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Wigner measures and observability for the Schrödinger equation on the disk

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Abstract We analyse the structure of semiclassical and microlocal Wigner measures for solutions to the linear Schrödinger equation on the disk, with Dirichlet boundary conditions. Our approach links the propagation of singularities beyond geometric optics with the completely integrable nature of the billiard in the disk. We prove a "structure theorem", expressing the restriction of the Wigner measures on each invariant torus in terms of *second-microlocal measures*. They are obtained by performing a finer localization in phase space around each of these tori, at the limit of the uncertainty principle, and are shown to propagate according to Heisenberg equations on the circle. Our construction yields as corollaries (a) that the disintegration of the Wigner measures is absolutely continuous in the angular variable, which is an expression of the dispersive properties of the equation; (b) an observability inequality, saying

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that the L^2-norm of a solution on any open subset intersecting the boundary (resp. the L^2-norm of the Neumann trace on any nonempty open set of the boundary) controls its full L^2-norm (resp. H^1-norm). These results show in particular that the energy of solutions cannot concentrate on periodic trajectories of the billiard flow other than the boundary.

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Contents

1 Introduction

1.1 Motivation

We consider the unit disk

$$\mathbb{D} = \{z = (x, y) \in \mathbb{R}^2, |z|^2 = x^2 + y^2 < 1\} \subset \mathbb{R}^2$$

and denote by Δ the euclidean Laplacian. We are interested in understanding dynamical properties of the (time-dependent) linear Schrödinger equation

$$\frac{1}{i}\frac{\partial u}{\partial t}(z,t) = \left(-\frac{1}{2}\Delta + V(t,z)\right)u(z,t), t \in \mathbb{R}, \quad z = (x,y) \in \mathbb{D}, \quad (1.1)$$

$$u\rceil_{t=0} = u^0 \in L^2(\mathbb{D}) \quad (1.2)$$

with Dirichlet boundary condition $u\rceil_{\partial\mathbb{D}} = 0$ (we shall write $\Delta = \Delta_D$ when we want to stress that we are using the Laplacian on $L^2(\mathbb{D})$ with that boundary condition; we shall also write Δ for the Laplacian on the whole $\mathbb{R}^2$ but this will be specified in the text). We assume that V is a smooth real-valued potential, say $V \in C^\infty\left(\mathbb{R} \times \overline{\mathbb{D}}; \mathbb{R}\right)$. We shall denote by $U_V(t)$ the (unitary) propagator starting at time 0, such that $u(\cdot, t) = U_V(t)u^0$ is the unique solution of (1.1)–(1.2).

This equation is aimed at describing the evolution of a quantum particle trapped in a disk-shaped cavity, $u(\cdot, t)$ being the wave-function at time t. The total L^2-mass of the solution is preserved: $\|u(\cdot,t)\|_{L^2(\mathbb{D})} = \|u^0\|_{L^2(\mathbb{D})}$ for all time $t \in \mathbb{R}$. Thus, if the initial datum is normalized, $\|u^0\|_{L^2(\mathbb{D})} = 1$, the quantity $|u(z,t)|^2 dz$ is, for every fixed t, a probability density on $\mathbb{D}$; given $\Omega \subset \mathbb{D}$, the expression:

$$\int_\Omega |u(z,t)|^2 dz$$

is the probability of finding the particle in the set Ω at time t. Having $\int_0^T \int_\Omega |u(z,t)|^2 dx dt \geq c_0 > 0$ for all solutions of (1.1) means that every quantum particle spends a positive fraction of time of the interval $(0, T)$ in the set Ω. A major issue in mathematical quantum mechanics is to describe the possible localization—or delocalization—properties of solutions to the Schrödinger equation (1.1), by which we mean the description of the distribution of the probability densities $|u(z,t)|^2dz$ for all solutions u. A more tractable problem consists in considering instead of single, fixed solutions, sequences $(u_n)_{n\in\mathbb{N}}$ of solutions to (1.1) and describe the asymptotic properties of the associated probability densities $|u_n(z,t)|^2dz$ or $|u_n(z,t)|^2dzdt$. This point of view still allows to deduce properties of single solutions u and their distributions $|u(z,t)|^2dz$, as we shall see in the sequel.

It is always possible to extract a subsequence that converges weakly:

$$\int_{\mathbb{D}\times\mathbb{R}} \phi(z,t)|u_n(z,t)|^2dzdt \longrightarrow \int_{\overline{\mathbb{D}}\times\mathbb{R}} \phi(z,t)\nu(dz,dt), \quad \text{for every } \phi \in C_c(\overline{\mathbb{D}}\times\mathbb{R}),$$

where ν is a nonnegative Radon measure on $\overline{\mathbb{D}}\times\mathbb{R}$ that describes the asymptotic mass distribution of the subsequence of solutions (u_n). One of the goals of this paper is to understand how the fact that (u_n) solves (1.1) influences the structure of the associated measures ν.

