical difficulties that one faces when addressing the *full* wave system (1.1). That analysis is postponed until the next section. Note that the coefficient $\frac{-1}{2i}$ is chosen here to fit the setting of System (1.1) (see Section 5.3.2 below) and has no importance here.

The first-order system (5.7) is well-posed for initial data $(u_1(0), u_2(0)) \in H^s(\Omega;\mathbb{C}^2)$ and a right hand-side $f \in L^1(\mathbb{R}; H^s(\Omega))$, giving rise to a unique solution $(u_1,u_2) \in \mathscr{C}^0(\mathbb{R}; H^s(\Omega;\mathbb{C}^2)) \cap \mathscr{C}^1(\mathbb{R}; H^{s-1}(\Omega;\mathbb{C}^2))$ (see for instance [23, Chapter 23.1]). Note that there is here no gain of regularity in the state space (as opposed to the full wave-system (1.1)). The associated observation problem is the following

$$\begin{cases} (\partial_t - i\lambda)v_1 = 0 & \text{in } (0,T)\times\Omega, \\ (\partial_t - i\lambda)v_2 + \frac{1}{2i}bv_1 = 0 & \text{in } (0,T)\times\Omega, \\ (v_1(0), v_2(0)) = (g,h) \in L^2(\Omega;\mathbb{C}^2), \end{cases} \tag{5.8}$$

together with the observability inequality

$$\|g\|^2_{L^2(\Omega)} + \|h\|^2_{L^2(\Omega)} \leqq C\int_0^T \|b_\omega v_2\|^2_{L^2(\Omega)}\,\mathrm{d}t. \tag{5.9}$$

In this setting, similar to what is done in Section 5.1, the controllability of System (5.7) is equivalent to estimate (5.9) for all $(g,h) \in L^2(\Omega;\mathbb{C}^2)$ and (v_1,v_2) associated solutions of (5.8).

We recall that the flow $(\varphi_t^+)_{t\in\mathbb{R}}$, used in the statement of Theorem 5.3, is defined by (2.9).

Theorem 5.3. *We have*

$$\int_0^T \|b_\omega v_2\|^2_{L^2(\Omega)}\,\mathrm{d}t = \left(\mathcal{G}_T^+(g,h),(g,h)\right)_{L^2(\Omega;\mathbb{C}^2)},$$

where $\mathcal{G}_T^+ \in \mathcal{L}(L^2(\Omega;\mathbb{C}^2))$ *is the Gramian operator of* (5.7).

Moreover, there exists $G_T^+ \in \Psi^0_{\mathrm{phg}}(\Omega;\mathbb{C}^{2\times 2})$*, and* R_T *an infinitely smoothing operator on* Ω *such that*

$$\mathcal{G}_T^+ = G_T^+ + R_T,$$

where the principal symbol of G_T^+ *is*

$$\sigma_0(G_T^+) = \begin{pmatrix} \frac{1}{4}\int_0^T b_\omega^2\circ\varphi_t^+\left(\int_0^t b\circ\varphi_\sigma^+ d\sigma\right)^2\mathrm{d}t & \frac{1}{2i}\int_0^T b_\omega^2\circ\varphi_t^+\left(\int_0^t b\circ\varphi_\sigma^+ d\sigma\right)\mathrm{d}t \\ -\frac{1}{2i}\int_0^T b_\omega^2\circ\varphi_t^+\left(\int_0^t b\circ\varphi_\sigma^+ d\sigma\right)\mathrm{d}t & \int_0^T b_\omega^2\circ\varphi_t^+\,\mathrm{d}t \end{pmatrix} \in S^0_{\mathrm{phg}}(T^*\Omega,\mathbb{C}^{2\times 2}).$$

In particular, we have

$$\det(\sigma_0(G_T^+)) = \frac{1}{8}\int_0^T\int_0^T (b_\omega^2\circ\varphi_{t_1}^+)(b_\omega^2\circ\varphi_{t_2}^+)\left(\int_{t_1}^{t_2} b\circ\varphi_\sigma^+ d\sigma\right)^2 \mathrm{d}t_1\,\mathrm{d}t_2 \in S^0_{\mathrm{phg}}(T^*\Omega).$$

The next corollary both proves the observability of (5.8) and characterizes the HUM operator. The connection between the HUM operator $\left(\mathcal{G}_T^+\right)^{-1}$ and the construction of the control of minimal L^2-norm for the present simplified half-wave model can be done as in Sections 5.1–5.2. This is left to the reader. Recall that the time $T^+_{\omega\to\mathcal{O}\to\omega}$ is defined in Definition 2.5.

Corollary 5.4. *Assume that both* ω *and* $\mathcal{O}$ *satisfy GCC and that* $T > T^+_{\omega\to\mathcal{O}\to\omega}$*. Then, we have the following properties:*

1. *The operator* $G_T^+ \in \Psi^0_{\mathrm{phg}}(\Omega,\mathbb{C}^{2\times 2})$ *is elliptic.*
2. *The operator* $\mathcal{G}_T^+$ *is coercive on* $L^2(\Omega;\mathbb{C}^2)$*:*

$$\int_0^T \|b_\omega v_2\|^2_{L^2(\Omega)}\,\mathrm{d}t = \left(\mathcal{G}_T^+(g,h),(g,h)\right)_{L^2(\Omega;\mathbb{C}^2)} \geqq C\|(g,h)\|^2_{L^2(\Omega;\mathbb{C}^2)}, \tag{5.10}$$

   for all $(g,h)\in L^2(\Omega;\mathbb{C}^2)$ *and* (v_1,v_2) *associated solutions of* (5.8).
3. *The operator* $\mathcal{G}_T^+$ *is invertible in* $\mathcal{L}(L^2(\Omega))$*. Its inverse* $(\mathcal{G}_T^+)^{-1}$*, the HUM operator, can be decomposed as* $(\mathcal{G}_T^+)^{-1} = \Lambda_T^+ + R_T$ *where* $R_T \in \mathcal{R}^\infty(\Omega)$ *and* $\Lambda_T^+ \in S^0_{\mathrm{phg}}(T^*\Omega,\mathbb{C}^{2\times 2})$*, with principal symbol*

$$\sigma_0(\Lambda_T^+) = \det(\sigma_0(G_T^+))^{-1} \begin{pmatrix} \int_0^T b_\omega^2\circ\varphi_t^+\,\mathrm{d}t & \frac{1}{2i}\int_0^T b_\omega^2\circ\varphi_t^+\left(\int_0^t b\circ\varphi_\sigma^+ d\sigma\right)\mathrm{d}t \\ -\frac{1}{2i}\int_0^T b_\omega^2\circ\varphi_t^+\left(\int_0^t b\circ\varphi_\sigma^+ d\sigma\right)\mathrm{d}t & \frac{1}{4}\int_0^T b_\omega^2\circ\varphi_t^+\left(\int_0^t b\circ\varphi_\sigma^+ d\sigma\right)^2\mathrm{d}t \end{pmatrix}.$$

4. *In particular, the HUM operator* $(\mathcal{G}_T^+)^{-1}$ *is an isomorphism of* $H^s(\Omega; \mathbb{C}^2)$ *for all* $s \geqq 0$ *and we have*

$$\mathrm{WF}^s((\mathcal{G}_T^+)^{-1}(f, g)) = \mathrm{WF}^s(f, g).$$

Recall that the notation $\mathcal{R}^\infty$ was introduced in Definition 2.2. The definition of $\mathrm{WF}^s(u)$, for $u \in \mathscr{D}'(\Omega)$, is for instance given in [30, Definition 1.2.21]. The wave-front set of a couple, or, more generally a k-tuple $(f_1, f_2, \dots, f_k) \in \mathscr{D}'(\Omega; \mathbb{C}^k)$, is defined by (see for instance [17])

$$\mathrm{WF}^s(f_1, f_2, \dots, f_k) = \bigcup_{j=1}^{k} \mathrm{WF}^s(f_j). \tag{5.11}$$

Corollary 5.4 is proved at the end of this section.

Proof of Theorem 5.3. Let us recall that the group $e^{it\lambda}$ is defined at the end of Section 2.1. The Duhamel formula in (5.8) gives the explicit representations

$$v_1(t) = e^{it\lambda}\Pi_+ g + \Pi_0 g,$$
$$v_2(t) = e^{it\lambda}\Pi_+ h + \Pi_0 h - \frac{1}{2i}\int_0^t \Big(e^{i(t-\sigma)\lambda}\Pi_+ b v_1(\sigma) + \Pi_0 b v_1(\sigma)\Big)\, d\sigma.$$

Developing this last expression, we have in particular,

$$v_2(t) = e^{it\lambda}\Pi_+ h - \frac{1}{2i}\int_0^t e^{i(t-\sigma)\lambda}\Pi_+ b e^{i\sigma\lambda}\Pi_+ g d\sigma + R_t(g, h),$$

where

$$R_t(g, h) =$$
$$\Pi_0 h - \frac{1}{2i} t \Pi_0 b \Pi_0 g + \frac{1}{2}\lambda^{-1}(e^{it\lambda} - 1)\Pi_+ b \Pi_0 g + \frac{1}{2}\Pi_0 b \lambda^{-1}(e^{it\lambda} - 1)\Pi_+ g.$$

Recall that $\lambda^{-1} \in \mathcal{L}(H_+^s(\Omega); H_+^{s+1}(\Omega))$ is defined at the end of Section 2.1.

Hence, R_t is a continuous family of infinitely smoothing operators, since $\Pi_0 \in \mathcal{L}(H^s(\Omega); \mathbb{C})$ for all $s \in \mathbb{R}$ and b and $e^{it\lambda}$ preserve the regularity.

Now, let us compute the observation

$$\int_0^T \|b_\omega v_2\|_{L^2(\Omega)}^2 \,\mathrm{d}t = \int_0^T \Big\| b_\omega e^{it\lambda}\Pi_+ h - \frac{1}{2i} b_\omega e^{it\lambda}\Pi_+ \int_0^t e^{-i\sigma\lambda} b e^{i\sigma\lambda} d\sigma \; \Pi_+ g + R_t(g, h)\Big\|_{L^2(\Omega)}^2 \mathrm{d}t. \tag{5.12}$$

According to the Egorov theorem (see Theorem 2.1), for any $N \in \mathbb{N}$, we have $e^{-i\sigma\lambda} b e^{i\sigma\lambda} - R_t^N \in \mathscr{C}^0\big(\mathbb{R}, \Psi_{\mathrm{phg}}^0(\Omega)\big)$, with principal symbol $b \circ \varphi_\sigma^+(x, \eta)$, where R_t^N is a continuous family of N-smoothing operators. Hence, the operator

$$B_t^+ := \int_0^t e^{-i\sigma\lambda} b e^{i\sigma\lambda} d\sigma \tag{5.13}$$

is in $\mathscr{C}^0\big(\mathbb{R}, \Psi^0_{\mathrm{phg}}(\Omega)\big)$ up to a continuous family of N-smoothing operators, and we have

$$\sigma_0(B_t^+)(x,\eta) = \int_0^t b \circ \varphi_\sigma^+(x,\eta) d\sigma, \qquad (x,\eta) \in T^*\Omega.$$

Coming back to (5.12) and developing the inner product, we obtain

$$\begin{aligned}\int_0^T \|b_\omega v_2\|^2_{L^2(\Omega)}\,\mathrm{d}t = &\int_0^T \Big(\Pi_+ e^{-it\lambda} b_\omega^2 e^{it\lambda}\Pi_+ h, h\Big)_{L^2(\Omega)}\,\mathrm{d}t\\ &-\frac{1}{2i}\int_0^T \Big(\Pi_+ e^{-it\lambda} b_\omega^2 e^{it\lambda}\Pi_+ B_t^+\Pi_+ g, h\Big)_{L^2(\Omega)}\,\mathrm{d}t\\ &+\frac{1}{2i}\int_0^T \Big(\Pi_+ (B_t^+)^*\Pi_+ e^{-it\lambda} b_\omega^2 e^{it\lambda}\Pi_+ h, g\Big)_{L^2(\Omega)}\,\mathrm{d}t\\ &+\frac{1}{4}\int_0^T \Big(\Pi_+ (B_t^+)^*\Pi_+ e^{-it\lambda} b_\omega^2 e^{it\lambda}\Pi_+ B_t^+\Pi_+ g, g\Big)_{L^2(\Omega)}\,\mathrm{d}t\\ &+\int_0^T \Big(\tilde{R}_t(g,h),(g,h)\Big)_{L^2(\Omega)}\,\mathrm{d}t,\end{aligned}$$

where $\tilde{R}_t$ is a continuous family of N-smoothing operators.

Setting

$$G_T^+ = \begin{pmatrix} \frac{1}{4}\int_0^T \Pi_+(B_t^+)^*\Pi_+ e^{-it\lambda} b_\omega^2 e^{it\lambda}\Pi_+ B_t^+\Pi_+\mathrm{d}t & \frac{1}{2i}\int_0^T \Pi_+(B_t^+)^*\Pi_+ e^{-it\lambda} b_\omega^2 e^{it\lambda}\Pi_+\mathrm{d}t \\ -\frac{1}{2i}\int_0^T \Pi_+ e^{-it\lambda} b_\omega^2 e^{it\lambda}\Pi_+ B_t^+\Pi_+\mathrm{d}t & \int_0^T \Pi_+ e^{-it\lambda} b_\omega^2 e^{it\lambda}\Pi_+\mathrm{d}t \end{pmatrix},$$

we find $G_T^+ \in \Psi^0_{\mathrm{phg}}(\Omega, \mathbb{C}^{2\times 2})$, since, according to the Egorov theorem, $e^{-it\lambda} b_\omega^2 e^{it\lambda} \in \Psi^0_{\mathrm{phg}}(\Omega)$ with principal symbol $b_\omega^2 \circ \varphi_t^+(x,\eta)$. The pseudodifferential calculus directly yields the principal symbol

$$\sigma_0(G_T^+) = \begin{pmatrix} \frac{1}{4}\int_0^T b_\omega^2 \circ \varphi_t^+ \left(\int_0^t b\circ\varphi_\sigma^+ d\sigma\right)^2 \mathrm{d}t & \frac{1}{2i}\int_0^T b_\omega^2\circ\varphi_t^+\left(\int_0^t b\circ\varphi_\sigma^+ d\sigma\right)\mathrm{d}t \\ -\frac{1}{2i}\int_0^T b_\omega^2\circ\varphi_t^+\left(\int_0^t b\circ\varphi_\sigma^+ d\sigma\right)\mathrm{d}t & \int_0^T b_\omega^2\circ\varphi_t^+\,\mathrm{d}t\end{pmatrix} \in S^0_{\mathrm{phg}}(T^*\Omega, \mathbb{C}^{2\times 2}).$$

We have thus obtained

$$\int_0^T \|b_\omega v_2\|^2_{L^2(\Omega)}\,\mathrm{d}t = \big((G_T^+ + R_T)(g,h),(g,h)\big)_{L^2(\Omega;\mathbb{C}^2)},$$

where R_T is an infinitely smoothing operator.

Computing $\det(\sigma_0(G_T^+))$, we find

$$\begin{aligned}\det\left(\sigma_0(G_T^+)\right) &= \frac{1}{4}\left(\int_0^T b_\omega^2\circ\varphi_t^+\,\mathrm{d}t\right)\left(\int_0^T b_\omega^2\circ\varphi_t^+\left(\int_0^t b\circ\varphi_\sigma^+ d\sigma\right)^2\mathrm{d}t\right)\\ &\quad -\frac{1}{4}\left(\int_0^T b_\omega^2\circ\varphi_t^+\left(\int_0^t b\circ\varphi_\sigma^+ d\sigma\right)\mathrm{d}t\right)^2\\ &= \frac{1}{4}\int_0^T\int_0^T (b_\omega^2\circ\varphi_{t_1}^+)(b_\omega^2\circ\varphi_{t_2}^+)\left(\int_0^{t_2} b\circ\varphi_\sigma^+ d\sigma\right)^2\mathrm{d}t_1\,\mathrm{d}t_2\\ &\quad -\frac{1}{4}\int_0^T\int_0^T (b_\omega^2\circ\varphi_{t_1}^+)(b_\omega^2\circ\varphi_{t_2}^+)\left(\int_0^{t_1} b\circ\varphi_\sigma^+ d\sigma\right)\\ &\qquad\qquad \times\left(\int_0^{t_2} b\circ\varphi_\sigma^+ d\sigma\right)\mathrm{d}t_1\,\mathrm{d}t_2.\end{aligned}$$

Given a function $f(t_1,t_2)$ defined on the square $[0,T]^2$, we notice that its mean value is equal to that of its symmetric part with respect to the diagonal:

$$\int_0^T\int_0^T f(t_1,t_2)\,\mathrm{d}t_1\,\mathrm{d}t_2 = \int_0^T\int_0^T \frac{1}{2}\left(f(t_1,t_2)+f(t_2,t_1)\right)\mathrm{d}t_1\,\mathrm{d}t_2.$$

Applying this remark to the function $(t_1,t_2)\mapsto (b_\omega^2\circ\varphi_{t_1}^+)(b_\omega^2\circ\varphi_{t_2}^+)\left(\int_0^{t_2} b\circ\varphi_\sigma^+ d\sigma\right)^2$, we obtain

$$\begin{aligned}4\det(\sigma_0(G_T^+)) &= \int_0^T\int_0^T (b_\omega^2\circ\varphi_{t_1}^+)(b_\omega^2\circ\varphi_{t_2}^+)\frac{1}{2}\left[\left(\int_0^{t_1} b\circ\varphi_\sigma^+ d\sigma\right)^2\right.\\ &\quad\left.+\left(\int_0^{t_2} b\circ\varphi_\sigma^+ d\sigma\right)^2\right]\mathrm{d}t_1\,\mathrm{d}t_2\\ &\quad -\int_0^T\int_0^T (b_\omega^2\circ\varphi_{t_1}^+)(b_\omega^2\circ\varphi_{t_2}^+)\left(\int_0^{t_1} b\circ\varphi_\sigma^+ d\sigma\right)\\ &\qquad\qquad\times\left(\int_0^{t_2} b\circ\varphi_\sigma^+ d\sigma\right)\mathrm{d}t_1\,\mathrm{d}t_2\\ &= \frac{1}{2}\int_0^T\int_0^T (b_\omega^2\circ\varphi_{t_1}^+)(b_\omega^2\circ\varphi_{t_2}^+)\left(\int_{t_1}^{t_2} b\circ\varphi_\sigma^+ d\sigma\right)^2\mathrm{d}t_1\,\mathrm{d}t_2.\end{aligned}$$

This concludes the proof of Theorem 5.3. □

Proof of Corollary 5.4. Let us first prove Item 1 Take $\rho = (x,\eta)\in S^*\Omega$. Since $T > T^+_{\omega\to\mathcal{O}\to\omega}$, there exists $0 < t_1(\rho) < t_{1,2}(\rho) < t_2(\rho) < T$ such that

$$\pi(\varphi^+_{t_1(\rho)}(\rho))\in\omega,\quad \pi(\varphi^+_{t_{1,2}(\rho)}(\rho))\in\mathcal{O},\quad \pi(\varphi^+_{t_2(\rho)}(\rho))\in\omega.$$

Since the functions b and b_ω are continuous this yields

$$\begin{aligned}&\det(\sigma_0(G_T^+))(\rho) =\\ &\frac{1}{8}\int_0^T\int_0^T (b_\omega^2\circ\varphi_{t_1}^+(\rho))(b_\omega^2\circ\varphi_{t_2}^+(\rho))\left(\int_{t_1}^{t_2} b\circ\varphi_\sigma^+(\rho)d\sigma\right)^2\mathrm{d}t_1\,\mathrm{d}t_2 > 0.\end{aligned}$$

With the manifold $S^*\Omega$ being compact, we have $\det(\sigma_0(G_T^+)) \geqq C > 0$ on $S^*\Omega$. The matrix $\sigma_0(G_T^+)$ has a non-negative trace and positive determinant, and is hence positive definite. Hence, G_T^+ is elliptic.