As an application, we aim at understanding the observability problem for the Schrödinger equation: given an open set $\Omega \subset \mathbb{D}$ and a time $T > 0$, does there exist a constant $C = C(\Omega, T) > 0$ such that we have:

$$\int_0^T \int_\Omega |u(z,t)|^2dzdt \geq C\|u^0\|^2_{L^2(\mathbb{D})}, \quad \text{for all } u^0 \in L^2(\mathbb{D}) \text{ and } u \text{ associated solution of (1.1)–(1.2)?} \tag{1.3}$$

If such an estimate holds, then every quantum particle must leave a trace on the set Ω during the time interval $(0,T)$; in other words: it is observable from $\Omega\times(0,T)$. This question is linked to that of understanding the structure of the limiting measures ν. Indeed, estimate (1.3) is *not* satisfied if and only if there exists a sequence of data (u_n^0) such that $\|u_n^0\|_{L^2(\mathbb{D})} = 1$ and $\int_0^T\int_\Omega |u_n(z,t)|^2dzdt \to 0$, where u_n is the solution of (1.1) issued from u_n^0. After the extraction of a subsequence, this holds if and only if the associated limit measure ν satisfies

$$\int_0^T\int_{\overline{\mathbb{D}}} \nu(dz,dt) = T, \quad \int_0^T\int_\Omega \nu(dz,dt) = 0.$$

The question of observability from $\Omega\times(0,T)$ may hence be reformulated as: can sequences of solution of (1.1) concentrate on sets which do not intersect

$\Omega \times (0, T)$? From the point of view of applications, it is of primary interest to understand which sets Ω do observe all quantum particles trapped in a disk. Moreover, the observability of (1.1) is equivalent to the controllability of the Schrödinger equation (see e.g. [39]), which means that it is possible to drive any initial condition to any final condition at time T, with a control (a forcing term in the right-hand side of (1.1)) located within Ω.

It is well-known that the space of position variables (z, t) does not suffice to describe the propagation properties of solutions to Schrödinger equations (or more generally wave equations) in the high frequency régime. One has to add the associated dual variables, $(\xi, H) \in \mathbb{R}^2 \times \mathbb{R}$ (momentum and energy) and lift the measure ν to the *phase space*, that is, the space of variables (z, t, ξ, H): this gives rise to the so-called *Wigner measures* [60]. We shall hence investigate the regularity and localization properties in position and momentum variables of the Wigner measures associated with sequences of normalized solutions of (1.1). They describe how the solutions are distributed over phase space. We shall develop both the *microlocal* and *semiclassical* points of view. These are two slightly different, but closely related, approaches to the problem: the semiclassical approach is more suitable when our initial data possess a well-defined oscillation rate, whereas the microlocal approach describes the singularities of solutions, independently of the choice of a scale of oscillation, at the price of giving slightly less precise results.

Our study fits in the régime of the "quantum-classical correspondence principle", which asserts that the high-frequency dynamics of the solutions to (1.1) are described in terms of the corresponding classical dynamics; in our context the underlying classical system is the billiard flow on $\mathbb{D}$. A well-known expression of the correspondence principle is that the Wigner measures are invariant by this flow.

Of course, one may consider similar questions for any bounded domain of $\mathbb{R}^d$ or any Riemannian manifold, and not only the disk $\mathbb{D}$. As a matter of fact, the answer to these questions depends strongly on the dynamics of the billiard flow (resp. the geodesic flow on a Riemannian manifold), and, to our knowledge, it is known only in few cases (see Sect. 1.6). For instance, on negatively curved manifolds, the related celebrated Quantum Unique Ergodicity conjecture remains to this day open. Two geometries for which the observation problem is well-understood, and the Wigner measures are rather well-described, are the torus $\mathbb{T}^d$ (see [6,12,16,32,33,35,46,47]) and the sphere $\mathbb{S}^d$, or more generally, manifolds all of whose geodesics are closed (see [4,34,44,45]), on which the classical dynamics is completely integrable. We shall later on compare these two situations with our results on the disk $\mathbb{D}$. We refer to the article [5] for a survey of recent results concerning Wigner measures associated to sequences of solutions to the time-dependent

Schrödinger equation in various geometries and to the review article [38] on the observability question.

1.2 Some consequences of our structure theorem

Our central results are Theorems 2.5 and 2.7 below, which provide a detailed structure of the Wigner measures associated to sequences of solutions to the Schrödinger equation, using notions of "*second-microlocal calculus*". As corollaries of these structure Theorems, we obtain:

- Corollary 1.1 (see also Theorem 2.10), which states a regularity property of the measures ν reflecting the dispersive character of the Schrödinger equation (1.1);
- Theorem 1.2 (resp. Theorem 1.3), which states the observability/controllability of the equation from any nonempty open set touching the boundary of the disk (resp. from any nonempty open set of the boundary).

Let us first state these corollaries in order to motivate the more technical results of this paper.