The proof of Item 2 consists in two steps (using a compactness–uniqueness argument as in Section 3 for the proof of Theorem 1.3). The first one is the following high-frequency observability estimate:

$$\begin{aligned}\int_0^T \|b_\omega v_2\|^2_{L^2(\Omega)}\,\mathrm{d}t &= \big(\mathcal{G}_T^+(g,h),(g,h)\big)_{L^2(\Omega;\mathbb{C}^2)}\\ &\geqq C\|(g,h)\|^2_{L^2(\Omega;\mathbb{C}^2)} - C'\|(g,h)\|^2_{H^{-\frac{1}{2}}(\Omega;\mathbb{C}^2)},\end{aligned} \tag{5.14}$$

for all $(g,h) \in L^2(\Omega;\mathbb{C}^2)$ and (v_1,v_2) associated solutions of (5.8).

Using the Gårding inequality for the operator $G_T^+ \in \Psi^0_{\mathrm{phg}}(\Omega)$ (see for instance [44, Chapter 2] or [14, Chapter 4]), this gives the existence of $C, C' > 0$ such that, for all $(g,h) \in L^2(\Omega;\mathbb{C}^2)$,

$$(G_T^+(g,h),(g,h))_{L^2(\Omega;\mathbb{C}^2)} \geqq C\|(g,h)\|^2_{L^2(\Omega;\mathbb{C}^2)} - C'\|(g,h)\|^2_{H^{-\frac{1}{2}}(\Omega;\mathbb{C}^2)}.$$

Recalling that R_T is 1-smoothing, that is in particular $R_T \in \mathcal{L}(H^{-1}(\Omega;\mathbb{C}^2); L^2(\Omega;\mathbb{C}^2))$, we have for all $\varepsilon > 0$,

$$\left|(R_T(g,h),(g,h))_{L^2(\Omega;\mathbb{C}^2)}\right| \leqq \frac{C''}{\varepsilon}\|(g,h)\|^2_{H^{-1}(\Omega;\mathbb{C}^2)} + \varepsilon\|(g,h)\|^2_{L^2(\Omega;\mathbb{C}^2)},$$

and hence

$$\begin{aligned}(\mathcal{G}_T^+(g,h),(g,h))_{L^2(\Omega;\mathbb{C}^2)} &= (G_T^+(g,h),(g,h))_{L^2(\Omega;\mathbb{C}^2)}\\ &\quad + (R_T(g,h),(g,h))_{L^2(\Omega;\mathbb{C}^2)}\\ &\geqq \big(C-\varepsilon\big)\|(g,h)\|^2_{L^2(\Omega;\mathbb{C}^2)}\\ &\quad - \left(C' + \frac{C'''}{\varepsilon}\right)\|(g,h)\|^2_{H^{-\frac{1}{2}}(\Omega;\mathbb{C}^2)}.\end{aligned}$$

Taking ε sufficiently small concludes the proof of (5.14).

The second step of the proof of Item 2 follows Section 3.2. We consider

$$\begin{aligned}\mathcal{N}(T) = \{(g,h) \in L^2(\Omega;\mathbb{C}^2) &\text{ such that the associated solution of (5.8)}\\ &\text{satisfies } v_2(t,x) = 0 \text{ for all } (t,x) \in (0,T)\times\omega\},\end{aligned}$$

and prove that $\mathcal{N}(T) = \{0\}$ for $T > T^+_{\omega\to\mathcal{O}\to\omega}$. Indeed, a proof similar to that of Lemma 3.4 yields that $\mathcal{N}(T)$ is a finite dimensional subspace of $H^1(\Omega;\mathbb{C}^2)$ using (5.14), stable by the action of the operator

$$\begin{pmatrix} -i\lambda & 0 \\ \frac{1}{2i}b & -i\lambda \end{pmatrix}.$$

Hence, it contains an eigenfunction of this operator, (φ_1,φ_2):

$$\begin{cases} -i\lambda\varphi_1 = \mu\varphi_1, \\ \frac{b}{2i}\varphi_1 - i\lambda\varphi_2 = \mu\varphi_2. \end{cases} \tag{5.15}$$

In particular, μ is then an eigenvalue of the skew-adjoint operator $-i\lambda$ and thus $\mu \in i\mathbb{R}$. Taking the $L^2(\Omega)$-inner product of the first line of (5.15) with φ_2, and that of the second line of (5.15) with φ_1, we obtain

$$\begin{cases} i(\lambda\varphi_2, \varphi_1)_{L^2(\Omega)} = -\mu(\varphi_2, \varphi_1)_{L^2(\Omega)}, \\ \frac{b}{2i}(\varphi_1, \varphi_1)_{L^2(\Omega)} - i(\lambda\varphi_2, \varphi_1)_{L^2(\Omega)} = \mu(\varphi_2, \varphi_1)_{L^2(\Omega)}. \end{cases}$$

Adding these two lines yields

$$(b\varphi_1, \varphi_1)_{L^2(\Omega)} = 0.$$

Since $b \geqq 0$ and b does not vanish identically, this proves that $\varphi_1 = 0$ on $\mathcal{O}$. As the first equation of (5.15) gives that $-\Delta\varphi_1 = \mu^2\varphi_1$, φ_1 is an eigenfunction of the Laplace operator vanishing on $\mathcal{O}$, a unique continuation result (see for instance the classical reference [6,7], the book [47] or the exposition article [25]) yields $\varphi_1 = 0$ on Ω.

Moreover, $(g, h) \in \mathcal{N}(T)$ yields $\varphi_2 = 0$ on ω. This proves that $\varphi_2 = 0$ on Ω, as φ_2 is then also an eigenfunction of the Laplace operator. This is similar to Remark 3.5. We then obtain $\mathcal{N}(T) = \{0\}$.

Inequality (5.14) is the analogue to the weak observability inequality of Proposition (3.2). Using the same contradiction argument as in Section 3.2 concludes the proof of the observability inequality (5.10).

The proof of Item 3 is inspired by [18, Theorem 4.1]. First, $\mathcal{G}_T^+$ is a bounded selfadjoint and coercive operator and is hence invertible according to the Riesz Theorem. Second, since $G_T^+ \in \Psi^0_{\mathrm{phg}}(\Omega, \mathbb{C}^{2\times 2})$ is elliptic, there exists (see for instance [23, Theorem 18.1.24]) a parametrix $\Lambda_T^+ \in \Psi^0_{\mathrm{phg}}(\Omega, \mathbb{C}^{2\times 2})$ such that

$$\Lambda_T^+ G_T^+ = \operatorname{Id} + R, \quad \text{with} \quad R \in \Psi^{-\infty}_{\mathrm{phg}}(\Omega, \mathbb{C}^{2\times 2}) \quad \text{and} \quad \sigma_0(\Lambda_T^+) = \sigma_0(G_T^+)^{-1}. \tag{5.16}$$

Hence, using the decomposition $\mathcal{G}_T^+ = G_T^+ + R_T$, with R_T infinitely smoothing, we have

$$\Lambda_T^+ \mathcal{G}_T^+ = \Lambda_T^+(G_T^+ + R_T) = \operatorname{Id} + R + \Lambda_T^+ R_T,$$

where $\Lambda_T^+ R_T$ is infinitely smoothing. Applying the operator $(\mathcal{G}_T^+)^{-1} \in \mathcal{L}(L^2(\Omega))$ to this identity, we obtain

$$(\mathcal{G}_T^+)^{-1} = \Lambda_T^+ - (R + \Lambda_T^+ R_T)(\mathcal{G}_T^+)^{-1}.$$

Observing that the operator $(R+\Lambda_T^+ R_T)(\mathcal{G}_T^+)^{-1}$ is in $\mathcal{R}^\infty$ concludes with (5.16) the proof of Item 3 (recall that the precise definition of $\mathcal{R}^\infty$ is given in Definition 2.2).

Finally, Item 4 is a direct consequence of Item 3. □

5.3.2. Characterization of the HUM Operator for Coupled Wave Equations The method used in this section follows that of Section 5.3.1. Yet the proof is more involved. In fact, here, there is a possible interaction between waves with positive and negative frequencies. For the same reason, some of the remainders that we shall obtain along the proof will only be 1-smoothing in this case, whereas we obtained infinitely smoothing remainder terms in Section 5.3.1. Note that for a scalar wave equation, such remainder terms are also 1-smoothing only. This can be improved by taking a time dependent control function $b_\omega(t, x)$ vanishing at all orders at times 0 and T (see [18, Theorem 4.1]). Due to the coupling of two waves in System (1.1) we shall see that the remainder terms that we shall obtain cannot be better than 1-smoothing, even if b_ω is chosen time dependent.

Recall that proving the observability inequality (1.5) for the adjoint system (1.6) is equivalent to proving the $L^2 - L^2$ observability inequality (2.13) for solutions of System (2.12):

$$\begin{cases} Pw_1 = 0 & \text{in } (0, T) \times \Omega, \\ Pw_2 = -b(x)\,(1-\Delta)^{\frac{1}{2}} w_1 & \text{in } (0, T) \times \Omega. \end{cases} \tag{5.17}$$

The symmetric setting of this system is, once again, simpler to handle, and we shall therefore work with $L^2 - L^2$ data. In the framework of Section 5.1 this corresponds to the case $s = 1$ and $\sigma = 1$ and the associated control system is (2.14).

Given the following initial data

$$(w_1(0), w_2(0), \partial_t w_1(0), \partial_t w_2(0)) = (w_1^0, w_2^0, w_1^1, w_2^1) \in L^2(\Omega; \mathbb{C}^2) \\ \times H^{-1}(\Omega; \mathbb{C}^2)$$

for System (5.17), we shall split them into their positive and negative parts following [18]. Concerning the state w_1, we set

$$g_\pm = \frac{1}{2}\left(\Pi_+ w_1^0 \mp i\lambda^{-1}\Pi_+ w_1^1\right) \in L^2_+(\Omega), \qquad g_0 = \Pi_0 w_1^0 \in \mathbb{C}, \\ g_1 = \Pi_0 w_1^1 \in \mathbb{C}, \tag{5.18}$$

so that

$$w_1(0) = w_1^0 = g_+ + g_- + g_0, \qquad \partial_t w_1(0) = w_1^1 = i\lambda(g_+ - g_-) + g_1.$$

In this splitting, the expression of the solution of System (5.17) is particularly simple:

$$w_1(t) = e^{it\lambda} g_+ + e^{-it\lambda} g_- + t g_1 + g_0. \tag{5.19}$$

We proceed to the same decomposition for w_2 and set

$$h_\pm = \frac{1}{2}\left(\Pi_+ w_2^0 \mp i\lambda^{-1}\Pi_+ w_2^1\right) \in L^2_+(\Omega), \qquad h_0 = \Pi_0 w_2^0 \in \mathbb{C}, \\ h_1 = \Pi_0 w_2^1 \in \mathbb{C}, \tag{5.20}$$

so that

$$w_2(0) = w_2^0 = h_+ + h_- + h_0, \qquad \partial_t w_2(0) = w_2^1 = i\lambda(h_+ - h_-) + h_1.$$

In this splitting, the expression of the solution of System (5.17) through the Duhamel formula is

$$\begin{aligned} w_2(t) = e^{it\lambda}h_+ + e^{-it\lambda}h_- + th_1 + h_0 \\ -\lambda^{-1}\int_0^t \sin((t-\sigma)\lambda)\Pi_+ b(1-\Delta)^{\frac{1}{2}} w_1(\sigma)d\sigma \\ -\int_0^t (t-\sigma)\Pi_0 b(1-\Delta)^{\frac{1}{2}} w_1(\sigma)d\sigma. \end{aligned} \tag{5.21}$$

We denote by Σ_+ the linear mapping

$$\begin{aligned} \Sigma_+ : H^s(\Omega;\mathbb{C}^2) \times H^{s-1}(\Omega;\mathbb{C}^2) &\to H_+^s(\Omega;\mathbb{C}^4) \\ (w_1^0, w_2^0, w_1^1, w_2^1) &\mapsto (g_+, h_+, g_-, h_-), \end{aligned}$$

corresponding to the splitting (5.18)–(5.20). Note that this mapping is onto but not injective since constants are lost. Note also that the order in (g_+, h_+, g_-, h_-) is important since we collect together data corresponding to the same wave frequencies. We also denote by Σ_0 the linear mapping associated with constant functions

$$\begin{aligned} \Sigma_0 : H^s(\Omega;\mathbb{C}^2) \times H^{s-1}(\Omega;\mathbb{C}^2) &\to \mathbb{C}^4 \\ (w_1^0, w_2^0, w_1^1, w_2^1) &\mapsto (g_0, h_0, g_1, h_1), \end{aligned}$$

corresponding to the splitting (5.18)–(5.20). Finally, we denote by Σ the *isomorphism* corresponding to the splitting (5.18)–(5.20):

$$\begin{aligned} &\Sigma : H^s(\Omega;\mathbb{C}^2) \times H^{s-1}(\Omega;\mathbb{C}^2) \to H_+^s(\Omega;\mathbb{C}^4) \times \mathbb{C}^4 \\ &\mathcal{W}^0 = (w_1^0, w_2^0, w_1^1, w_2^1) \mapsto (\Sigma_+\mathcal{W}^0, \Sigma_0\mathcal{W}^0) = (g_+, h_+, g_-, h_-, g_0, h_0, g_1, h_1). \end{aligned}$$

We recall the natural duality bracket

$$\begin{aligned} \left\langle (u_1^0, u_2^0, u_1^1, u_2^1), (w_1^0, w_2^0, w_1^1, w_2^1) \right\rangle_* = (u_1^1, w_1^0)_{L^2(\Omega)} - \left\langle u_1^0, w_1^1 \right\rangle_{H^1(\Omega), H^{-1}(\Omega)} \\ + (u_2^1, w_2^0)_{L^2(\Omega)} - \left\langle u_2^0, w_2^1 \right\rangle_{H^1(\Omega), H^{-1}(\Omega)}, \end{aligned}$$

as used in Section 5.1 describing the Hilbert Uniqueness Method. In the case $s = 0$, the transpose operator of Σ with respect to this duality bracket

$$\Sigma^* : L_+^2(\Omega;\mathbb{C}^4) \times \mathbb{C}^4 \to H^1(\Omega;\mathbb{C}^2) \times L^2(\Omega;\mathbb{C}^2),$$

is given by

$$\begin{aligned} &\left\langle \Sigma^* H, \mathcal{W}^0 \right\rangle_* = (H, \Sigma\mathcal{W}^0)_{L_+^2(\Omega;\mathbb{C}^4)\times\mathbb{C}^4}, \\ &\qquad \text{for all } H \in L_+^2(\Omega;\mathbb{C}^4) \times \mathbb{C}^4,\ \mathcal{W}^0 \in L^2(\Omega;\mathbb{C}^2) \times H^{-1}(\Omega;\mathbb{C}^2). \end{aligned}$$

We can now state the analogue of Theorem 5.3 for full wave systems, providing a characterization of the Gramian operator (in the wave splitting (5.18)–(5.20)).

Theorem 5.5. *Denoting by $\mathcal{W}^0 = (w_1^0, w_2^0, w_1^1, w_2^1)$ the initial data for System* (5.17)*, we have*

$$\int_0^T \|b_\omega w_2\|^2_{L^2(\Omega)} \,\mathrm{d}t = \left(\mathcal{G}_T \Sigma \mathcal{W}^0, \Sigma \mathcal{W}^0\right)_{L^2_+(\Omega;\mathbb{C}^4)\times\mathbb{C}^4}, \tag{5.22}$$

where $\mathcal{L}_T = \Sigma^* \mathcal{G}_T \Sigma \in \mathcal{L}\big(L^2(\Omega;\mathbb{C}^2)\times H^{-1}(\Omega;\mathbb{C}^2); H^1(\Omega;\mathbb{C}^2)\times L^2(\Omega;\mathbb{C}^2)\big)$ is the Gramian operator of (5.17)*, and $\mathcal{G}_T \in \mathcal{L}(L^2_+(\Omega;\mathbb{C}^4)\times\mathbb{C}^4)$ is its representation in the splitting* (5.18)–(5.20).

Moreover, there exist $G_T \in \Psi^0_{\mathrm{phg}}(\Omega;\mathbb{C}^{4\times4}) \cap \bigcap_{s\in\mathbb{R}} \mathcal{L}(H^s_+(\Omega;\mathbb{C}^4))$, $R_T \in \bigcap_{s\in\mathbb{R}} \mathcal{L}(H^s_+(\Omega;\mathbb{C}^4))$ a 1-smoothing operator, $R_T^0 \in \mathcal{L}(\mathbb{C}^4)$, and $\tilde{R}_T \in \bigcap_{s\in\mathbb{R}} \mathcal{L}(\mathbb{C}^4; H^s_+(\Omega;\mathbb{C}^4))$ such that $\mathcal{G}_T$ on $L^2_+(\Omega;\mathbb{C}^4)\times\mathbb{C}^4$ is given by (a 8×8 operator matrix)

$$\mathcal{G}_T = \begin{pmatrix} G_T + R_T & \tilde{R}_T \\ \tilde{R}_T^* & R_T^0 \end{pmatrix}. \tag{5.23}$$

The principal symbol of G_T is (*a 4×4 symbol matrix*)

$$\sigma_0(G_T) = \begin{pmatrix} \sigma_0(G_T^+) & 0 \\ 0 & \sigma_0(G_T^-) \end{pmatrix}, \tag{5.24}$$

with $\sigma_0(G_T^\pm) \in S^0_{\mathrm{phg}}(T^*\Omega, \mathbb{C}^{2\times2})$,

$$\sigma_0(G_T^\pm) = \begin{pmatrix} \frac14 \int_0^T b_\omega^2 \circ \varphi_t^\pm \left(\int_0^t b\circ\varphi_\sigma^\pm d\sigma\right)^2 \mathrm{d}t & \pm\frac{1}{2i}\int_0^T b_\omega^2\circ\varphi_t^\pm \left(\int_0^t b\circ\varphi_\sigma^\pm d\sigma\right) \mathrm{d}t \\ \mp\frac{1}{2i}\int_0^T b_\omega^2\circ\varphi_t^\pm \left(\int_0^t b\circ\varphi_\sigma^\pm d\sigma\right)\mathrm{d}t & \int_0^T b_\omega^2\circ\varphi_t^\pm\,\mathrm{d}t \end{pmatrix}. \tag{5.25}$$

In particular, we have

$$\det(\sigma_0(G_T)) = \det(\sigma_0(G_T^+))\det(\sigma_0(G_T^-)),$$

with

$$\det(\sigma_0(G_T^\pm)) = \frac18 \int_0^T\int_0^T (b_\omega^2\circ\varphi_{t_1}^\pm)(b_\omega^2\circ\varphi_{t_2}^\pm) \times \left(\int_{t_1}^{t_2} b\circ\varphi_\sigma^\pm d\sigma\right)^2 \mathrm{d}t_1\,\mathrm{d}t_2 \in S^0_{\mathrm{phg}}(T^*\Omega).$$

From (5.22), if the operator $\mathcal{L}_T = \Sigma^*\mathcal{G}_T\Sigma$ is invertible then the HUM operator $\mathcal{H}_T$ is precisely its inverse (see Section 5.2). The following corollary provides the microlocal structure of $\mathcal{H}_T$ within the splitting framework. In particular, this provides a second proof of the observability of System (5.17) under the appropriate geometric conditions.

Corollary 5.6. *Assume that ω and $\mathcal{O}$ satisfy GCC and that $T > T_{\omega\to\mathcal{O}\to\omega}$. Then, we have the following properties:*

1. *The operator $G_T \in \Psi^0_{\mathrm{phg}}(\Omega, \mathbb{C}^{4\times 4})$ is elliptic.*
2. *The operator $\mathcal{G}_T$ is coercive on $L^2_+(\Omega;\mathbb{C}^4)\times\mathbb{C}^4$:*

$$\int_0^T \|b_\omega w_2\|^2_{L^2(\Omega)}\,\mathrm{d}t = \Big(\mathcal{G}_T\Sigma\mathcal{W}^0, \Sigma\mathcal{W}^0\Big)_{L^2_+(\Omega;\mathbb{C}^4)\times\mathbb{C}^4} \geqq C\|\Sigma\mathcal{W}^0\|^2_{L^2_+(\Omega;\mathbb{C}^4)\times\mathbb{C}^4} \geqq C'\|\mathcal{W}^0\|^2_{L^2(\Omega;\mathbb{C}^2)\times H^{-1}(\Omega;\mathbb{C}^2)}, \tag{5.26}$$

for all $\mathcal{W}^0 \in L^2(\Omega;\mathbb{C}^2)\times H^{-1}(\Omega;\mathbb{C}^2)$ and (w_1, w_2) associated solutions of System (5.17).

3. *The operator $\mathcal{G}_T$ is invertible on $\mathcal{L}(L^2_+(\Omega;\mathbb{C}^4)\times\mathbb{C}^4)$. The HUM operator (in the splitting (5.18)–(5.20)) is its inverse $\mathcal{G}_T^{-1}$ and can be decomposed on $L^2_+(\Omega;\mathbb{C}^4)\times\mathbb{C}^4$ as (the 8×8 operator matrix)*

$$\mathcal{G}_T^{-1} = \begin{pmatrix} \Lambda_T + S_T & \tilde{S}_T \\ \tilde{S}_T^* & S_T^0 \end{pmatrix},$$

where $S_T \in \mathcal{R}^1$, $S_T^0 \in \mathcal{L}(\mathbb{C}^4)$, $\tilde{S}_T \in \bigcap_{s\in\mathbb{R}} \mathcal{L}(\mathbb{C}^4; H^s_+(\Omega;\mathbb{C}^4))$ and $\Lambda_T \in S^0_{\mathrm{phg}}(T^*\Omega, \mathbb{C}^{4\times 4}) \cap \bigcap_{s\in\mathbb{R}} \mathcal{L}(H^s_+(\Omega;\mathbb{C}^4))$, with principal symbol

$$\sigma_0(\Lambda_T) = \sigma_0(G_T)^{-1} = \begin{pmatrix} \sigma_0(G_T^+)^{-1} & 0 \\ 0 & \sigma_0(G_T^-)^{-1} \end{pmatrix}.$$

4. *In particular, the HUM operator $\mathcal{G}_T^{-1}$ is an isomorphism of $H^s_+(\Omega;\mathbb{C}^4)\times\mathbb{C}^4$ for all $s \geqq 0$ and we have*

$$\mathrm{WF}^s((\mathcal{G}_T)^{-1}\Sigma\mathcal{W}^0) = \mathrm{WF}^s(\Sigma\mathcal{W}^0).$$

Recall that the definition of the H^s-wavefront set of a k-tuple is given in (5.11).

Note that in Theorem 5.5, we do not use any sign assumption on the coupling term b, but only that b is real valued. As a consequence, we can state in this more general case a criterion for the high-frequency controllability of System (1.1). Compare with Theorem 1.3.

Definition 5.7. We say that (ω, b, T) satisfies the Polarization Control Condition (PCC) if, for any $\rho \in S^*\Omega$, there exist $0 < t_1^\pm < t_2^\pm < T$ such that

$$\varphi_{t_1^\pm}^\pm(\rho) \in \omega, \qquad \varphi_{t_2^\pm}^\pm(\rho) \in \omega, \qquad \text{and} \quad \int_{t_1}^{t_2} b\circ\varphi_\sigma^\pm(\rho)\,d\sigma \neq 0.$$

In particular, this requires that both ω and the set $\{b \neq 0\}$ satisfy GCC. Note that if $b \geqq 0$, we have that (ω, b, T) satisfies PCC if and only if ω and $\mathcal{O} := \{b > 0\}$ both satisfy GCC and $T > T_{\omega\to\mathcal{O}\to\omega}$. With this definition, we have the following result.

Corollary 5.8. 1. *The weak observability inequality*

$$\int_0^T \|b_\omega w_2\|^2_{L^2(\Omega)}\,\mathrm{d}t \geqq C\|\mathcal{W}^0\|^2_{L^2(\Omega;\mathbb{C}^2)\times H^{-1}(\Omega;\mathbb{C}^2)} - \|\mathcal{W}^0\|^2_{H^{-1}(\Omega;\mathbb{C}^2)\times H^{-2}(\Omega;\mathbb{C}^2)},$$

holds for all $\mathcal{W}^0 \in L^2(\Omega;\mathbb{C}^2)\times H^{-1}(\Omega;\mathbb{C}^2)$ and (w_1, w_2) associated solutions of System (5.17), if and only if (ω, b, T) satisfies PCC.

2. *Suppose that (ω, b, T) satisfies PCC and that no eigenfunction (φ_1, φ_2) of the operator*

$$\begin{pmatrix} -\Delta & 0 \\ b(1-\Delta)^{\frac{1}{2}} & -\Delta \end{pmatrix},$$

satisfy $\varphi_2|_\omega = 0$. Then, we have the observability inequality

$$\int_0^T \|b_\omega w_2\|^2_{L^2(\Omega)}\,\mathrm{d}t \geqq C\|\mathcal{W}^0\|^2_{L^2(\Omega;\mathbb{C}^2)\times H^{-1}(\Omega;\mathbb{C}^2)},$$

for all $\mathcal{W}^0 \in L^2(\Omega;\mathbb{C}^2)\times H^{-1}(\Omega;\mathbb{C}^2)$ and (w_1, w_2) associated solutions of System (5.17) (that is System (1.1) is controllable).

Note that the additional unique continuation assumption is valid in the following two particular cases

- $b \geqq 0$ on Ω and $\{b > 0\} \neq \emptyset$ (see Section 3.2 and Remark 3.5);
- $\omega \cap \{b \neq 0\} \neq \emptyset$ (see [26, Proposition 5.1]).

The question seems to be open in the general case.

Proof of Theorem 5.5. Here, we mostly follow the proof of Theorem 5.3. In fact, several additional terms appear in the calculations that we have to deal with.

In the matrix $\mathcal{G}_T$, given in (5.23), each one of the four blocks is a 4×4 matrix of operators (or simply scalars numbers). In a first step, we check that all blocks have the announced form, excluding the first block. In a second step, we shall focus on this first block that contains all the high-frequency of $\mathcal{G}_T$. We wish to compute $\int_0^T \|b_\omega w_2\|^2_{L^2(\Omega)}\,\mathrm{d}t$ and to cast it in the form of the right hand-side of (5.22), that is $\big(\mathcal{G}_T H, H\big)_{L^2_+(\Omega;\mathbb{C}^4)\times\mathbb{C}^4)}$, with $H = \Sigma\mathcal{W}^0$.

Focusing on (5.19)–(5.21), we first remark that $(\Pi_0 w_2(t), e_0)_{L^2(\Omega)} \in \mathscr{C}^\infty(\mathbb{R};\mathbb{C})$. Hence, for all $t\in\mathbb{R}$, the map

$$H = \Sigma\mathcal{W}^0 \mapsto (\Pi_0 w_2(t), e_0)_{L^2(\Omega)}$$

is an infinitely smoothing operator. As a consequence, it suffices to carry out our computation with w_2 replaced by

$$\Pi_+ w_2(t) = e^{it\lambda}h_+ + e^{-it\lambda}h_- - \lambda^{-1}\int_0^t \sin((t-\sigma)\lambda)\Pi_+ b(1-\Delta)^{\frac{1}{2}} \times\left(e^{i\sigma\lambda}g_+ + e^{-i\sigma\lambda}g_- + \sigma g_1 + g_0\right) d\sigma,$$

that is, ignoring the components along e_0, which contribute to the smoothing operators in the expression of $\mathcal{G}_T$ in (5.23). In the previous expression, the term

$$\lambda^{-1}\int_0^t \sin((t-\sigma)\lambda)\Pi_+ b(1-\Delta)^{\frac{1}{2}}(\sigma g_1 + g_0)\,d\sigma,$$

is in $\mathscr{C}^\infty(\mathbb{R}\times\Omega)$ since the functions $g_1 = g_1 e_0$ and $g_0 = g_0 e_0$ are smooth. As a consequence, we have

$$\begin{aligned}
&\int_0^T \|b_\omega w_2\|^2_{L^2(\Omega)}\,\mathrm{d}t \\
&= \int_0^T (R_t H, H)_{L^2(\Omega)}\,\mathrm{d}t + \int_0^T \Big\| b_\omega e^{it\lambda}h_+ + b_\omega e^{-it\lambda}h_- b_\omega \lambda^{-1} \\
&\times \int_0^t \sin((t-\sigma)\lambda)\Pi_+ b(1-\Delta)^{\frac{1}{2}}\left(e^{i\sigma\lambda}g_+ + e^{-i\sigma\lambda}g_-\right)d\sigma \Big\|^2_{L^2(\Omega)}\,\mathrm{d}t. \quad (5.27)
\end{aligned}$$

Now, we focus on the first block of the operator matrix in (5.23), that is a 4×4 matrix of operators on Ω, that yields the bilinear form applied to $(g_+, h_+, g_-, h_-) = \Sigma_+(w_1^0, w_2^0, w_1^1, w_2^1)$. It is associated with the last term in the expression (5.27), given by

$$\begin{aligned}
\int_0^T \Big\| b_\omega \Big(e^{it\lambda}h_+ + e^{-it\lambda}h_- \frac{\lambda^{-1}}{2i}\int_0^t \left(e^{i(t-\sigma)\lambda} - e^{-i(t-\sigma)\lambda}\right)\Pi_+ b(1-\Delta)^{\frac{1}{2}} \\
\times \left(e^{i\sigma\lambda}g_+ + e^{-i\sigma\lambda}g_-\right)d\sigma \Big) \Big\|^2_{L^2(\Omega)}\,\mathrm{d}t.
\end{aligned}$$

We only compute the first two lines of this matrix (as operating on (g_+, h_+, g_-, h_-)) and denote by $\mathcal{G}_T(\cdot,\cdot)$ the associated term in this matrix. The computation of the last two lines is similar and is left to the reader.

Let us start with the (simpler) second line. We have

$$\begin{aligned}
\mathcal{G}_T(h_+, h_+) &= \int_0^T \left(b_\omega e^{it\lambda}h_+, b_\omega e^{it\lambda}h_+\right)_{L^2(\Omega)}\,\mathrm{d}t \\
&= \left(\int_0^T e^{-it\lambda} b_\omega^2 e^{it\lambda}\,\mathrm{d}t\; h_+, h_+\right)_{L^2(\Omega)}. \quad (5.28)
\end{aligned}$$

Next, we compute

$$\begin{aligned}
\mathcal{G}_T(h_+, h_-) &= \int_0^T \left(b_\omega e^{it\lambda}h_+, b_\omega e^{-it\lambda}h_-\right)_{L^2(\Omega)}\,\mathrm{d}t \\
&= \left(\int_0^T e^{it\lambda} b_\omega^2 e^{it\lambda}\,\mathrm{d}t\; h_+, h_-\right)_{L^2(\Omega)} = (R_T h_+, h_-)_{L^2(\Omega)}, \quad (5.29)
\end{aligned}$$

for some 1-smoothing[2] operator R_T, according to Lemma A.1 (see Appendix A). Similarly, we compute

$$
\begin{aligned}
-\mathcal{G}_T(h_+, g_+) &= \int_0^T \Big(b_\omega e^{it\lambda} h_+, b_\omega \frac{\lambda^{-1}}{2i} \int_0^t \Big(e^{i(t-\sigma)\lambda} \\
&\qquad - e^{-i(t-\sigma)\lambda} \Big) \Pi_+ b (1-\Delta)^{\frac{1}{2}} e^{i\sigma\lambda} d\sigma \; g_+ \Big)_{L^2(\Omega)} \, \mathrm{d}t \\
&= \int_0^T \Big(b_\omega e^{it\lambda} h_+, b_\omega \frac{1}{2i} \int_0^t \Big(e^{i(t-\sigma)\lambda} \\
&\qquad - e^{-i(t-\sigma)\lambda} \Big) \Pi_+ b e^{i\sigma\lambda} d\sigma \; g_+ + R_t g_+ \Big)_{L^2(\Omega)} \, \mathrm{d}t,
\end{aligned}
$$

for some continuous family of 1-smoothing operators R_t, since $(1-\Delta)^{\frac{1}{2}} - \lambda \in \Psi^0_{\mathrm{phg}}(\Omega)$ and $[b, \lambda] \in \Psi^0_{\mathrm{phg}}(\Omega)$. We obtain

$$
\begin{aligned}
&\mathcal{G}_T(h_+, g_+) \\
&= \int_0^T \Big(\frac{1}{2i} \Pi_+ \Big(\int_0^t e^{-i\sigma\lambda} b e^{i\sigma\lambda} d\sigma \Big) \Pi_+ e^{-it\lambda} b_\omega^2 e^{it\lambda} h_+, g_+ \Big)_{L^2(\Omega)} \, \mathrm{d}t \\
&\quad - \int_0^T \Big(\frac{1}{2i} \Pi_+ \Big(\int_0^t e^{-i\sigma\lambda} b e^{-i\sigma\lambda} d\sigma \Big) \Pi_+ e^{it\lambda} b_\omega^2 e^{it\lambda} h_+, g_+ \Big)_{L^2(\Omega)} \, \mathrm{d}t \\
&\quad + (R_T h_+, g_+)_{L^2(\Omega)},
\end{aligned}
$$

where R_T is again a continuous family of 1-smoothing operators. In the central term of the right-hand side, we notice that $\int_0^t e^{-i\sigma\lambda} b e^{-i\sigma\lambda} d\sigma$ is a continuous family of 1-smoothing operators[3] according to Lemma A.1. With the Egorov Theorem (Theorem 2.1), we define as in (5.13) the pseudodifferential operators

$$
B_t^{\pm} := \int_0^t e^{\mp i\sigma\lambda} b e^{\pm i\sigma\lambda} d\sigma \in \Psi^0_{\mathrm{phg}}(\Omega), \tag{5.30}
$$

with principal symbol

$$
\sigma_0(B_t^{\pm})(x, \eta) = \int_0^t b \circ \varphi_\sigma^{\pm}(x, \eta) d\sigma, \quad (x, \eta) \in T^*\Omega.
$$

We now have

$$
\begin{aligned}
\mathcal{G}_T(h_+, g_+) &= \Big(\frac{1}{2i} \int_0^T \Pi_+ B_t^+ \Pi_+ e^{-it\lambda} b_\omega^2 e^{it\lambda} \, \mathrm{d}t \; h_+, g_+ \Big)_{L^2(\Omega)} \\
&\quad + (R_T h_+, g_+)_{L^2(\Omega)},
\end{aligned} \tag{5.31}
$$

---

[2] This smoothing term can be made infinitely smoothing if b_ω is chosen vanishing at infinite order at $t = 0$ and $t = T$ (see [18]).