Corollary 1.1 *Let (u_n^0) be a sequence in $L^2(\mathbb{D})$, such that $\|u_n^0\|_{L^2(\mathbb{D})} = 1$ for all n. Consider the sequence of nonnegative Radon measures ν_n on $\overline{\mathbb{D}} \times \mathbb{R}$, defined by*

$$\nu_n(dz, dt) = |U_V(t)u_n^0(z)|^2 dz dt. \tag{1.4}$$

Then, for any weak-$*$ limit ν of the sequence (ν_n), there exists $\theta \in [0, 1]$ such that for almost every t, there is an absolutely continuous *probability measure $\tilde{\nu}_t$ on $\mathbb{D}$ such that*

$$\nu(dz, dt) = \nu_t(dz)dt, \quad \textit{with} \quad \nu_t = \theta\tilde{\nu}_t + (1 - \theta)(2\pi)^{-1}\delta_{\partial\mathbb{D}}.$$

In particular, for almost every t, $\nu_t\rceil_{\mathbb{D}}$ is absolutely continuous. *Above, $(2\pi)^{-1}\delta_{\partial\mathbb{D}}$ denotes the unique probability measure on $\mathbb{R}^2$ that is concentrated on the boundary of the disk and is invariant by rotations around the origin.*

This result shows in particular that the weak-$*$ accumulation points of the densities (1.4) possess some regularity in the interior of the disk. We shall see that one can always exhibit sequences of solutions that concentrate singularly on the boundary, corresponding to $\theta = 0$ in Corollary 1.1: the so-called whispering-gallery modes, see Sect. 1.3 and Remark 2.12. Note also that θ

does not depend on time (this is a consequence of Proposition 2.9); this simply means that the mass of the measure on the boundary and that in the interior of the disk remain constant in time.

In Theorem 2.10 below, we present a stronger version of Corollary 1.1 describing (in phase space) the regularity of microlocal lifts of such limit measures ν. This precise description (as well as all results of this paper) relies on the complete integrability of the billiard flow on the disk. Its statement needs the introduction of action angle coordinates and associated invariant tori, and is postponed to Sect. 2.7.

The second class of results mentioned above is related to unique continuation-type properties of the Schrödinger equation (1.1). We consider the following condition on an open set $\Omega \subset \overline{\mathbb{D}}$, a time $T > 0$ and a potential V:

$$\left(u^0 \in L^2(\mathbb{D}), \quad U_V(t)u^0\rceil_{(0,T)\times\Omega} = 0\right) \Longrightarrow u^0 = 0. \qquad (\mathrm{UCP}_{V,\Omega,T})$$

As a consequence of Theorem 2.7, we shall also prove the following quantitative version of ($\mathrm{UCP}_{V,\Omega,T}$).

Theorem 1.2 *Let $\Omega \subset \overline{\mathbb{D}}$ be an open set such that $\Omega \cap \partial\mathbb{D} \neq \emptyset$ and $T > 0$. Assume one of the following statements holds*:

- *the potential $V \in C^\infty([0,T] \times \overline{\mathbb{D}}; \mathbb{R})$, the time T, and the open set Ω satisfy* ($\mathrm{UCP}_{V,\Omega,T}$),
- *the potential $V \in C^\infty(\overline{\mathbb{D}}; \mathbb{R})$ does not depend on t.*

Then there exists $C = C(V, \Omega, T) > 0$ such that:

$$\left\|u^0\right\|^2_{L^2(\mathbb{D})} \leq C \int_0^T \left\|U_V(t)u^0\right\|^2_{L^2(\Omega)} dt, \qquad (1.5)$$

for every initial datum $u^0 \in L^2(\mathbb{D})$.

Roughly speaking, this means that any set Ω touching $\partial\mathbb{D}$ observes all quantum particles trapped in the disk. As we shall see, these are the only sets satisfying this property (see Sect. 1.3 and Remark 2.12).

We are also interested in the boundary analogue of ($\mathrm{UCP}_{V,\Omega,T}$) for a given potential V, a time $T > 0$ and an open set $\Gamma \subset \partial\mathbb{D}$:

$$\left(u^0 \in H^1_0(\mathbb{D}), \quad \partial_n(U_V(t)u^0)\rceil_{(0,T)\times\Gamma} = 0\right) \Longrightarrow u^0 = 0, \qquad (\mathrm{UCP}_{V,\Gamma,T})$$

where $\partial_n = \frac{\partial}{\partial n}$ denotes the exterior normal derivative to $\partial\mathbb{D}$. As a consequence of Theorem 2.7, we shall also prove the following quantitative version of ($\mathrm{UCP}_{V,\Gamma,T}$).

Theorem 1.3 *Let Γ be any nonempty subset of $\partial\mathbb{D}$ and $T > 0$. Suppose one of the following holds*:

- *the potential* $V \in C^\infty([0,T] \times \overline{\mathbb{D}}; \mathbb{R})$, *the time* T *and the set* Γ *satisfy* ($\mathrm{UCP}_{V,\Gamma,T}$),
- $V \in C^\infty(\overline{\mathbb{D}}; \mathbb{R})$ *does not depend on* t.

Then there exists $C = C(V, \Gamma, T) > 0$ *such that*:

$$\left\|u^0\right\|^2_{H^1(\mathbb{D})} \leq C \int_0^T \left\|\partial_n(U_V(t)u^0)\right\|^2_{L^2(\Gamma)} dt, \tag{1.6}$$

for every initial datum $u^0 \in H^1_0(\mathbb{D})$.