[3] It seems to us that no particular choice of b (except $b = 0$) can improve this smoothing property.

where R_T is a continuous family of 1-smoothing operators. Similarly, we compute

$$\begin{aligned}
&\mathcal{G}_T(h_+, g_-)\\
&= -\int_0^T \left(b_\omega e^{it\lambda}h_+, b_\omega \frac{1}{2i}\int_0^t \left(e^{i(t-\sigma)\lambda} - e^{-i(t-\sigma)\lambda}\right)\Pi_+ b e^{-i\sigma\lambda} g_- + R_t g_-\right)_{L^2(\Omega)} \mathrm{d}t\\
&= \left(\frac{1}{2i}\int_0^T \Pi_+ \left(\int_0^t e^{i\sigma\lambda} b e^{i\sigma\lambda} d\sigma\right)\Pi_+ e^{-it\lambda} b_\omega^2 e^{it\lambda}\,\mathrm{d}t\; h_+, g_-\right)_{L^2(\Omega)}\\
&\quad -\left(\frac{1}{2i}\int_0^T \Pi_+ \left(\int_0^t e^{+i\sigma\lambda} b e^{-i\sigma\lambda} d\sigma\right)\Pi_+ e^{it\lambda} b_\omega^2 e^{it\lambda}\,\mathrm{d}t\; h_+, g_-\right)_{L^2(\Omega)}\\
&\quad + (R_T h_+, g_-)_{L^2(\Omega)}.
\end{aligned}$$

According to Lemma A.1, the operator $\int_0^t e^{i\sigma\lambda} b e^{i\sigma\lambda} d\sigma$ is a continuous family of 1-smoothing operators, and so is the first term in the right-hand side of this expression. The second term is also a continuous family of 1-smoothing operators, using Lemma A.2. We thus obtain

$$\mathcal{G}_T(h_+, g_-) = (R_T h_+, g_-)_{L^2(\Omega)}, \tag{5.32}$$

where R_T is 1-smoothing.

We have already computed the second line of the first block of the matrix $\mathcal{G}_T$. Let us now compute the first line. First, we have

$$\mathcal{G}_T(g_+, g_+) = \int_0^T \left\| b_\omega \frac{1}{2i}\int_0^t \left(e^{i(t-\sigma)\lambda} - e^{-i(t-\sigma)\lambda}\right)\Pi_+ b e^{i\sigma\lambda} g_+ + R_t g_+ \right\|^2_{L^2(\Omega)} \mathrm{d}t,$$

and we notice that, according to Lemma A.1, $b_\omega \frac{1}{2i}\int_0^t e^{-i(t-\sigma)\lambda}\Pi_+ b e^{i\sigma\lambda}\,\mathrm{d}\sigma$ is a continuous family of 1-smoothing operators. Hence, we have (with a continuous family of 1-smoothing operators $\tilde{R}_t$)

$$\begin{aligned}
\mathcal{G}_T(g_+, g_+) &= \int_0^T \left\| b_\omega \frac{1}{2i}\int_0^t e^{i(t-\sigma)\lambda}\Pi_+ b e^{i\sigma\lambda} g_+ + \tilde{R}_t g_+ \right\|^2_{L^2(\Omega)} \mathrm{d}t,\\
&= \frac{1}{4}\left(\Pi_+ \int_0^T (B_t^+)^* \Pi_+ e^{-it\lambda} b_\omega^2 e^{it\lambda}\Pi_+ B_t^+ \,\mathrm{d}t\, \Pi_+ g_+, g_+\right)_{L^2(\Omega)}\\
&\quad + (R_T g_+, g_+)_{L^2(\Omega)},
\end{aligned} \tag{5.33}$$

as in the proof of Theorem 5.3.

Then, the operator arising in the term $\mathcal{G}_T(g_+, h_+)$ is the adjoint of that of $\mathcal{G}_T(h_+, g_+)$, given by (5.31).

Next

$$\begin{aligned}
\mathcal{G}_T(g_+, h_-) &= \int_0^T \Bigg(e^{it\lambda} b_\omega^2 \frac{\lambda^{-1}}{2i}\int_0^t \left(e^{i(t-\sigma)\lambda} - e^{-i(t-\sigma)\lambda}\right)\\
&\qquad\qquad \times \Pi_+ b(1-\Delta)^{\frac{1}{2}} e^{i\sigma\lambda} d\sigma\; g_+, h_-\Bigg)_{L^2(\Omega)} \mathrm{d}t\\
&= \frac{1}{2i}\int_0^T \left(e^{it\lambda} b_\omega^2 e^{it\lambda}\int_0^t e^{-i\sigma\lambda}\Pi_+ b e^{i\sigma\lambda} d\sigma\; g_+ + R_t g_+, h_-\right)_{L^2(\Omega)} \mathrm{d}t
\end{aligned}$$

since $\int_0^t e^{-i(t-\sigma)\lambda}\Pi_+ b e^{i\sigma\lambda}\,\mathrm{d}\sigma$ is a continuous family of 1-smoothing operators according to Lemma A.1. Lemma A.2 then yields

$$\mathcal{G}_T(g_+, h_-) = (R_T g_+, h_-)_{L^2(\Omega)} \tag{5.34}$$

for R_T some 1-smoothing operator.

It only remains to compute

$$\begin{aligned}
&\mathcal{G}_T(g_+, g_-)\\
&= -\int_0^T \Bigg(b_\omega \frac{\lambda^{-1}}{2i}\int_0^t \left(e^{i(t-\sigma)\lambda} - e^{-i(t-\sigma)\lambda}\right)\Pi_+ b(1-\Delta)^{\frac{1}{2}} e^{i\sigma\lambda} d\sigma\ g_+,\\
&\quad b_\omega \frac{\lambda^{-1}}{2i}\int_0^t \left(e^{i(t-\sigma)\lambda} - e^{-i(t-\sigma)\lambda}\right)\Pi_+ b(1-\Delta)^{\frac{1}{2}} e^{-i\sigma\lambda} d\sigma\ g_-\Bigg)_{L^2(\Omega)} \mathrm{d}t\\
&= \int_0^T \left(b_\omega \frac{1}{2i}\int_0^t e^{i(t-\sigma)\lambda}\Pi_+ b e^{i\sigma\lambda} d\sigma\ g_+, b_\omega \frac{1}{2i}\int_0^t e^{-i(t-\sigma)\lambda}\Pi_+ b e^{-i\sigma\lambda} d\sigma\ g_-\right)_{L^2(\Omega)} \mathrm{d}t\\
&\quad + (R_T g_+, g_-)_{L^2(\Omega)},
\end{aligned}$$

after having used twice Lemma A.1. This can be rewritten as

$$\begin{aligned}
\mathcal{G}_T(g_+, g_-) &= \int_0^T \left(b_\omega \frac{1}{2i} e^{it\lambda} B_t^+ g_+, b_\omega \frac{1}{2i} e^{-it\lambda} B_t^- g_-\right)_{L^2(\Omega)} \mathrm{d}t\\
&= \frac{1}{4}\left(\int_0^T (B_t^-)^* e^{it\lambda} b_\omega^2 \frac{1}{2i} e^{it\lambda} B_t^+ \,\mathrm{d}t\ g_+, g_-\right)_{L^2(\Omega)}.
\end{aligned}$$

Using Lemma A.2, we finally obtain

$$\mathcal{G}_T(g_+, g_-) = (R_T g_+, g_-)_{L^2(\Omega)}, \tag{5.35}$$

for R_T some 1-smoothing operator.

Finally, combining (5.28), (5.29), (5.31), (5.32), (5.33), (5.34) and (5.35), we obtain the first two lines of the first block in (5.23) (that is the term $G_T + R_T$), with symbols according to (5.24)–(5.25). The last two lines can be computed similarly. The determinant of $\sigma_0(G_T)$ is then given by Theorem 5.3. This concludes the proof of Theorem 5.5. □

Proof of Corollary 5.6. The proof of Corollary 5.6 is very similar to that of Corollary 5.4, although more technical. The proof of Item 1 is the same as that of Item 1 in Corollary 5.4.

Then, the observability inequality of Item 2 is again proved in two steps. The ellipticity of G_T together with the Gårding inequality first yield the weak observability estimate

$$\begin{aligned}
\int_0^T \|b_\omega w_2\|^2_{L^2(\Omega)} \mathrm{d}t &= \left(\mathcal{G}_T \Sigma\mathcal{W}^0, \Sigma\mathcal{W}^0\right)_{L^2_+(\Omega;\mathbb{C}^4)\times\mathbb{C}^4}\\
&\geqq C\|\Sigma\mathcal{W}^0\|^2_{L^2_+(\Omega;\mathbb{C}^4)\times\mathbb{C}^4} - C'\|\Sigma\mathcal{W}^0\|^2_{H_+^{-\frac{1}{2}}(\Omega;\mathbb{C}^4)\times\mathbb{C}^4},
\end{aligned} \tag{5.36}$$

for all $\mathcal{W}^0 \in L^2(\Omega;\mathbb{C}^2) \times H^{-1}(\Omega;\mathbb{C}^2)$ and (w_1, w_2) associated solutions of System (5.17). Then, the proof of the observability inequality (5.26) assuming the weak observability inequality (5.36) is already done in Section 3.2.

Next, to prove Items 3 and 4, we first remark that $\mathcal{G}_T$ is invertible on $\mathcal{L}(L^2_+(\Omega;\mathbb{C}^4) \times \mathbb{C}^4)$, as for the proof of Item 3 in Corollary 5.4. This yields in particular that the operator $L_T := G_T + R_T \in \mathcal{L}(H^s_+(\Omega;\mathbb{C}^4))$ (for any $s \in \mathbb{R}$) is invertible in $\mathcal{L}(L^2_+(\Omega;\mathbb{C}^4))$. Let us show (by induction) that its inverse L_T^{-1} in $\mathcal{L}(L^2_+(\Omega;\mathbb{C}^4)$ is in $\mathcal{L}(H^s_+(\Omega;\mathbb{C}^4))$ for any $s \geqq 0$. This is true for $s = 0$. Now, suppose $L_T^{-1} \in \mathcal{L}(H^{s-1}_+(\Omega;\mathbb{C}^4))$ and take $f \in H^s_+(\Omega;\mathbb{C}^4)$. Hence the equation $L_T u = f$ has a solution $u \in H^{s-1}_+(\Omega;\mathbb{C}^4)$, satisfying

$$G_T u = f - R_T u \in H^s_+(\Omega;\mathbb{C}^4),$$

since R_T is a 1-smoothing operator preserving any H^s_+. Using the ellipticity of the operator $G_T \in \Psi^0_{\mathrm{phg}}(\Omega, \mathbb{C}^{4\times 4})$, proved in Item 1, this yields $u \in H^s(\Omega;\mathbb{C}^4)$, and hence $L_T^{-1} \in \mathcal{L}(H^s_+(\Omega;\mathbb{C}^4))$. By induction, we have

$$L_T^{-1} \in \mathcal{L}(H^s_+(\Omega;\mathbb{C}^4)), \quad \text{for all } s \geqq 0. \tag{5.37}$$

Now, we come back to the description of the inverse operator $\mathcal{G}_T^{-1}$ of $\mathcal{G}_T$ in $\mathcal{L}(L^2_+(\Omega;\mathbb{C}^4) \times \mathbb{C}^4)$, which takes the form

$$\mathcal{G}_T^{-1} = \begin{pmatrix} \tilde{\Lambda}_T & \tilde{S}_T \\ \tilde{S}_T^* & S_T^0 \end{pmatrix},$$

for some $\tilde{\Lambda}_T \in \mathcal{L}(L^2_+(\Omega;\mathbb{C}^4))$, $S_T^0 \in \mathcal{L}(\mathbb{C}^4)$, and $\tilde{S}_T \in \mathcal{L}(\mathbb{C}^4; L^2_+(\Omega;\mathbb{C}^4))$. Now, writing $\mathcal{G}_T\mathcal{G}_T^{-1} = \mathrm{Id}$ yields, in particular

$$L_T\tilde{\Lambda}_T + \tilde{R}_T\tilde{S}_T^* = \mathrm{Id}_{\mathcal{L}(L^2_+(\Omega;\mathbb{C}^4))} \quad \text{and} \quad L_T\tilde{S}_T + \tilde{R}_T S_T^0 = 0_{\mathcal{L}(\mathbb{C}^4; L^2_+(\Omega;\mathbb{C}^4))}.$$

The first of these two identities together with (5.37) yields

$$\tilde{\Lambda}_T = L_T^{-1} - L_T^{-1}\tilde{R}_T\tilde{S}_T^* \in \mathcal{L}(H^s_+(\Omega;\mathbb{C}^4)), \quad \text{for all } s \geqq 0. \tag{5.38}$$

Similarly, the second expression together with (5.37) gives

$$\tilde{S}_T = -L_T^{-1}\tilde{R}_T S_T^0 \in \bigcap_{s\in\mathbb{R}} \mathcal{L}(\mathbb{C}^4; H^s_+(\Omega;\mathbb{C}^4)).$$

We finally prove that $\tilde{\Lambda}_T$ takes the form claimed in the statement of the corollary. Since $G_T \in \Psi^0_{\mathrm{phg}}(\Omega, \mathbb{C}^{4\times 4})$ is elliptic, there exists (see for instance [23, Theorem 18.1.24]) a parametrix $\Lambda_T \in \Psi^0_{\mathrm{phg}}(\Omega, \mathbb{C}^{2\times 2})$ such that

$$\Lambda_T G_T = \mathrm{Id} + R, \quad \text{with} \quad R \in \Psi^{-\infty}_{\mathrm{phg}}(\Omega, \mathbb{C}^{4\times 4}) \quad \text{and} \quad \sigma_0(\Lambda_T) = \sigma_0(G_T)^{-1}.$$

Applying this parametrix to the equation $(G_T + R_T)\tilde{\Lambda}_T + \tilde{R}_T\tilde{S}_T^* = \mathrm{Id}$ yields

$$\tilde{\Lambda}_T = \Lambda_T - \Lambda_T R_T\tilde{\Lambda}_T - \Lambda_T\tilde{R}_T\tilde{S}_T^* - R\tilde{\Lambda}_T.$$

Then, the operator $-\Lambda_T \tilde{R}_T \tilde{S}_T^* - R\tilde{\Lambda}_T \in \mathcal{R}^\infty$ and $\Lambda_T R_T \tilde{\Lambda}_T \in \mathcal{R}^1$ according to (5.38) and as R_T is 1-smoothing. This concludes the proof of Item 3 and the first part of Item 4.

Finally, the second part of Item 4 (the wavefront set identity) is a consequence of the pseudodifferential nature of the principal part of the operator $\mathcal{G}_T^{-1}$. □

Proof of Corollary 5.8. The key point here is that Condition PCC is equivalent to the fact that $\rho \mapsto \det(\sigma_0(G_T))(\rho)$ does not vanish for $\rho \in S^*\Omega$. If PCC is satisfied, then the operator G_T is elliptic and we can follow the proof of Corollary 5.4. Similarly Item 2 only concerns the low-frequency problem and its proof follows Section 3.2.

Conversely, if PCC is not satisfied, there exists $\nu_0 \in S^*\Omega$ such that $\det(\sigma_0(G_T)(\nu_0)) = 0$. Hence, there exists a vector $v \in \mathbb{C}^4 \setminus \{0\}$ such that

$$\sigma_0(G_T)(\nu_0)v = 0. \tag{5.39}$$

After a linear change of coordinates in $\mathbb{C}^4$, we may assume that $v = (1, 0, 0, 0)$. Consider now a sequence $(w_0^k)_{k\in\mathbb{N}}$ of scalar functions on Ω such that

$$\lim_{k\to\infty} \|w_0^k\|_{L^2(\Omega)} = 1, \qquad w_0^k \rightharpoonup 0 \quad \text{in } L^2(\Omega; \mathbb{C}),$$

and $(w_0^k)_{k\in\mathbb{N}}$ is pure and admits the microlocal defect measure $\delta_{(x,\eta)=\nu_0}$ (such a sequence is constructed in Appendix B.3, see Equation (B.5)). The vectorial sequence $(u^k)_{k\in\mathbb{N}}$ given by $u^k := \Pi_+ w_0^k v = (\Pi_+ w_0^k, 0, 0, 0)$ satisfies (since $\Pi_0 w_0^k \to 0$ because of the weak convergence)

$$\lim_{k\to\infty} \|u^k\|_{L^2(\Omega;\mathbb{C}^4)} = 1, \qquad u^k \rightharpoonup 0 \quad \text{in } L^2(\Omega; \mathbb{C}^4), \tag{5.40}$$

is also pure, and admits the microlocal defect measure

$$\mu = \begin{pmatrix} \delta_{\nu_0} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

We compute

$$\begin{aligned} \lim_{k\to\infty} \left(\mathcal{G}_T \begin{pmatrix} u^k \\ 0_{\mathbb{C}^4} \end{pmatrix}, \begin{pmatrix} u^k \\ 0_{\mathbb{C}^4} \end{pmatrix} \right)_{L^2_+(\Omega;\mathbb{C}^4)\times\mathbb{C}^4} &= \lim_{k\to\infty} \left((G_T + R_T)u^k, u^k \right)_{L^2_+(\Omega;\mathbb{C}^4)} \\ &= \lim_{k\to\infty} (G_T u^k, u^k)_{L^2(\Omega;\mathbb{C}^4)}, \end{aligned}$$

since R_T is 1-smoothing. Moreover, we have

$$(G_T u^k, u^k)_{L^2(\Omega;\mathbb{C}^4)} \to \int_{S^*\Omega} \operatorname{tr}\{\sigma_0(G_T)(\rho)\mu(\mathrm{d}\rho)\}.$$

Writing $\sigma_0(G_T)(\rho) = \sigma_{ij}(\rho)$, for $i, j \in \{1, 2, 3, 4\}$, this yields

$$(G_T u^k, u^k)_{L^2(\Omega;\mathbb{C}^4)} \to \left\langle \delta_{\nu_0}, \sigma_{11} \right\rangle_{S^*\Omega} = \sigma_{11}(\nu_0).$$

Also, condition (5.39) then gives $\sigma_{1j}(\nu_0) = 0$ for $j \in \{1, 2, 3, 4\}$, so that

$$\left(\mathcal{G}_T \begin{pmatrix} u^k \\ 0_{\mathbb{C}^4} \end{pmatrix}, \begin{pmatrix} u^k \\ 0_{\mathbb{C}^4} \end{pmatrix}\right)_{L^2_+(\Omega;\mathbb{C}^4)\times\mathbb{C}^4} \to 0, \quad \text{as } k \to \infty.$$

In view of (5.40), this disproves the observability inequality

$$(\mathcal{G}_T H, H)_{L^2_+(\Omega;\mathbb{C}^4)\times\mathbb{C}^4} \geqq C\|H\|^2_{L^2_+(\Omega;\mathbb{C}^4)\times\mathbb{C}^4} - C'\|H\|^2_{H^{-1}_+(\Omega;\mathbb{C}^4)\times\mathbb{C}^4}, \\ \text{for all } H \in L^2_+(\Omega;\mathbb{C}^4)\times\mathbb{C}^4.$$

Recalling Theorem 5.5, this concludes the proof of Corollary 5.8. □

6. Coupled Waves with Different Speeds

Here we consider a constant coefficient $\gamma > 0$, $\gamma \neq 1$ and the following system

$$\begin{cases} Pu_1 + b(x)\, u_2 = 0 & \text{in } (0,T)\times\Omega, \\ P_\gamma u_2 = b_\omega(x)\, f & \text{in } (0,T)\times\Omega, \end{cases} \tag{6.1}$$

with $P = \partial_t^2 - \Delta$ as in the previous sections and $P_\gamma = \partial_t^2 - \gamma^2\Delta$.

We start with the proof of Theorem 1.4 and we then provide a proof for Theorem 1.8.

Proof of Theorem 1.4. First, we split the second equation in System (1.7) into

$$\begin{cases} P_\gamma \Pi_+ u_2 = \Pi_+ F & \text{in } \mathbb{R}\times\Omega, \\ (\Pi_+ u_2, \partial_t \Pi_+ u_2)|_{t=0} = (\Pi_+ u_2^0, \Pi_+ u_2^1) \in H^{s+1}_+(\Omega)\times H^s_+(\Omega), & \end{cases}$$

and the null-frequency part

$$\begin{cases} P_\gamma \Pi_0 u_2 = \Pi_0 F & \text{in } \mathbb{R}\times\Omega, \\ (\Pi_0 u_2, \partial_t \Pi_0 u_2)|_{t=0} = (\Pi_0 u_2^0, \Pi_0 u_2^1) \in \mathbb{C}^2. & \end{cases}$$

Since $P_\gamma \Pi_0 u_2 = \partial_t^2 \Pi_0 u_2$, we have explicitly

$$\Pi_0 u_2(t) = \Pi_0 u_2^0 + t\Pi_0 u_2^1 + \int_0^t (t-s)\Pi_0 F(s)\, \mathrm{d}s, \tag{6.2}$$

and hence $\Pi_0 u_2 \in \mathscr{C}^1(\mathbb{R};\mathbb{C})$. Similarly, we decompose u_1 as $u_1 = \Pi_0 u_1 + \Pi_+ u_1$. We have

$$\begin{cases} P\Pi_0 u_1 = -\Pi_0 b u_2 & \text{in } \mathbb{R}\times\Omega, \\ (\Pi_0 u_1, \partial_t \Pi_0 u_1)|_{t=0} = (\Pi_0 u_1^0, \Pi_0 u_1^1) \in \mathbb{C}^2. & \end{cases}$$

As $u_2 \in \mathscr{C}^1(\mathbb{R}; H^s(\Omega))$, we obtain $\partial_t^2 \Pi_0 u_1 = -\Pi_0 b u_2 \in \mathscr{C}^1(\mathbb{R}; \mathbb{C})$, and hence $\Pi_0 u_1 \in \mathscr{C}^3(\mathbb{R}; \mathbb{C})$. We again split $\Pi_+ u_1$ into $\Pi_+ u_1 = v_1 + w_1$, with

$$\begin{cases} P v_1 = -\Pi_+ b \Pi_0 u_2 & \text{in } \mathbb{R} \times \Omega, \\ (v_1, \partial_t v_1)|_{t=0} = (\Pi_+ u_1^0, \Pi_+ u_1^1) \in H_+^{s+3}(\Omega) \times H_+^{s+2}(\Omega), \end{cases}$$

and

$$\begin{cases} P w_1 = -\Pi_+ b \Pi_+ u_2 & \text{in } \mathbb{R} \times \Omega, \\ (w_1, \partial_t w_1)|_{t=0} = (0, 0). \end{cases} \tag{6.3}$$

We have, directly, $v_1 \in \mathscr{C}^0(\mathbb{R}; H_+^{s+3}(\Omega)) \cap \mathscr{C}^1(\mathbb{R}; H_+^{s+2}(\Omega))$, since $\Pi_+ b \Pi_0 u_2 \in \mathscr{C}^0(\mathbb{R}; H_+^k(\Omega))$ for all $k \in \mathbb{N}$.

We now focus on $w_1 = \Pi_+ w_1$. We use the splitting introduced in Section 5.3.2

$$h_+ = \frac{1}{2}(\Pi_+ u_2^0 - i\lambda^{-1} \Pi_+ u_2^1) \in H_+^{s+1}(\Omega),$$
$$h_- = \frac{1}{2}(\Pi_+ u_2^0 + i\lambda^{-1} \Pi_+ u_2^1) \in H_+^{s+1}(\Omega),$$

so that we have the explicit Duhamel formula for $\Pi_+ u_2$

$$\begin{aligned} \Pi_+ u_2(s) &= e^{is\gamma\lambda} h_+ + e^{-is\gamma\lambda} h_- \\ &\quad + \frac{1}{2i}(\gamma\lambda)^{-1} \int_0^s \left(e^{i(s-\sigma)\gamma\lambda} - e^{-i(s-\sigma)\gamma\lambda} \right) \Pi_+ F(\sigma) d\sigma \\ &= e^{is\gamma\lambda}(h_+ + F_+(s)) + e^{-is\gamma\lambda}(h_- + F_-(s)), \end{aligned}$$

where

$$F_+(s) = \frac{1}{2i}(\gamma\lambda)^{-1} \int_0^s e^{-i\sigma\gamma\lambda} \Pi_+ F(\sigma) d\sigma \quad \in \mathscr{C}^0(\mathbb{R}; H_+^{s+1}(\Omega)),$$
$$F_-(s) = -\frac{1}{2i}(\gamma\lambda)^{-1} \int_0^s e^{i\sigma\gamma\lambda} \Pi_+ F(\sigma) d\sigma \quad \in \mathscr{C}^0(\mathbb{R}; H_+^{s+1}(\Omega)).$$

The Duhamel formula for (6.3) gives

$$\begin{aligned} 2i\lambda w_1(t) &= -\int_0^t \left(e^{i(t-s)\lambda} - e^{-i(t-s)\lambda} \right) \Pi_+ b \Pi_+ u_2(s)\, \mathrm{d}s \\ &= -e^{it\lambda} \int_0^t e^{-is\lambda} \Pi_+ b \left(e^{is\gamma\lambda}(h_+ + F_+(s)) + e^{-is\gamma\lambda}(h_- + F_-(s)) \right) \mathrm{d}s \\ &\quad + e^{-it\lambda} \int_0^t e^{is\lambda} \Pi_+ b \left(e^{is\gamma\lambda}(h_+ + F_+(s)) + e^{-is\gamma\lambda}(h_- + F_-(s)) \right) \mathrm{d}s \\ &= -e^{it\lambda} \int_0^t e^{-is\lambda} \Pi_+ b e^{is\gamma\lambda}\, \mathrm{d}s\, h_+ - e^{it\lambda} \int_0^t e^{-is\lambda} \Pi_+ b e^{is\gamma\lambda} F_+(s)\, \mathrm{d}s \\ &\quad - e^{it\lambda} \int_0^t e^{-is\lambda} \Pi_+ b e^{-is\gamma\lambda}\, \mathrm{d}s\, h_- - e^{it\lambda} \int_0^t e^{-is\lambda} \Pi_+ b e^{-is\gamma\lambda} F_-(s)\, \mathrm{d}s \\ &\quad + e^{-it\lambda} \int_0^t e^{is\lambda} \Pi_+ b e^{is\gamma\lambda}\, \mathrm{d}s\, h_+ + e^{-it\lambda} \int_0^t e^{is\lambda} \Pi_+ b e^{is\gamma\lambda} F_+(s)\, \mathrm{d}s \end{aligned}$$

$$+e^{-it\lambda}\int_0^t e^{is\lambda}\Pi_+ b e^{-is\gamma\lambda}\,ds\,h_- + e^{-it\lambda}\int_0^t e^{is\lambda}\Pi_+ b e^{-is\gamma\lambda} F_-(s)\,ds. \tag{6.4}$$

The data $h_\pm$ are in $H_+^{s+1}(\Omega)$, therefore, using Lemma A.1 (with $\Pi_+ b \in \Psi^0_{\mathrm{phg}}(\Omega)$), all the corresponding terms above belong to $\mathscr{C}^0(\mathbb{R}; H_+^{s+2}(\Omega))$. Similarly, using Lemma A.3, all terms including $F_\pm$ above also belong to $\mathscr{C}^0(\mathbb{R}; H_+^{s+2}(\Omega))$. As a consequence, we have $w_1 \in \mathscr{C}^0(\mathbb{R}; H_+^{s+3}(\Omega))$. Differentiating with respect to time expression (6.4) and using again lemmata A.1 and A.3, we obtain $\partial_t w_1 \in \mathscr{C}^0(\mathbb{R}; H_+^{s+2}(\Omega))$.

Recalling that $u_1 = \Pi_0 u_1 + v_1 + w_1$ and using the regularity properties of each of these three terms, we obtain $u_1 \in \mathscr{C}^0(\mathbb{R}; H^{s+3}(\Omega)) \cap \mathscr{C}^1(\mathbb{R}; H^{s+2}(\Omega))$. Coming back to the equation

$$\partial_t^2 u_1 = \Delta u_1 - b\,u_2 \in \mathscr{C}^0(\mathbb{R}; H^{s+1}(\Omega)) \cap \mathscr{C}^1(\mathbb{R}; H^s(\Omega)),$$

we finally obtain $u_1 \in \bigcap_{k=0}^{3} \mathscr{C}^k(\mathbb{R}; H^{s+3-k}(\Omega))$.

Estimate (1.8) comes from the estimates of lemmata A.1 and A.3 applied to all terms in (6.4). □

The next result shows that the situation becomes worse if $\overline{\omega}\cap\overline{\mathcal{O}}$ does not satisfy GCC. In this case one cannot hope to obtain control in any Sobolev space.

Proposition 6.1. *Assume that $\overline{\omega}\cap\overline{\mathcal{O}}$ does not satisfy GCC. Then, for all $s \geqq 0$, there exists $(u_1^0, u_1^1) \in H^{s+1}(\Omega)\times H^s(\Omega)$, such that the solution to System (6.1) for all $T > 0$ and all $f \in L^2((0,T)\times\Omega)$, along with the following initial conditions,*

$$(u_1, \partial_t u_1)|_{t=0} = (u_1^0, u_1^1), \quad (u_2, \partial_t u_2)|_{t=0} = (0,0)$$

satisfies

$$(u_1(T), \partial_t u_1(T), u_2(T), \partial_t u_2(T)) \neq (0,0,0,0).$$

Remark 6.2. 1. This result proves the first item of Theorem 1.8 using the invariance of (6.1) by $t \mapsto T - t$.

2. The result stated above can be improved in many ways. The case of two different metrics, g_1 and g_2, can also be addressed with the ideas of the proof, assuming only that $\eta \mapsto (g_1 - g_2)_x(\eta,\eta)$ is a nondegenerate quadratic form everywhere on Ω. Here we restrict ourselves to the simple situation $g_2 = \gamma^2 g_1$, $\gamma \neq 1$ for the sake of exposition.

Proof of Proposition 6.1. We recall that the operators L_+, L_- are defined in Section 2.1.

As $\overline{\omega}\cap\overline{\mathcal{O}}$ does not satisfy GCC there exists a bicharacterisitic that never meets $\pi^{-1}(\overline{\omega}\cap\overline{\mathcal{O}})$ in T^*M. Without any loss of generality let us assume that $\Gamma \subset \operatorname{Char} L_+$.

We have

$$P_\gamma u_2 = b_\omega(x)\ f, \qquad (u_2, \partial_t u_2)|_{t=0} = (0,0).$$

Extending f by zero outside $[0, T]$ we have $\mathrm{supp}(u_2) \subset \{t \geqq 0\}$. For u_1^0 and u_1^1 to be defined below we introduce w and v in the following way:

$$\begin{aligned} Pv &= -b(x)\, u_2, \qquad (v, \partial_t v)|_{t=0} = (0,0),\\ Pw &= 0, \qquad (w, \partial_t w)|_{t=0} = (u_1^0, u_1^1). \end{aligned}$$

The linearity of the equations yields $u_1 = v + w$.

Study of w. Since P is elliptic at $(t, x, 1, 0)$ we have

$$(t, x, 1, 0) \notin \mathrm{WF}(w) \quad \forall\, t, x. \tag{6.5}$$

Let $\rho_0 = (0, x_0, \tau_0, \eta_0) \in \Gamma \cap \{t = 0\}$. We choose[4] $u_1^0 \in H^{s+1}(\Omega)$ such that $\{(x_0, \mathbb{R}_+^* \eta_0)\} = \mathrm{WF}(u_1^0)$. We then set $u_1^1 = i\lambda u_1^0 \in H^s(\Omega)$.

We set $z = L_+ w$ and because of the precise choice we made for (u_1^0, u_1^1) we observe that $z|_{t=0} = 0$. It follows that

$$L_- z = -Pw = 0, \qquad z|_{t=0} = 0,$$

which gives $z = 0$ (see for example Theorem 23.1.2 in [23]) and therefore

$$L_+ w = 0, \qquad w|_{t=0} = u_1^0.$$

In a local chart we consider $\chi(\tau, \eta)$ a symbol of order 0 with support in V and equal to one in V', for $V' \Subset V$ neighborhoods of $\{\eta = 0\}$. Then $(1 - \mathrm{op}(\chi))L_+$ is a pseudo-differential operator of order one in all variables [23, Theorem 18.1.35].

We have $(1 - \mathrm{op}(\chi))L_+ w = 0$ which, by (6.5) and choosing V sufficiently small, implies

$$\mathrm{WF}(w) \subset \mathrm{Char}(L_+).$$

Since $(x_0, \eta_0) \in \mathrm{WF}(w|_{t=0})$, by Theorem 8.2.4 in [22] we have that $(0, x_0, \tau, \eta_0) \in \mathrm{WF}(w)$ for some $\tau \in \mathbb{R}$ and necessarily $\tau = \tau_0$, that is, $\rho_0 \in \mathrm{WF}(w)$. Since $Pw = 0$, the singularity propagation theorem of Hörmander [24, Theorem 26.1.1] implies that

$$\Gamma \subset \mathrm{WF}(w). \tag{6.6}$$

[4] To choose u_1^0 we can invoke the constructive approach of Theorem 8.1.4 in [24] that yields at first a distribution with $\{(x_0, \mathbb{R}_+^* \eta_0)\}$ as its wavefront set. Since Ω is compact, this distribution has finite order and belongs to $H^\sigma(\Omega)$ for some $\sigma \in \mathbb{R}$, by the Paley-Wiener theorem. Finally, we can apply an appropriate power of the Laplace operator, an elliptic pseudo-differential operator on Ω (according to [41] or [42, Theorem 11.2]) that preserves the wavefront set, to yield the proper function in the Sobolev space $H^{s+1}(\Omega)$.