Note that the unique continuation properties ($\mathrm{UCP}_{V,\Omega,T}$) and ($\mathrm{UCP}_{V,\Gamma,T}$) are known to hold, for instance, when V is analytic in (t, z), as a consequence of the Holmgren uniqueness theorem as stated by Hörmander in [28, Theorem 5.3.1]. The regularity of V for ($\mathrm{UCP}_{V,\Omega,T}$) and ($\mathrm{UCP}_{V,\Gamma,T}$) to hold can actually be lowered to V bounded and depending analytically on the variable t, according to the Tataru–Robbiano–Zuily–Hörmander Theorem [31,55,59] as stated by Hörmander (see [31] Theorem 5.1 together with the Remark p. 205).

The three above results express a delocalization property of the energy of solutions to (1.1). The observation of the L^2-norm restricted to *any* open set of the disk touching the boundary is sufficient to recover linearly the norm of the data. In particular, the L^2-mass of solutions cannot concentrate on periodic trajectories of the billiard (except those contained in the boundary). The observability inequalities (1.5) and (1.6) are especially relevant in control theory [11,39,41]: in turn, they imply a controllability result from the set Ω or Γ.

As a consequence of the observability inequality 1.5, we have the following result (where we use the notation of Corollary 1.1).

Corollary 1.4 *For every open set* $\Omega \subset \overline{\mathbb{D}}$ *touching the boundary, for every* $T > 0$, *there exists a constant* $C(T, \Omega) > 0$ *such that for any initial data* (u_n^0) *and any weak-$*$ limit* ν *of the sequence* (ν_n) *as in Corollary* 1.1, *we have*

$$\int_0^T \nu_t(\Omega) dt \geq \frac{1}{C(T, \Omega)}.$$

Again, this translates the fact that any solution has to leave positive mass on any set Ω touching the boundary $\partial\mathbb{D}$ during the time interval $(0, T)$.

1.3 Stationary solutions to (1.1): eigenfunctions on the disk

If the potential $V(t, z)$ does not depend on the time variable t, we have as particular solutions of the Schrödinger equation the “stationary solutions”, those with initial data given by eigenfunctions of the elliptic operator $-\Delta_D + 2V(z)$.

In the absence of a potential, i.e. if $V = 0$, these solutions are well understood: the eigenfunctions of $-\Delta_D$ on $\mathbb{D}$ are the functions whose (non-normalized) expression in polar coordinates ($x = -r\sin u$, $y = r\cos u$) is

$$\psi^{\pm}_{n,k}(re^{iu}) = J_n(\alpha_{n,k} r)e^{\pm inu}, \tag{1.7}$$

where n, k are non-negative integers, J_n is the n-th Bessel function, and the $\alpha_{n,k}$ are its positive zeros indexed increasingly by the integer k. The corresponding eigenvalue is $\alpha^2_{n,k}$. Putting then $u^0 = \psi^{\pm}_{n,k}$ gives a time-periodic solution $u(\cdot, t) = e^{it\frac{\alpha^2_{n,k}}{2}}\psi^{\pm}_{n,k}$ to (1.1)–(1.2). Moreover, the eigenvalues of $-\Delta_D$ have multiplicity two. This is a consequence of a celebrated result by Siegel [57], showing that J_n, J_m have no common zeroes for $n \neq m$. In particular, the limit measures associated to sequences of eigenfunctions are explicitly computable in terms of the limits of the stationary distributions:

$$\frac{|\psi^{\pm}_{n,k}(z)|^2}{\|\psi^{\pm}_{n,k}\|^2_{L^2(\mathbb{D})}} dz = \frac{|J_n(\alpha_{n,k} r)|^2}{\|\psi^{\pm}_{n,k}\|^2_{L^2(\mathbb{D})}} r\,dr\,du,$$

as the frequency $\alpha_{n,k}$ tends to infinity (this expression has to be slightly modified when considering linear combinations of the two eigenfunctions $\psi^{+}_{n,k}$ and $\psi^{-}_{n,k}$, corresponding to the same eigenvalue, with n fixed and k tending to infinity). Let us recall some particular cases of this construction. For fixed k and for $n \to \infty$, it is classical [36, Lemma 3.1] that

$$\frac{|\psi^{\pm}_{n,k}(z)|^2}{\|\psi^{\pm}_{n,k}\|^2_{L^2(\mathbb{D})}} dz \rightharpoonup (2\pi)^{-1}\delta_{\partial\mathbb{D}},$$

which corresponds to the so-called whispering gallery modes. On the other hand, letting $k, n \to \infty$ with n/k being constant, one may obtain for any $\gamma \in [0, 1)$ depending on the ratio n/k [51, Section 4.1]

$$\frac{|\psi^{\pm}_{n,k}(z)|^2}{\|\psi^{\pm}_{n,k}\|^2_{L^2(\mathbb{D})}} dz \rightharpoonup \frac{1}{2\pi(1-\gamma^2)^{1/2}} \frac{1}{(|z|^2-\gamma^2)^{1/2}} \mathbb{1}_{(\gamma,1)}(|z|)dz.$$

Except the Dirac measure on the boundary, these measures all belong to $L^p(\mathbb{D})$ for any $p < 2$ (hence satisfying Corollary 1.1) and are invariant by rotation and positive on the boundary (hence satisfying Corollary 1.4). These measures in fact enjoy more regularity and symmetry than those asserted by Corollaries 1.1 and 1.4.