Study of v. As $\operatorname{supp}(bu_2) \subset \{t \geqq 0\}$ we have

$$v = 0 \text{ in } \{t \leqq 0\}. \tag{6.7}$$

From $P_\gamma u_2 = b_\omega(x)\, f$, we have $\mathrm{WF}(u_2) \subset \mathrm{Char}(P_\gamma) \cup \mathrm{WF}(b_\omega(x)\, f)$ by Theorem 18.1.28 in [23]. As Γ does not meet $\pi^{-1}(\overline{\omega} \cap \overline{\mathcal{O}})$ and $\Gamma \cap \mathrm{Char}(P_\gamma) = \emptyset$ we find

$$\Gamma \cap \mathrm{WF}(bu_2) \subset \Gamma \cap \pi^{-1}(\overline{\mathcal{O}}) \cap \pi^{-1}(\overline{\omega}) = \emptyset. \tag{6.8}$$

The singularity propagation theorem of Hörmander [24, Theorem 26.1.1] implies that $\mathrm{WF}(v) \setminus \mathrm{WF}(bu_2)$ is invariant by the hamiltonian vector field H_p. Since Γ is invariant by H_p and with (6.8) and (6.7) we obtain that

$$\Gamma \cap \mathrm{WF}(v) = \emptyset. \tag{6.9}$$

Conclusion. From (6.6) and (6.9) we find that $\Gamma \subset \mathrm{WF}(u_1)$. It follows that $(u_1, \partial_t u_1, u_2, \partial_t u_2)$ cannot vanish at the final control time T. In fact it would first imply $u_2 = 0$ in $\{t \geqq T\}$ where it satisfies $P_\gamma u_2 = 0$. Second, we would have $P u_1 = 0$ in $\{t \geqq T\}$ implying $u_1 = 0$ in $\{t \geqq T\}$, which obviously does not hold. □

To conclude the study of System (6.1), we prove an "almost converse" of Theorem 1.4 and Proposition 6.1. We prove the controllability of System (6.1) in the space $(H^3(\Omega) \times H^2(\Omega)) \times (H^1(\Omega) \times L^2(\Omega))$, if $\omega \cap \mathcal{O}$ satisfies GCC.

Proposition 6.3. *Assume that $\omega \cap \mathcal{O}$ satisfies GCC and that $T > \max\{T_{\omega\cap\mathcal{O}}(1), T_\omega(\gamma)\}$. Then, System (6.1) is controllable in the space $H^3(\Omega) \times H^2(\Omega)$ in time T in the sense given in Section 1.2.3.*

Remark that $T_\omega(\gamma)$ is the time needed to control the second component of System (6.1), which is directly controlled, whereas $T_{\omega\cap\mathcal{O}}(1)$ is the time needed to control the first component of System (6.1).

Note also that for $T < T_\omega(\gamma)$, the second equation is not controllable. Similarly, if $T < T_{\overline{\omega}\cap\overline{\mathcal{O}}}(1)$ the first equation is not controllable: one can find a bicharacteristic curve Γ that does not meet $\pi^{-1}(\overline{\omega} \cap \overline{\mathcal{O}})$ in the time interval $[0, T]$ and the same construction as in the proof of Proposition 6.1 applies. We thus obtain the second item of Theorem 1.8.

Proof. According to Proposition 5.1 (case $s = 2, \sigma = 0$) and Theorem 1.4, the result is equivalent to proving the observability inequality

$$E_{-2}(v_1(0)) + E_0(v_2(0)) \leqq C \int_0^T \|b_\omega v_2\|^2_{L^2(\Omega)}\, \mathrm{d}t, \tag{6.10}$$

for all $(v_1, v_2) \in \big(\mathscr{C}^0(\mathbb{R}; H^{-2}(\Omega)) \cap \mathscr{C}^1(\mathbb{R}; H^{-3}(\Omega))\big) \times \big(\mathscr{C}^0(\mathbb{R}; L^2(\Omega)) \cap \mathscr{C}^1(\mathbb{R}; H^{-1}(\Omega))\big)$ solutions of (5.3). Setting $w_1 = (1 - \Delta)^{-1} v_1$, $w_2 = v_2$, this observability inequality is equivalent to proving

$$E_0(w_1(0)) + E_0(w_2(0)) \leqq C \int_0^T \|b_\omega w_2\|^2_{L^2(\Omega)}\, \mathrm{d}t, \tag{6.11}$$

for all $(w_1, w_2) \in \mathscr{C}^0(\mathbb{R}; L^2(\Omega; \mathbb{C}^2)) \cap \mathscr{C}^1(\mathbb{R}; H^{-1}(\Omega; \mathbb{C}^2))$ solutions to

$$\begin{cases} P w_1 = 0 & \text{in } (0, T) \times \Omega, \\ P_\gamma w_2 = -b(x)(1 - \Delta) w_1 & \text{in } (0, T) \times \Omega. \end{cases} \tag{6.12}$$

Recall that System (6.12) is well-posed in this space, according to Theorem 1.4.

As in Section 3, we prove (6.11) in two steps with a compactness–uniqueness strategy. The first step is to prove the relaxed observability inequality

$$E_0(w_1(0)) + E_0(w_2(0)) \leqq C \left(\int_0^T \int_\Omega |b_\omega w_2|^2 \, dx \, dt + E_{-1}(w_1(0)) + E_{-1}(w_2(0)) \right), \tag{6.13}$$

for all $(w_1, w_2) \in \mathscr{C}^0(\mathbb{R}; L^2(\Omega; \mathbb{C}^2)) \cap \mathscr{C}^1(\mathbb{R}; H^{-1}(\Omega; \mathbb{C}^2))$ solutions to (6.12).

We proceed by contradiction and suppose that the observability inequality (6.13) is not satisfied. Thus, there exists a sequence $(w_1^k, w_2^k)_{k \in \mathbb{N}}$ of $\mathscr{C}^0(0, T; L^2(\Omega)) \cap \mathscr{C}^1(0, T; H^{-1}(\Omega))$-solutions of (6.12) such that

$$E_0(w_1^k(0)) + E_0(w_2^k(0)) = 1, \tag{6.14}$$

$$\int_0^T \int_\Omega |b_\omega w_2^k|^2 \, dx \, dt \to 0, \quad k \to \infty, \tag{6.15}$$

$$E_{-1}(w_1^k(0)) + E_{-1}(w_2^k(0)) \to 0, \quad k \to \infty. \tag{6.16}$$

According to (6.14) and to (6.16), together with the continuity of the solution with respect to the initial data given by Theorem 1.4, the sequence (w_1^k, w_2^k) is bounded in $L^2(M_T; \mathbb{C}^2)$, and converges to zero in $H^{-1}(M_T; \mathbb{C}^2)$. It follows that

$$(w_1^k, w_2^k) \rightharpoonup (0, 0) \quad \text{in } L^2(M_T; \mathbb{C}^2).$$

As a consequence of [21, Theorem 1], there exists a subsequence of $(w_1^k, w_2^k)_{k \in \mathbb{N}}$ (still denoted $(w_1^k, w_2^k)_{k \in \mathbb{N}}$ in what follows) and two associated microlocal defect measures

$$\mu_1 \in \mathcal{M}_+(S^* M_T), \qquad \mu_2 \in \mathcal{M}_+(S^* M_T),$$

such that for any $A \in \Psi^0_{\mathrm{phg}}(M_T)$,

$$\lim_{k \to \infty} (A w_1^k, w_1^k)_{L^2(M_T; \mathbb{C})} = \langle \mu_1, \sigma_0(A) \rangle_{S^* M_T},$$
$$\lim_{k \to \infty} (A w_2^k, w_2^k)_{L^2(M_T; \mathbb{C})} = \langle \mu_2, \sigma_0(A) \rangle_{S^* M_T}.$$

Note that in this case (as opposed to the case $\gamma = 1$ treated in Section 3) we shall not use the coupling of the two waves and the associated measure μ_{12}.

The first equation of (6.12) yields, as in Lemma 3.3 that

$$\mathrm{supp}(\mu_1) \subset \mathrm{Char}(P), \quad \text{and} \quad \langle \mu_1, H_p a \rangle_{S^* M_T} = 0, \tag{6.17}$$

for any $a \in S^{-1}_{\mathrm{phg}}(T^*M_T;\mathbb{C})$. Moreover, (6.15) gives

$$\operatorname{supp}(\mu_2) \cap \pi^{-1}((0,T)\times\omega) = \emptyset.$$

This gives, for any $\chi \in \mathscr{C}^\infty_c\big((0,T)\times(\mathcal{O}\cap\omega)\big)$,

$$((1-\Delta)^{-2}\chi P_\gamma w_2^k, \chi P_\gamma w_2^k)_{L^2(M_T;\mathbb{C})} \to \left\langle \mu_2, p_\gamma^2 |\eta|_x^{-4}\chi^2\right\rangle_{S^*M_T} = 0.$$

Using the second equation of (6.12), we now have

$$\begin{aligned} 0 &= \lim_{k\to+\infty} ((1-\Delta)^{-2}\chi b(1-\Delta)w_1^k, \chi b(1-\Delta)w_1^k)_{L^2(M_T;\mathbb{C})} \\ &= \left\langle \mu_1, \chi^2 b^2\right\rangle_{S^*M_T}. \end{aligned}$$

As a consequence, we have

$$\operatorname{supp}(\mu_1) \cap \pi^{-1}((0,T)\times\mathcal{O}\cap\omega) = \emptyset.$$

This, together with the free propagation of μ_1 given in (6.17), and the assumption that $\mathcal{O}\cap\omega$ satisfies GCC, implies that μ_1 vanishes identically on $(0,T)\times\Omega$, as soon as $T > T_{\omega\cap\mathcal{O}}(1)$. Hence, we have $w_1^k \to 0$ in $L^2_{\mathrm{loc}}(M_T)$.

It remains to study w_2^k. We have $P_\gamma w_2^k = -b(x)(1-\Delta)w_1^k \to 0$ in $H^{-2}_{\mathrm{loc}}(M_T)$ since $w_1^k \to 0$ in $L^2_{\mathrm{loc}}(M_T)$. This yields $\operatorname{supp}(\mu_2) \subset \operatorname{Char}(P_\gamma)$.

Let $\varepsilon > 0$ and choose $\chi_1 \in S^0_{\mathrm{phg}}(M_T)$, $0 \leqq \chi_1$, such that

- $\chi_1 = 1$ in a neighborhood of $\operatorname{Char}(P_\gamma)\cap(\varepsilon, T-\varepsilon)$,
- $\chi_1 = 0$ in a neighborhood of $\operatorname{Char}(P)$,

and let $\Xi_1 \in \Psi^0_{\mathrm{phg}}(M_T)$ an operator such that $\sigma_0(\Xi_1) = \chi_1$. There exists a parametrix $Q \in \Psi^{-2}_{\mathrm{phg}}(M_T)$ such that (see for instance [23, Theorem 18.1.24]),

$$QP = \Xi_1 + R, \qquad R \in \Psi^{-\infty}_{\mathrm{phg}}(M_T). \tag{6.18}$$

Applying this parametrix to the first equation of (6.12) gives

$$\Xi_1 w_1^k + R w_1^k = 0.$$

In particular, we have $\Xi_1 w_1^k \to 0$ in $H^s_{\mathrm{loc}}(M_T)$ for all $s \in \mathbb{R}$.

Consider now χ_2 satisfying the same properties as χ_1 with, moreover, $\chi_1 = 1$ on a neighborhood of $\operatorname{supp}(\chi_2)$. Let $\Xi_2 \in \Psi^0_{\mathrm{phg}}(M_T)$ an operator such that $\sigma_0(\Xi_2) = \chi_2$.

Writing $w_2^k = (1-\Xi_2)w_2^k + \Xi_2 w_2^k$, we directly have $(1-\Xi_2)w_2^k \to 0$ in $L^2_{\mathrm{loc}}((\varepsilon, T-\varepsilon)\times M)$ as $\operatorname{supp}(\mu_2) \subset \operatorname{Char}(P_\gamma)$. Below, we shall prove the following convergence.

Lemma 6.4. $P_\gamma \Xi_2 w_2^k \to 0$ *in* $H^{-1}_{\mathrm{loc}}(M_T)$.

This implies that $\langle \mu_2, H_{p_\gamma}(\chi a)\rangle_{S^*M_T} = \langle \mu_2, |\chi_2|^2 H_{p_\gamma}(\chi a)\rangle_{S^*M_T} = 0$ for $\chi(t) \in \mathscr{C}_c^\infty(\varepsilon, T-\varepsilon)$ and for any $a \in S^{-1}_{\mathrm{phg}}(T^*M_T)$. Hence, the measure μ_2 satisfies a free propagation relation along the bicharacteristic flow of P_γ for $t \in (\varepsilon, T-\varepsilon)$. Since $\varepsilon > 0$ is arbitrary and since $\mathrm{supp}(\mu_2) \cap \pi^{-1}((0,T)\times\omega) = \emptyset$, the measure μ_2 vanishes identically on S^*M_T as soon as $T > T_\omega(\gamma)$. This proves that $(w_1^k, w_2^k) \to (0,0)$ in $L^2_{\mathrm{loc}}(M_T;\mathbb{C}^2)$, and, following the end of the proof of Proposition 3.2, that $E_0(w_1^k(0)) + E_0(w_2^k(0)) \to 0$. This yields a contradiction with (6.14), and concludes the proof of the relaxed observability inequality (6.13).

We conclude the proof of (6.11) as in Section 3.2. We define the set of invisible solutions

$$\mathcal{N}(T) = \{\mathscr{W} = (w_1^0, w_2^0, w_1^1, w_2^1) \in \mathscr{H} \text{ such that the associated solution of (6.12)} \\ \text{satisfies } w_2(t,x) = 0 \text{ for all } (t,x) \in (0,T)\times\omega\}.$$

As in Section 3.2, proving that $\mathcal{N}(T) = \{0\}$ implies that (6.11) is true. Again, proving that $\mathcal{N}(T) = \{0\}$ is equivalent to proving that there is no eigenfunction (φ_1, φ_2) of the operator

$$\begin{pmatrix} -\Delta & 0 \\ b(x)(1-\Delta) & -\gamma^2\Delta \end{pmatrix}$$

such that $\varphi_2|_\omega = 0$. Indeed, letting μ be the associated eigenvalue, we have

$$-\gamma^2\Delta\varphi_2 + b(x)(1-\Delta)\varphi_1 = \mu\varphi_2, \quad \text{and} \quad \varphi_2|_{\mathcal{O}\cap\omega} = 0. \tag{6.19}$$

This yields in particular $(1-\Delta)\varphi_1|_{\mathcal{O}\cap\omega} = 0$. As $-\Delta\varphi_1 = \mu\varphi_1$, the function $(1-\Delta)\varphi_1$ is an eigenfunction of $-\Delta$, vanishing on the nonempty open set $\mathcal{O}\cap\omega$, and thus vanishes identically (see for instance [6,7,25]). Hence, we have $\varphi_1 = 0$. Coming back to (6.19), and using the same argument, we obtain $\varphi_2 = 0$. Finally, this proves that $\mathcal{N}(T) = \{0\}$.

Note that a quantitative version of this unique continuation result is proved in [26, Proposition 5.1]. This concludes the proof of the observability inequality (6.11), and hence that of Theorem 6.3. □

Proof of Lemma 6.4. We have

$$P_\gamma \Xi_2 w_2^k = \Xi_2 P_\gamma w_2^k + [P_\gamma, \Xi_2]w_2^k = -\Xi_2 b(x)(1-\Delta)w_1^k + [P_\gamma, \Xi_2]w_2^k. \tag{6.20}$$

We write

$$\Xi_2 b(x)(1-\Delta)w_1^k = \Xi_2 b(x)(1-\Delta)\Xi_1 w_1^k + \Xi_2 b(x)(1-\Delta)(1-\Xi_1)w_1^k.$$

Since the supports of χ_2 and $1-\chi_1$ are disjoint the operator $\Xi_2 b(x)(1-\Delta)(1-\Xi_1)$ is regularizing and since $\Xi_1 w_1^k \to 0$ in $H^s_{\mathrm{loc}}(M_T)$ for all $s \in \mathbb{R}$ then the same holds for $\Xi_2 b(x)(1-\Delta)w_1^k$. For the second term in (6.20) we write

$$[P_\gamma, \Xi_2]w_2^k = [P_\gamma, \Xi_2]\Xi_3 w_2^k + [P_\gamma, \Xi_2](1-\Xi_3)w_2^k,$$

where $\Xi_3 \in \Psi^0_{\mathrm{phg}}(M_T)$ is an operator such that $\sigma_0(\Xi_3) = \chi_3$ with χ_3 satisfying the same properties as χ_2 with, moreover, $\chi_2 = 1$ on a neighborhood of $\mathrm{supp}(\chi_3)$. We then have $[P_\gamma, \Xi_2]\Xi_3$ regularizing. Since $(1-\Xi_3)w_2^k \to 0$ in $L^2_{\mathrm{loc}}((\varepsilon, T-\varepsilon)\times M)$ as $\mathrm{supp}(\mu_2) \subset \mathrm{Char}(P_\gamma)$ and $[P_\gamma, \Xi_2]$ is of order one, we find that $[P_\gamma, \Xi_2]w_2^k \to 0$ in $H^{-1}_{\mathrm{loc}}((\varepsilon, T-\varepsilon)\times M)$. This concludes the proof. □

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Appendix A. 1-Smoothing Properties

In this section, we state and prove three lemmata concerning the 1-smoothing properties of some families of operators. These properties are used in a crucial way in the proofs of Theorems 5.3, 5.5 and 1.4.