The observability question for eigenfunctions can also be simply handled in account of the bounded multiplicity of the spectrum. For any non-empty open set $\Omega_{I_1,I_2} = \{re^{iu}, r \in I_1, u \in I_2\} \subset \mathbb{D}$ (where I_1 is an open subset of [0, 1], I_2 an open interval of $\mathbb{S}^1$), for any eigenfunction ψ of $-\Delta_D$, one has:

$$\|\psi\|_{L^2(\Omega_{I_1,\mathbb{S}^1})} \leq C(|I_2|)\|\psi\|_{L^2(\Omega_{I_1,I_2})}$$

where $C(|I_2|)$ is a positive constant depending only on the size of I_2. On the other hand, if Ω_{I_1,I_2} touches the boundary ($1 \in I_1$) it automatically satisfies the geometric control condition as defined in [11,39]. The results in those references imply that:

$$\|\psi\|_{L^2(\mathbb{D})} \leq C'(I_1)\|\psi\|_{L^2(\Omega_{I_1,\mathbb{S}^1})}.$$

Therefore, for such Ω_{I_1,I_2}, we have

$$\|\psi\|_{L^2(\mathbb{D})} \leq C(|I_2|)C'(I_1)\|\psi\|_{L^2(\Omega_{I_1,I_2})}.$$

It is not known to the authors whether or not any of the results of the present article could be deduced directly from the result for eigenfunctions, even when the potential vanishes identically. This does not seem to appear in the literature. On flat tori, proving observability or regularity of Wigner measures associated to the Schrödinger equation from the explicit expression of the solutions in terms of Fourier series requires a careful analysis of the distribution of lattice points on paraboloids [7,12,33] or sophisticated arguments on lacunary Fourier series [32,35]. On the disk, and in absence of a potential, to find a proof similar to that of [12,33] one would need to expand the kernel of $e^{-it\Delta_D/2}$ in terms of Bessel functions:

$$e^{-it\Delta_D/2} = \sum_{n,k,\pm} e^{it\alpha_{n,k}^2/2}|\psi_{n,k}^{\pm}\rangle\langle\psi_{n,k}^{\pm}|$$

and to work with this explicit expression. Such an approach would anyway require some very technical work on the spacings between the $\alpha_{n,k}$.

Here, instead, we establish directly the links between the completely integrable nature of the dynamics of the billiard flow and the delocalization and dispersion properties of the solutions to the Schrödinger equation. Note that all results of this paper also hold for eigenfunctions of the operator $-\Delta_D + 2V(z)$ (as stationary solutions to (1.1)). As a matter of fact, our approach is more general for it applies as well to quasimodes and clusters of eigenfunctions of the operator $-\Delta_D + 2V(z)$. The reader is referred to [3] and Remark 2.6 for more details on this matter.

1.4 The semiclassical viewpoint

In spite of the fact that our statements and proofs are formulated exclusively in terms of the non-semiclassical Schrödinger equation (1.1), our results do have an interpretation in the light of the semiclassical limit for the Schrödinger equation. Suppose that v_h solves the semiclassical Schrödinger equation:

$$\frac{h}{i}\frac{\partial v_h}{\partial t}(z,t) = \left(-\frac{h^2}{2}\Delta + h^2 V(ht,z)\right) v_h(z,t), \quad v_h\rceil_{t=0} = u^0. \tag{1.8}$$

Then $u(\cdot,t) := v_h(\cdot,t/h)$ is in fact the solution to the (non-semiclassical) Schrödinger equation (1.1) with initial datum u^0. As a consequence, describing properties of solutions to (1.1) on time intervals of size of order 1 amounts to describing properties of solutions to the semiclassical Schrödinger equation (1.8) up to times of order $1/h$. Our results show that the semiclassical approximation breaks down in time $1/h$. For instance, if we take as initial datum in (1.8) a coherent state localized at $(z_0,\xi_0) \in \mathbb{D}\times\mathbb{R}^2$,

$$u_n^0 = \frac{1}{h_n^\alpha}\rho\left(\frac{z-z_0}{h_n^\alpha}\right) e^{\frac{i}{h_n} z\cdot\xi_0}, \quad \rho\in C_c^\infty(\mathbb{D}), \quad \rho(0)=1, \quad \alpha\in(0,1), \quad h_n\to 0,$$

our results imply that the associated solution of (1.8) on the time interval $(0,1/h_n)$ is no longer concentrated on the billiard trajectory issued from (z_0,ξ_0). Instead, we show that it spreads on the disk $\mathbb{D}$ (the associated measure is absolutely continuous) and it leaves a positive mass on any set touching the boundary (even if the trajectory of the billiard issued from (z_0,ξ_0) avoids this set).

Thus, our analysis goes far beyond the well-understood semiclassical limit for times of order 1, or even of order $\log(1/h)$ (known as the Ehrenfest time, see [13]). Such a long time analysis is possible thanks to the complete integrability of the system. In fact, in the paper [1], which deals with the Schrödinger equation (and more general completely integrable systems) on the flat torus, it is shown that the time scale $1/h$ is exactly the one at which the delocalization of solutions takes place; for chaotic systems on the contrary, the semiclassical approximation is expected to break down at the Ehrenfest time [5,8,9].