The proofs given in this section are elementary and are inspired by [18]. These properties, however, deeply rely on the Fourier integral operator property of propagators of type $e^{it\lambda}$.

Lemma A.1. *Let $s \in \mathbb{R}$, $\gamma, \delta \in \mathbb{R}$ such that $\gamma \neq \delta$, and $b \in \Psi^0_{\mathrm{phg}}(\Omega)$. Then, the operator defined by*

$$A(t) = \int_0^t e^{-iz\gamma\lambda} b e^{iz\delta\lambda}\, dz,$$

satisfies $A \in \mathscr{C}^0(\mathbb{R}; \mathcal{L}(H^s(\Omega), H^{s+1}(\Omega)))$. In particular, for all $t \in \mathbb{R}$, $A(t)$ is 1-smoothing.

Lemma A.1 can be seen as a corollary of Lemma A.2, and hence we omit its proof. We chose to state Lemma A.1 separately since in the main part of the article, regularizing properties are most often used under this simpler form.

Lemma A.2. *Let $s \in \mathbb{R}$, $\gamma, \delta \in \mathbb{R}$ such that $\gamma \neq \delta$. Let also $b \in \Psi^0_{\mathrm{phg}}(\Omega)$, and consider $m, \tilde{m} \in \mathscr{C}^0(\mathbb{R}, \Psi^0_{\mathrm{phg}}(\Omega))$ two continuous families of operators. Then, the operator defined by*

$$A(t) = \int_0^t \left(\int_0^z m(\sigma)d\sigma\right) e^{-iz\gamma\lambda} b e^{iz\delta\lambda} \left(\int_0^z \tilde{m}(\sigma)d\sigma\right) dz,$$

satisfies $A \in \mathscr{C}^0(\mathbb{R}; \mathcal{L}(H^s(\Omega), H^{s+1}(\Omega)))$. In particular, for all $t \in \mathbb{R}$, $A(t)$ is 1-smoothing.

The next lemma is only used in the proof of Theorem 1.4 and is proved at the end of this section.

Lemma A.3. *Let $s \in \mathbb{R}$, $T > 0$, $\gamma, \delta \in \mathbb{R}$ such that $\gamma \neq \delta$. Let also $b \in \Psi^0_{\mathrm{phg}}(\Omega)$, and suppose that $F \in L^1(0, T; H^s(\Omega))$. Then, the function defined by*

$$\mathscr{F}(t) = \int_0^t e^{-iz\gamma\lambda} b e^{iz\delta\lambda} \left(\int_0^z F(\sigma) d\sigma \right) dz,$$

satisfies $\mathscr{F} \in \mathscr{C}^0(0, T; H^{s+1}(\Omega))$, and

$$\|\mathscr{F}\|_{L^\infty(0,T;H^{s+1})} \leqq C \|F\|_{L^1(0,T;H^s)}, \tag{A.1}$$

for some $C = C(s, T, \gamma, \delta, b) > 0$.

Proof of Lemma A.2. We first notice that $A(t) \in \mathscr{C}^0(\mathbb{R}; \mathcal{L}(H^s(\Omega)))$ since all operators in $A(t)$ preserve the regularity. It suffices to prove that $\lambda A(t) \in \mathscr{C}^0(\mathbb{R}; \mathcal{L}(H^s(\Omega)))$. For this, we compute

$$\begin{aligned} -i\gamma\lambda A(t) &= -i\gamma\lambda \int_0^t \left(\int_0^z m(\sigma) d\sigma \right) e^{-iz\gamma\lambda} b e^{iz\delta\lambda} \left(\int_0^z \tilde{m}(\sigma) d\sigma \right) dz \\ &= \int_0^t \left(\int_0^z m(\sigma) d\sigma \right) (-i\gamma\lambda) e^{-iz\gamma\lambda} b e^{iz\delta\lambda} \left(\int_0^z \tilde{m}(\sigma) d\sigma \right) dz \\ &\quad + \int_0^t \left[(-i\gamma\lambda), \left(\int_0^z m(\sigma) d\sigma \right) \right] e^{-iz\gamma\lambda} b e^{iz\delta\lambda} \left(\int_0^z \tilde{m}(\sigma) d\sigma \right) dz, \end{aligned}$$

and we notice that the last term

$$\int_0^t \left[(-i\gamma\lambda), \left(\int_0^z m(\sigma) d\sigma \right) \right] e^{-iz\gamma\lambda} b e^{iz\delta\lambda} \left(\int_0^z \tilde{m}(\sigma) \right.$$
$$\left. d\sigma \right) dz \in \mathscr{C}^0(\mathbb{R}; \mathcal{L}(H^s(\Omega), H^s(\Omega))),$$

since $\left[(-i\gamma\lambda), \left(\int_0^z m(\sigma) d\sigma \right) \right] = -i\gamma \int_0^z \left[\lambda, m(\sigma) \right] d\sigma \in \mathscr{C}^0\left(\mathbb{R}, \Psi^0_{\mathrm{phg}}(\Omega)\right)$. As a consequence, we can write, for some $R_t \in \mathscr{C}^0(\mathbb{R}; \mathcal{L}(H^s(\Omega)))$,

$$-i\gamma\lambda A(t) = \int_0^t \left(\int_0^z m(\sigma) d\sigma \right) \partial_z (e^{-iz\gamma\lambda}) b e^{iz\delta\lambda} \left(\int_0^z \tilde{m}(\sigma) d\sigma \right) dz + R_t.$$

After an integration by parts, this gives

$$\begin{aligned} -i\gamma\lambda A(t) = &- \int_0^t \left(\int_0^z m(\sigma) d\sigma \right) e^{-iz\gamma\lambda} b \, \partial_z (e^{iz\delta\lambda}) \left(\int_0^z \tilde{m}(\sigma) d\sigma \right) dz \\ &- \int_0^t m(z) e^{-iz\gamma\lambda} b e^{iz\delta\lambda} \left(\int_0^z \tilde{m}(\sigma) d\sigma \right) dz \\ &- \int_0^t \left(\int_0^z m(\sigma) d\sigma \right) e^{-iz\gamma\lambda} b e^{iz\delta\lambda} \tilde{m}(z) \, dz \\ &+ \left[\left(\int_0^z m(\sigma) d\sigma \right) e^{-iz\gamma\lambda} b e^{iz\delta\lambda} \left(\int_0^z \tilde{m}(\sigma) d\sigma \right) \right]_0^t + R_t. \end{aligned}$$

Except for the first term in the right hand-side of this expression, all terms are clearly in $\mathscr{C}^0(\mathbb{R}; \mathcal{L}(H^s(\Omega)))$, so that we can write, for some $\tilde{R}_t \in \mathscr{C}^0(\mathbb{R}; \mathcal{L}(H^s(\Omega)))$,

$$\begin{aligned}
&-i\gamma\lambda A(t) \\
&= -\int_0^t \left(\int_0^z m(\sigma)d\sigma\right) e^{-iz\gamma\lambda} b\, \partial_z(e^{iz\delta\lambda}) \left(\int_0^z \tilde{m}(\sigma)d\sigma\right) dz + \tilde{R}_t \\
&= -\int_0^t \left(\int_0^z m(\sigma)d\sigma\right) e^{-iz\gamma\lambda} b(i\delta\lambda) e^{iz\delta\lambda} \left(\int_0^z \tilde{m}(\sigma)d\sigma\right) dz + \tilde{R}_t \\
&= -\int_0^t \left(\int_0^z m(\sigma)d\sigma\right) (i\delta\lambda) e^{-iz\gamma\lambda} b e^{iz\delta\lambda} \left(\int_0^z \tilde{m}(\sigma)d\sigma\right) dz \\
&\quad - \int_0^t \left(\int_0^z m(\sigma)d\sigma\right) e^{-iz\gamma\lambda}[b, (i\delta\lambda)] e^{iz\delta\lambda} \left(\int_0^z \tilde{m}(\sigma)d\sigma\right) dz + \tilde{R}_t.
\end{aligned}$$

Again, we notice that the central term in the right hand-side of this expression is in $\mathscr{C}^0(\mathbb{R}; \mathcal{L}(H^s(\Omega)))$, since $[b, (i\delta\lambda)] \in \Psi^0_{\mathrm{phg}}(\Omega)$. As a consequence, we can write, for some $\tilde{\tilde{R}}_t \in \mathscr{C}^0(\mathbb{R}; \mathcal{L}(H^s(\Omega)))$,

$$\begin{aligned}
&-i\gamma\lambda A(t) \\
&= -\int_0^t \left(\int_0^z m(\sigma)d\sigma\right) (i\delta\lambda) e^{-iz\gamma\lambda} b e^{iz\delta\lambda} \left(\int_0^z \tilde{m}(\sigma)d\sigma\right) dz + \tilde{\tilde{R}}_t \\
&= -i\delta\lambda A(t) - \int_0^t \left[\left(\int_0^z m(\sigma)d\sigma\right), (i\delta\lambda)\right] e^{-iz\gamma\lambda} b e^{iz\delta\lambda} \left(\int_0^z \tilde{m}(\sigma)d\sigma\right) dz + \tilde{\tilde{R}}_t.
\end{aligned}$$

This gives

$$\begin{aligned}
&i(\delta - \gamma)\lambda A(t) \\
&\quad = -\int_0^t \left[\left(\int_0^z m(\sigma)d\sigma\right), (i\delta\lambda)\right] e^{-iz\gamma\lambda} b e^{iz\delta\lambda} \left(\int_0^z \tilde{m}(\sigma)d\sigma\right) dz + \tilde{\tilde{R}}_t,
\end{aligned}$$

and in particular $(\delta - \gamma)\lambda A(t) \in \mathscr{C}^0(\mathbb{R}; \mathcal{L}(H^s(\Omega)))$. This concludes the proof of Lemma A.2. □

Proof of Lemma A.3. The definition of $\mathscr{F}$ directly gives $\mathscr{F} \in \mathscr{C}^0(0, T; H^s(\Omega))$ (with the associated estimate). We hence only have to check that $\lambda\mathscr{F} \in \mathscr{C}^0(0, T; H^s(\Omega))$. For this, we compute

$$\begin{aligned}
-i\gamma\lambda\mathscr{F}(t) &= \int_0^t (-i\gamma\lambda) e^{-iz\gamma\lambda} b e^{iz\delta\lambda} \left(\int_0^z F(\sigma)d\sigma\right) dz \\
&= \left[e^{-iz\gamma\lambda} b e^{iz\delta\lambda} \left(\int_0^z F(\sigma)d\sigma\right)\right]_0^t \\
&\quad - \int_0^t e^{-iz\gamma\lambda} b \partial_z \left[e^{iz\delta\lambda} \left(\int_0^z F(\sigma)d\sigma\right)\right] dz \qquad \text{(A.2)}
\end{aligned}$$

after an integration by parts. Furthermore,

$$\begin{aligned}
&-\int_0^t e^{-iz\gamma\lambda} b\partial_z \left[e^{iz\delta\lambda} \left(\int_0^z F(\sigma)d\sigma \right) \right] dz \\
&= -i\delta \int_0^t e^{-iz\gamma\lambda} b\lambda e^{iz\delta\lambda} \left(\int_0^z F(\sigma)d\sigma \right) dz \\
&\quad - \int_0^t e^{-iz\gamma\lambda} b e^{iz\delta\lambda} F(z)\, dz \\
&= -i\delta\lambda \mathscr{F}(t) \\
&\quad -i\delta \int_0^t e^{-iz\gamma\lambda} [b,\lambda] e^{iz\delta\lambda} \left(\int_0^z F(\sigma)d\sigma \right) dz \\
&\quad - \int_0^t e^{-iz\gamma\lambda} b e^{iz\delta\lambda} F(z)\, dz. \qquad \text{(A.3)}
\end{aligned}$$

Putting (A.2) in (A.3), we obtain

$$\begin{aligned}
&-i(\gamma-\delta)\lambda\mathscr{F}(t) = e^{-it\gamma\lambda} b e^{it\delta\lambda} \left(\int_0^t F(\sigma)d\sigma \right) \\
&-i\delta \int_0^t e^{-iz\gamma\lambda} [b,\lambda] e^{iz\delta\lambda} \left(\int_0^z F(\sigma)d\sigma \right) dz - \int_0^t e^{-iz\gamma\lambda} b e^{iz\delta\lambda} F(z)\, dz. \qquad \text{(A.4)}
\end{aligned}$$

The first and the last terms in (A.4) are in $\in \mathscr{C}^0(0,T; H^s(\Omega))$ as $z \to e^{iz\delta\lambda} \in \mathscr{C}^0(\mathbb{R}; \mathcal{L}(H^s(\Omega)))$. Finally the second term in (A.4) is also in $\mathscr{C}^0(0,T; H^s(\Omega))$ since $[b,\lambda] \in \Psi^0_{\text{phg}}(\Omega)$. Estimate (A.1) comes from the $L^\infty(0,T; H^s(\Omega))$ estimate of (A.4). □

Appendix B. Proofs of Some Technical Results

B.1. Some Facts on Hamilton Flows

For a function $q(x,\xi)$ defined on T^*X, with X a smooth manifold, the Hamilton vector field associated with q is given by, in local coordinates, $H_q = (\nabla_x q, -\nabla_\xi q)$. The Hamiltonian flow χ_s is then given by

$$\frac{\mathrm{d}}{\mathrm{d}s}\chi_s(x,\xi) = H_q\left(\chi_s(x,\xi)\right), \quad \chi_0(x,\xi) = (x,\xi),$$

for $(x,\xi) \in T^*X \setminus 0$. We have the following lemmata.

Lemma B.1. *Let $F : \mathbb{R} \to \mathbb{R}$ be a smooth monotonous function. For $(x,\xi) \in T^*X$ the map $s \mapsto \chi_{F'(q(x,\xi))s}(x,\xi)$ is the flow associated with the Hamiltonian $F(q)$. In particular, for $\mu \neq 0$, the map $s \mapsto \chi_{\mu s}$ is the flow associated with the Hamiltonian μq.*

Proof. We have $H_{F(q)} = F'(q)H_q$. Let $(x,\xi) \in T^*X$. As q is constant along the flow $s \mapsto \chi_s$ we have also $q(\chi_s(x,\xi)) = q(x,\xi)$, $s \in \mathbb{R}$. We set $\mu = F'(q(x,\xi))$. Hence,

$$\frac{\mathrm{d}}{\mathrm{d}s}\chi_{\mu s}(x,\xi) = \mu \frac{\mathrm{d}}{\mathrm{d}t}\chi_t(x,\xi)|_{t=\mu s} = \mu H_q\left(\chi_{\mu s}(x,\xi)\right) = H_{F(q)}\left(\chi_{\mu s}(x,\xi)\right),$$

which yields the result. □

Lemma B.2. *Let $\lambda \neq 0$ and M_λ be the map such that $M_\lambda(x,\xi) = (x,\lambda\xi)$, that is a multiplication by λ in the fiber. Assume that $q(x,\xi)$ is homogeneous of degree k, that is, $q \circ M_\lambda = \lambda^k q$ for $\lambda > 0$. Then*

$$M_\lambda \circ \chi_{\lambda^{k-1}s} = \chi_s \circ M_\lambda.$$

In particular, if $k = 1$, the operators M_λ and χ_s commute.