1.5 The structure theorem

We would like to stress the fact that all these results are obtained as consequences of our main theorem (not yet stated), Theorem 2.5 (or its variant Theorem 2.7) that gives a precise description of the structure of Wigner measures arising from solutions to (1.1). This theorem provides a unified framework from which to derive simultaneously the absolute continuity of projections of semiclassical measures (a fact that is related to dispersive effects) on the one hand, and, on the other hand, the observability estimates (1.5) and (1.6), which are quantitative unique continuation properties. Since a precise statement requires the introduction of many other objects, we postpone it to Sects. 2.4 and 2.5 (semiclassical and microlocal formulations respectively), and only give a rough idea of the method for the moment.

The standard construction of the Wigner measures, outlined in Sect. 2.5, allows to lift a measure ν to a measure μ^∞ on phase space (or μ_{sc} in the semiclassical setting): these are the associated microlocal defect measures [23]. The law of propagation of singularities for Eq. (1.1) implies that μ^∞ is invariant by the billiard flow in the disk, and we want to exploit the complete integrability of this flow.

For this, we use action-angle coordinates to integrate the dynamics of the billiard flow and describe associated invariant tori (Sect. 3.1). The angular momentum J of a point (z, ξ) in phase space is preserved by the flow, and so is the Hamiltonian $E = |\xi|$. The actions J and E are in involution and independent, except at the points of $\partial\mathbb{D}$ with tangent momentum. The angle α that a trajectory makes when bouncing on the boundary is a also preserved quantity (in fact a function of J/E). The key point of our proof is to analyze in detail the possible concentration of sequences of solutions to (1.1) on the sets $\mathcal{I}_{\alpha_0} = \{\alpha = \alpha_0\}$ of all points of phase space sharing a common incidence/reflection angle α_0. To this aim, we perform a second microlocalization on this set, in the spirit of [1,6,46]. We decompose a Wigner measure as a sum of measures supported on these invariant sets. The case $\alpha_0 \notin \pi\mathbb{Q}$ corresponds to trajectories hitting the boundary in a dense set, and is trivial for us since it supports only one invariant measure. We focus on those $\mathcal{I}_{\alpha_0}$ for which $\alpha_0 \in \pi\mathbb{Q}$. Any trajectory of the billiard having this angle is periodic. We wish to "zoom" on this torus to describe the concentration of the associated measure. Assuming that the initial sequence has a typical oscillation scale of order $1/h$, we perform a second microlocalization at scale 1, which is the limit of the Heisenberg uncertainty principle. Roughly speaking, the idea is to relocalize in the action variable J at scale 1 (i.e. h times $1/h$), so that the Heisenberg uncertainty principle implies delocalization in the conjugated angle variable. We obtain two limit objects, interpreted as *second-microlocal measures*. The first one captures the part of our sequence of solutions whose derivatives in directions "transverse to the flow" remain bounded; the second one captures the part of the solution rapidly oscillating in these directions. Understanding the notion of transversality adapted to this problem is achieved by constructing a flow that interpolates between the billiard flow (generated by the Hamiltonian E) and the rotation flow (generated by the Hamiltonian J). The second measure is a usual microlocal/semiclassical measure whereas the first one is a less usual operator-valued measure taking into account non-oscillatory phenomena. We prove that both second-microlocal measures enjoy additional invariance properties: the first one is invariant by the rotation flow, whereas the second one propagates through a Heisenberg equation on the circle. This translates, respectively, into Theorem 2.5 (ii) and (iii).

This program was already completed in [1,6,46] for the Schrödinger equation on flat tori, but carrying it out in the disk induces considerable additional difficulties. Our phase space does not directly come equipped with its action-angle coordinates, so that we need first to change variables. This requires in particular to build a Fourier Integral Operator to switch from (z, ξ)-variables to action-angle coordinates. These coordinates are very nice to understand the dynamics and are necessary to perform the second microlocalization, but they are extremely nasty to treat the boundary condition, for which the use of polar coordinates is more suitable. It seems that we cannot avoid having to go back and forth between the two sets of coordinates. Our approach to that particular technical aspect is inspired by [25]; however, the second-microlocal nature of the problem requires to perform the asymptotic expansions of [25] one step further.

1.6 Relations to other works

1.6.1 *Regularity of semiclassical measures*

This work pertains to the longstanding study of the so-called "quantum-classical correspondence", which aims at understanding the links between high frequency solutions of the Schrödinger equation and the dynamics of the underlying billiard flow (see for instance the survey article [5]).

More precisely, it is concerned with a case of completely integrable billiard flow. This particular dynamical situation has already been addressed in [46] and [6] in the case of flat tori, and in [1] for more general integrable systems (without boundary). These three papers use in a central way a "second microlocalization" to understand the concentration of measures on invariant tori. The main tools are second-microlocal semiclassical measures, introduced in the local Euclidean setting in [19–21,48–50], and defined in [1,6,46] as global objects.

On the sphere $\mathbb{S}^d$, or more generally, on a manifold with periodic geodesic flow, the situation is radically different. The geodesic flow for this type of geometries is still completely integrable, but it is known [4,34,44,45] that every invariant measure is a Wigner measure; those are not necessarily absolutely continuous when projected in the position space. The difference with the previous situation is that the underlying dynamical system, though completely integrable, is *degenerate*. What was evidenced in [1] is that a sufficient and necessary condition for the absolute continuity of Wigner measures, is that the hamiltonian be a *strictly convex/concave* function of the action variables – a condition that is even stronger than non-degeneracy. In the case of the disk, the complete integrability of the billiard flow on $\mathbb{D}$ degenerates on the boundary. There, both actions coincide, which allows for the concentration of solutions on the invariant torus at the boundary (as was the case with the aforementioned whispering gallery modes).

Note that on the torus and on the disk, it remains an open question to fully characterize the set of Wigner measures associated to sequences of solutions to the time-dependent Schrödinger equation. In the case of flat tori, the papers [7,33] provide additional information about the regularity of these measures.

1.6.2 Observability of the Schrödinger equation

Since the pioneering work of Lebeau [39], it is known that observability inequalities like (1.5)–(1.6) always hold if all trajectories of the billiard enter the observation region Ω or Γ in finite time (at a "non-diffractive point" in the boundary observation case). However, since [27,32], we know that this strong geometric control condition is not necessary: (1.5) holds on the two-torus as soon as $\Omega \neq \emptyset$; for different proofs and extensions of this result see [6,10,15,16,35,47]. These properties seem to deeply depend on the global dynamics of the billiard flow.

On manifolds with periodic geodesic flow, it is *necessary* that $\overline{\Omega}$ meets all geodesics for an observation inequality as (1.5) to hold [47]. This is due to the strong stability properties of the geodesic flow.

To our knowledge, apart from the case of flat tori, few results are known concerning the observability of the Schrödinger equation in situations where the geometric control condition fails. The paper [1] extends [6] to general completely integrable systems under a convexity assumption for the hamiltonian. Note also that the boundary observability (1.6) holds in the square if (and only if) the observation region Γ contains both a horizontal and a vertical nonempty segments [54]. Finally, for chaotic systems, the observability inequality (1.5) is also valid on manifolds with negative curvature if the set of uncontrolled trajectories is sufficiently small [8,9].

Our Theorems 1.2 and 1.3 provide a (necessary and) sufficient condition for the observability of the Schrödinger group on the disk. The necessity of the condition is clear in the case of boundary observability, and in the case of internal observability, if $\Omega \subset \mathbb{D}$ is such that $\Omega \cap \partial\mathbb{D} = \emptyset$, the observability inequality (1.5) fails. When $V = 0$ this comes from the existence of whispering-gallery modes, see Sect. 1.3, and this remains true for any V, as proved in Remark 2.12.

Let us conclude this introduction with a few more remarks.

Remark 1.5 In this article, we only treat the case of Dirichlet boundary conditions. The extension of our method to the Neumann or mixed boundary condition deserves further investigation.

Remark 1.6 Let us comment on the regularity required on the potential V. Arguments developed in [6] show that all the results of this paper could actually be weakened to $V \in C^0\left(\mathbb{R} \times \overline{\mathbb{D}}; \mathbb{R}\right)$ or even to the case where V is continuous outside a set of zero measure. Corollary 1.1 in fact also holds for any $V \in L^2_{\text{loc}}(\mathbb{R}; \mathcal{L}(L^2(\mathbb{D})))$, and in particular for any bounded complex valued potentials. See also Remark 2.6 below.

Remark 1.7 Our results directly yield a polynomial decay rate for the energy of the damped wave equation $(\partial_t^2 - \Delta + b(z)\partial_t)u = 0$ with Dirichlet Boundary conditions on the disk. More precisely, [2, Theorem 2.3] and Theorem 1.2 imply that if $b \geq 0$ is positive on an open set Ω such that $\Omega \cap \partial\mathbb{D} \neq \emptyset$, then the $H_0^1 \times L^2$ norm of solutions decays at rate $1/\sqrt{t}$ for data in $(H^2 \cap H_0^1) \times H_0^1$. This rate is better than the a priori logarithmic decay rate given by the Lebeau theorem [40]. The latter is however optimal if $\text{supp}(b) \cap \partial\mathbb{D} = \emptyset$ as a consequence of the whispering gallery phenomenon (see e.g. [43]).

2 The structure theorem

In this section, we give the main definitions used in the article and state our structure theorems. We first define microlocal and semiclassical Wigner measures (which are the central objects discussed in the paper) in Sect. 2.1. We then briefly describe the billiard flow and introduce adapted action-angle coordinates in Sect. 2.2. This allows us to formulate our main results (Sects. 2.4 and 2.5), both in the semiclassical (Theorem 2.5) and in the microlocal (Theorem 2.7) framework. Next, in Sect. 2.7, we define various measures at the boundary of the disk, that will be useful in the proofs, and explain their links with the Wigner measures in the interior.