Proof. Let $(x_0,\xi_0) \in T^*X \setminus 0$. We set $(x(s),\xi(s)) = \chi_s(x_0,\xi_0)$ and

$$\gamma(s) = M_\lambda \circ \chi_{\lambda^{k-1}s}(x_0,\xi_0) = (x,\lambda\xi)(\lambda^{k-1}s).$$

We have, in local coordinates,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}s}\gamma(s) &= \left(\lambda^{k-1}\frac{\mathrm{d}}{\mathrm{d}s}x, \lambda^k\frac{\mathrm{d}}{\mathrm{d}s}\xi\right)(\lambda^{k-1}s)\\ &= \left(\lambda^{k-1}\nabla_\xi q(x,\xi), -\lambda^k\nabla_x q(x,\xi)\right)(\lambda^{k-1}s)\\ &= H_q(x,\lambda\xi)(\lambda^{k-1}s)\\ &= H_q(\gamma(s)),\end{aligned}$$

by homogeneity. As $\gamma(0) = M_\lambda(x_0,\xi_0)$ we obtain $\gamma(s) = \chi_s \circ M_\lambda(x_0,\xi_0)$. □

Lemma B.3. *Let σ be the map such that $\sigma(x,\xi) = (x,-\xi)$. Assume that q is such that $q \circ \sigma = q$, then*

$$\sigma \circ \chi_s = \chi_{-s} \circ \sigma.$$

Proof. Observe first that $\nabla_x q \circ \sigma = \nabla_x q$ and $\nabla_\xi q \circ \sigma = -\nabla_\xi q$. Let $(x_0,\xi_0) \in \mathcal{T}^*X \setminus 0$. We set

$$\gamma(s) = \sigma\left(\chi_{-s}(x_0,\xi_0)\right).$$

We have, in local coordinates,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}s}\gamma(s) &= -\sigma\frac{\mathrm{d}}{\mathrm{d}t}\chi_t(x_0,\xi_0)|_{t=-s} = -\sigma\left((\nabla_\xi q, -\nabla_x q)(\chi_{-s}(x_0,\xi_0))\right)\\ &= (-\nabla_\xi q, -\nabla_x q)(\chi_{-s}(x_0,\xi_0)) = (\nabla_\xi q, -\nabla_x q)(\gamma(s)) = H_q(\gamma(s)).\end{aligned}$$

As $\gamma(0) = (x_0,-\xi_0)$ we obtain $\gamma(s) = \chi_s \circ \sigma(x_0,\xi_0)$. □

B.2. Proof of Lemma 3.3

The fact that the measures μ_1 and μ_2 are real and non-negative is a direct consequence of the first two equations of (3.7), tested on selfadjoint non-negative operators. Note also that the measures μ_1 and μ_2 are microlocal defect measures associated with the scalar sequences $(w_1^k)_{k\in\mathbb{N}}$ and $(w_2^k)_{k\in\mathbb{N}}$ respectively.
Let us prove that $\operatorname{supp}(\mu_2) \subset \operatorname{Char}(P)$. Proving that $\operatorname{supp}(\mu_1) \subset \operatorname{Char}(P)$ follows the same steps. Recalling that the symbol $\tilde{\lambda} \in S^1_{\text{phg}}(T^*M;\mathbb{C})$ is defined in (2.2), there exists $\tilde{\Lambda}_{-2} \in \Psi^{-2}_{\text{phg}}(M;\mathbb{C})$ that satisfies $\sigma_{-2}(\tilde{\Lambda}_{-2}) = \tilde{\lambda}^{-2}$. Take $a \in S^0_{\text{phg}}(T^*M_T;\mathbb{C})$ and $A \in \Psi^0_{\text{phg}}(M_T;\mathbb{C})$ as an associated operator. The second equation of (3.2) gives

$$(A\tilde{\Lambda}_{-2}Pw_2^k, w_2^k)_{L^2(M_T;\mathbb{C})} = -(A\tilde{\Lambda}_{-2}Bw_1^k, w_2^k)_{L^2(M_T;\mathbb{C})}.$$

Since $A\tilde{\Lambda}_{-2}B \in \Psi^{-1}_{\text{phg}}(M_T;\mathbb{C})$, we have $A\tilde{\Lambda}_{-2}Bw_1^k \to 0$ in $L^2(M_T;\mathbb{C})$ so that we obtain

$$(A\tilde{\Lambda}_{-2}Pw_2^k, w_2^k)_{L^2(M_T;\mathbb{C})} \to 0.$$

The second equation of (3.7) then gives

$$0 = \left\langle \mu_2, a\tilde{\lambda}^{-2}p \right\rangle_{S^*M_T} = \langle \mu_2, ap\rangle_{S^*M_T},$$

which is satisfied for all $a \in S^0_{\text{phg}}(T^*M_T;\mathbb{C})$ if and only if $\operatorname{supp}(\mu_2) \subset \operatorname{Char}(P)$.
Now, let $a \in S^0_{\text{phg}}(T^*M_T;\mathbb{C})$ and take $A \in \Psi^0_{\text{phg}}(M_T;\mathbb{C})$ such that $\sigma_0(A) = a$. Then, we have

$$(Aw_1^k, Aw_2^k)_{L^2(M_T;\mathbb{C})} = (A^*Aw_1^k, w_2^k)_{L^2(M_T;\mathbb{C})} \to \left\langle \mu_{12}, |a|^2 \right\rangle_{S^*M_T}.$$

Similarly, the Cauchy–Schwarz inequality gives

$$\begin{aligned} |(Aw_1^k, Aw_2^k)_{L^2(M_T;\mathbb{C})}| &\leqq \|Aw_1^k\|_{L^2(M_T;\mathbb{C})}\|Aw_2^k\|_{L^2(M_T;\mathbb{C})} \\ &\to \left(\left\langle \mu_1, |a|^2 \right\rangle_{S^*M_T}\right)^{\frac{1}{2}} \left(\left\langle \mu_2, |a|^2 \right\rangle_{S^*M_T}\right)^{\frac{1}{2}}. \end{aligned}$$

This finally yields for all $a \in S^0_{\text{phg}}(T^*M_T;\mathbb{C})$,

$$\left| \left\langle \mu_{12}, |a|^2 \right\rangle_{S^*M_T} \right|^2 \leqq \left\langle \mu_1, |a|^2 \right\rangle_{S^*M_T} \left\langle \mu_1, |a|^2 \right\rangle_{S^*M_T},$$

and hence $\operatorname{supp}(\mu_{12}) \subset \operatorname{supp}(\mu_1) \cap \operatorname{supp}(\mu_2)$.
Next, we define the operator

$$\mathcal{Q} = \begin{pmatrix} P & B^* \\ 0 & P \end{pmatrix} \in \Psi^2_{\text{phg}}(M, \mathbb{C}^{2\times 2}),$$

where B^* denotes the adjoint of the operator B. Remark that, for all

$$\mathcal{A} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \in \Psi_{\mathrm{phg}}^{-1}(M_T, \mathbb{C}^{2\times 2}),$$

we have

$$\begin{aligned}(\mathcal{AP} - \mathcal{QA}) &= \begin{pmatrix} [A_{11}, P] + A_{12}B - B^*A_{21} & [A_{12}, P] - B^*A_{22} \\ [A_{21}, P] + A_{22}B & [A_{22}, P] \end{pmatrix} \\ &\in \Psi_{\mathrm{phg}}^{0}(M_T, \mathbb{C}^{2\times 2}). \end{aligned} \tag{B.1}$$

Hence, we have

$$(\mathcal{AP} - \mathcal{QA})W^k \in L^2(M_T; \mathbb{C}^2),$$

together with

$$\mathcal{AP}W^k = 0.$$

In particular, these last two identities yield

$$\mathcal{QA}W^k \in L^2(M_T; \mathbb{C}^2). \tag{B.2}$$

For smooth data $W^k|_{t=0}$, we can integrate by parts and have

$$\left(\mathcal{QA}W^k, W^k\right)_{L^2(M_T;\mathbb{C}^2)} = \left(\mathcal{A}W^k, \mathcal{P}W^k\right)_{L^2(M_T;\mathbb{C}^2)} = 0,$$

since the operator $\mathcal{A}$ has a kernel with compact support (and hence the boundary terms at times 0 and T vanish). A density argument together with (B.2) then gives $\left(\mathcal{QA}W^k, W^k\right)_{L^2(M_T;\mathbb{C}^2)} = 0$ for all $W^k \in L^2(M_T; \mathbb{C}^2)$. Hence, we have

$$0 = \left((\mathcal{AP} - \mathcal{QA})W^k, W^k\right)_{L^2(M_T;\mathbb{C}^2)}. \tag{B.3}$$

As a consequence of (3.6), (B.3) and (B.1), we obtain

$$0 = \int_{S^*M_T} \mathrm{tr}\left\{\begin{pmatrix} \frac{1}{i}\{a_{11}, p\} + b|\eta|_x(a_{12} - a_{21}) & \frac{1}{i}\{a_{12}, p\} - b|\eta|_x a_{22} \\ \frac{1}{i}\{a_{21}, p\} + b|\eta|_x a_{22} & \frac{1}{i}\{a_{22}, p\} \end{pmatrix}\begin{pmatrix} \mu_1 & \mu_{12} \\ \overline{\mu}_{12} & \mu_2 \end{pmatrix}(\mathrm{d}\rho)\right\},$$

with $a_{jl} = \sigma_{-1}(A_{jl})$, $j, l = 1, 2$. Since the application $A \to \sigma_{-1}(A)$ is from $\Psi_{\mathrm{phg}}^{-1}(M_T)$ onto $S_{\mathrm{phg}}^{-1}(T^*M_T; \mathbb{C})$, this is equivalent to

$$\begin{cases} \left\langle \mu_1, \frac{1}{i}\{a_{11}, p\}\right\rangle_{S^*M_T} = 0, \\ \left\langle \mu_2, \frac{1}{i}\{a_{22}, p\}\right\rangle_{S^*M_T} + \langle 2i\,\mathrm{Im}(\mu_{12}), b|\eta|_x a_{22}\rangle_{S^*M_T} = 0, \\ \langle \mu_1, b|\eta|_x(a_{12} - a_{21})\rangle_{S^*M_T} + \left\langle \mu_{12}, \frac{1}{i}\{a_{21}, p\}\right\rangle_{S^*M_T} + \left\langle \overline{\mu}_{12}, \frac{1}{i}\{a_{12}, p\}\right\rangle_{S^*M_T} = 0, \end{cases} \tag{B.4}$$

for any $a_{11}, a_{22}, a_{12}, a_{21} \in S_{\mathrm{phg}}^{-1}(T^*M_T; \mathbb{C})$. Taking successively $a_{21} = a_{12}$ and $a_{21} = -a_{12}$ in the last identity of (B.4) yields

$$\left\langle \mathrm{Re}(\mu_{12}), \frac{1}{i}\{a_{12}, p\}\right\rangle_{S^*M_T} = 0 \quad \text{and}$$
$$\langle \mu_1, b|\eta|_x a_{12}\rangle_{S^*M_T} - \langle \mathrm{Im}(\mu_{12}), \{a_{12}, p\}\rangle_{S^*M_T} = 0.$$

These equations together with the first two identities of (B.4) give (3.8), which concludes the proof of the first part of the lemma.
Moreover, (3.4) yields

$$0 = \lim_{k\to\infty} (|b_\omega|^2 w_2^k, w_2^k)_{L^2(M_T;\mathbb{C})} = \left\langle \mu_2, |b_\omega|^2 \right\rangle_{S^*M_T},$$

which directly gives $\pi(\mathrm{supp}(\mu_2)) \cap \big((0,T)\times\omega\big) = \emptyset$ as $\mu_2 \geqq 0$ and $\{b_\omega > 0\} = \omega$. This concludes the proof of the second part of the lemma as $\mathrm{supp}(\mu_{12}) \subset \mathrm{supp}(\mu_2)$.

B.3. Proof of Lemma 4.3

A proof of this lemma can be found in the semiclassical setting in [9, Section 4.2] (see also [11]). We can parametrize the bicharacteristic Γ as

$$\Gamma \cap \pi^{-1}(M_T) = \{\phi_s(\rho_0), s \in (0,T)\},$$

with

$$\rho_0 = (0, x_0, \tau_0, \eta_0) \in \Gamma \cap \{t = 0\}.$$

We have $\tau_0^2 = |\eta_0|_x^2$. Let us assume that $\tau_0 > 0$. The case $\tau_0 < 0$ can be treated similarly. We set

$$\nu_0 = \left(x_0, \frac{\eta_0}{|\eta_0|_x}\right) \in S^*\Omega.$$

There exists a local chart (U_κ, κ) of Ω such that $x_0 \in U_\kappa$. We denote by (y_0, ξ_0) the coordinates of ν_0 in this chart.
We choose $\psi \in \mathscr{C}_c^\infty(\mathbb{R}^n)$ such that $\mathrm{supp}(\psi) \subset \kappa(U_\kappa)$, and $\psi = 1$ in a neighborhood of y_0. Next we define

$$v^k(y) = C_0 k^{\frac{n}{4}} e^{ik\varphi(y)}\psi(y), \quad \text{with } \varphi(y) = y\cdot\xi_0 + i(y-y_0)^2 \text{ and } C_0 > 0.$$

Now, we set

$$w_0^k = \kappa^* v^k \in \mathscr{C}_c^\infty(\Omega). \tag{B.5}$$

We have $w_0^k \rightharpoonup 0$ in $L^2(\Omega)$, $\lim_{k\to\infty} \|w_0^k\|_{L^2(\Omega)} = 1$ for an appropriate choice of C_0. Moreover, a classical computation on $(v^k)_{k\in\mathbb{N}}$ shows that $(w_0^k)_{k\in\mathbb{N}}$ is pure and admits the microlocal defect measure $m = \delta_{(x,\eta)=\nu_0}$.
We set $w_1^k = i\lambda w_0^k \in$ and let w^k be the solution of

$$\begin{cases} Pw^k = 0 & \text{in } (0,T)\times\Omega, \\ (w^k, \partial_t w^k)|_{t=0} = (w_0^k, w_1^k) & \text{on } \Omega. \end{cases} \tag{B.6}$$

For the sequence $(w^k)_{k\in\mathbb{N}}$, (4.4) is satisfied. We recall that L_+, L_-, λ... are defined in Section 2.1. Setting $z^k = L_+ w^k$, we have $L_- z^k = 0$ and

$$z^k|_{t=0} = \left(\frac{1}{i}\partial_t w^k - \lambda w^k\right)|_{t=0} = \frac{1}{i} w_1^k - \lambda w_0^k = 0.$$

Therefore, $z^k = 0$ in $\mathbb{R}^+ \times \Omega$, that is w^k solves the first-order equation

$$\begin{cases} L_+ w^k = 0 & \text{in } (0,T)\times\Omega, \\ w^k|_{t=0} = w_0^k & \text{on } \Omega. \end{cases} \tag{B.7}$$

System (B.7) is well-posed, and the sequence $(w^k)_{k\in\mathbb{N}}$ is bounded in $L^2(M_T)$ (see [23, Section 23.1]), weakly converging to zero and hence admits a microlocal defect measure μ (up to a subsequence).

Now, we prove that $\operatorname{supp}(\mu) \subset \Gamma$. Since w^k solves (B.6), the measure μ is preserved along Γ which is also a bicharacteristic curve of $\ell_+ = \tau - |\eta|_x$. Hence, it suffices to prove that any point $(0, x_1, \tau_1, \eta_1) \in \operatorname{Char}(P) \cap S^*M$ different from $(0, x_0, \tau_0, \eta_0)$ is not in $\operatorname{supp}(\mu)$.

Let $a \in S^0_{\mathrm{phg}}(T^*\Omega; \mathbb{C})$ be a zero-order homogeneous symbol, with compact support near x_1, such that $a = 1$ in a conic neighborhood of (x_1, η_1) and $a = 0$ in a conic neighborhood of (x_0, η_0). Taking $A \in \Psi^0_{\mathrm{phg}}(\Omega; \mathbb{C})$ such that $\sigma_0(A) = a$, we have

$$A w_0^k \to 0 \quad \text{in } L^2(\Omega),$$

since $m = \delta_{(x,\eta)=\nu_0}$. We shall now use the flows ϕ_s^+ and φ_s^+, defined in Section 2.2. We define the tangential polyhomogeneous symbol[5]

$$q(t, x, \eta) = a(\varphi^+_{-t}(x, \eta)) \in S^0_{\mathcal{T}}((-T, T) \times T^*\Omega; \mathbb{C}),$$

which satisfies

$$\begin{cases} H_{\ell_+} q = 0, \\ q(0, x, \eta) = a(x, \eta). \end{cases}$$

We denote by $\Psi^m_{\mathcal{T}}((-T, T) \times \Omega; \mathbb{C})$ the set of tangential operators of order m. We take $Q \in \Psi^0_{\mathcal{T}}((-T, T) \times \Omega; \mathbb{C})$ such a tangential operator satisfying $\sigma_0(Q) = q$. The commutator $[L_+, Q]$ satisfies $[L_+, Q] \in \Psi^0_{\mathcal{T}}((-T, T) \times \Omega; \mathbb{C})$ with principal symbol $\frac{1}{i}\{\ell_+, q\} = \frac{1}{i} H_{\ell_+} q = 0$. Hence, we have $[L_+, Q] \in \Psi^{-1}_{\mathcal{T}}((-T, T) \times \Omega; \mathbb{C})$. Now, we compute

$$\begin{cases} L_+ Q w^k = [L_+, Q] w^k \to 0 & \text{in } L^2(M_T), \\ Q w^k|_{t=0} = A w_0^k \to 0 & \text{in } L^2(\Omega). \end{cases}$$

Then, applying the hyperbolic energy inequality to this first-order system, we obtain

$$\|Q w^k\|_{L^\infty(0,T;L^2(\Omega))} \leqq C\left(\|A w_0^k\|_{L^2(\Omega)} + \|[L_+, Q] w^k\|_{L^1(0,T;L^2(M_T))}\right) \to 0.$$

Finally, let $r \in S^0_{\mathrm{phg}}(T^*M; \mathbb{C})$ be such that $r = 1$ in a neighborhood of $(0, x_1, \tau_1, \eta_1)$, $r = 0$ for $|\eta|_x < \frac{|\tau|}{2}$, and r has a compact support in the time variable t, included in $(-\frac{T}{2}, \frac{T}{2})$. Taking $R \in \Psi^0_{\mathrm{phg}}(M; \mathbb{C})$ such that $\sigma_0(R) = r$

[5] For tangential symbols $S^m_{\mathcal{T}}((-T, T) \times T^*\Omega; \mathbb{C})$ and associated operators $\Psi^m_{\mathcal{T}}((-T, T) \times \Omega; \mathbb{C})$, we refer to [23, page 94 (bottom)]. Here, t stands for the parameter upon which the symbols depend.

the operator RQ is in the class $\Psi^0_{\mathrm{phg}}(M;\mathbb{C})$, according to [23, Theorem 18.1.35]. Moreover, RQ is elliptic at $(0, x_1, \tau_1, \eta_1)$ since $r(0, x_1, \tau_1, \eta_1)q(0, x_1, \eta_1) = 1$ and we have

$$\|RQw^k\|_{L^2(M_T)} \leqq C\|Qw^k\|_{L^2(M_T)} \to 0.$$

As a consequence, $(0, x_1, \tau_1, \eta_1) \notin \mathrm{supp}(\mu)$. The invariance of μ along the bicharacteristic flow finally gives $\mathrm{supp}(\mu) \subset \Gamma$, which concludes the proof of Lemma 4.3. □

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