2.1 Wigner measures: microlocal versus semiclassical points of view

Let $T^*\mathbb{R}^2 = \mathbb{R}^2 \times \mathbb{R}^2$ be the cotangent bundle over $\mathbb{R}^2$, and $T^*\mathbb{R} = \mathbb{R} \times \mathbb{R}$ be the cotangent bundle over $\mathbb{R}$. We shall denote by $z \in \mathbb{R}^2$ (resp. $t \in \mathbb{R}$) the space (resp. time) variable and $\xi \in \mathbb{R}^2$ (resp. $H \in \mathbb{R}$) the associated frequency.

Our main results can be formulated in two different and complementary settings. We first introduce the symbol class needed to formulate their microlocal version, allowing to define *microlocal* Wigner distributions. We then define *semiclassical* Wigner distributions and briefly compare these two objects.

Definition 2.1 Let us call $\mathcal{S}_0$ the space of functions $a = a(z, \xi, t, H) \in C^\infty(T^*\mathbb{R}^2 \times T^*\mathbb{R})$ such that

(a) a is compactly supported in the variables z, t.
(b) a is homogeneous at infinity in (ξ, H) in the following sense: there exists $R_0 > 0$ such that

$$a(z, \xi, t, H) = a(z, \lambda\xi, t, \lambda^2 H), \quad \text{for } |\xi|^2 + |H| > R_0 \quad \text{and } \lambda \geq 1. \tag{2.1}$$

Equivalently, there is $a_{\hom} \in C^\infty(T^*\mathbb{R}^2 \times T^*\mathbb{R}\backslash\{(\xi, H) = (0, 0)\})$ satisfying (2.1) for all $\lambda > 0$, such that

$$a(z, \xi, t, H) = a_{\hom}(z, \xi, t, H), \quad \text{for } |\xi|^2 + |H| > R_0.$$

Such a homogeneous function $a_{\hom}$ is entirely determined by its restriction to the set $\{|\xi|^2 + 2|H| = 2\} \subset \mathbb{R}^2 \times \mathbb{R}$, which is homeomorphic to a 2-dimensional sphere $\mathbb{S}^2$. Thus we may (and will, when convenient) identify $a_{\hom}$ with a function in the space $C^\infty(\mathbb{R}^2_z \times \mathbb{R}_t \times \mathbb{S}^2_{\xi,H})$.

Note that the different homogeneities with respect to the H and ξ variables is adapted to the scaling of the Schrödinger operator.

Let (u_n^0) be a sequence in $L^2(\mathbb{D})$, such that $\|u_n^0\|_{L^2(\mathbb{D})} = 1$ for all n. For $z \in \mathbb{D}$ and $t \in \mathbb{R}$ we denote $u_n(z, t) = U_V(t)u_n^0(z)$. In what follows (e.g. in formula (2.3) below), we shall systematically extend the functions u_n, a priori defined on $\mathbb{D}$, by the value 0 outside $\mathbb{D}$ as done in [25], where semiclassical Wigner measures for boundary value problems were first considered. The extended sequence (still denoted u_n) now satisfies the equation

$$\left(-\frac{1}{2}\Delta + V - \frac{1}{i}\frac{\partial}{\partial t}\right)u_n = \frac{1}{2}\frac{\partial u_n}{\partial n} \otimes \delta_{\partial\mathbb{D}}, \quad (z, t) \in \mathbb{R}^2 \times \mathbb{R}, \tag{2.2}$$

where Δ denotes the Laplacian on $\mathbb{R}^2$. Remark that the term $\frac{\partial u_n}{\partial n}\rceil_{\partial\mathbb{D}}$ has no straightforward meaning at this level of regularity. We shall see below how to give a signification to this equation, both in the semiclassical (see Remark 2.4) and in the microlocal (see Sect. 2.8.3) settings.

The *microlocal Wigner distributions* associated to (u_n) act on symbols $a \in \mathcal{S}_0$ by

$$W_{u_n}(a) := \langle u_n, \mathrm{Op}_1(a)u_n\rangle_{L^2(\mathbb{R}^2_z \times \mathbb{R}_t)}, \tag{2.3}$$

where $\mathrm{Op}_1(a) = a(z, D_z, t, D_t)$ (with the standard notation $D = -i\partial$) is a pseudodifferential operator obtained by the standard quantization procedure: in what follows, $\mathrm{Op}_\epsilon(a) = a(z, \epsilon D_z, t, \epsilon D_t)$ will stand for the operator acting on $L^2(\mathbb{R}^2 \times \mathbb{R})$ by:

$$\big(\mathrm{Op}_\epsilon(a)u\big)(z,t) = \frac{1}{(2\pi\epsilon)^3}\int_{\mathbb{R}^2\times\mathbb{R}}\int_{\mathbb{R}^2\times\mathbb{R}} e^{\frac{i\xi\cdot(z-z')+iH(t-t')}{\epsilon}} a(z,\xi,t,H)\, u(z',t')\,dz'dt'\,d\xi dH. \tag{2.4}$$

In particular if a depends only on (z,t), we have

$$W_{u_n}(a) = \int_{\mathbb{D}} a(z,t)|u_n(z,t)|^2 dzdt.$$

Remark 2.2 It is not clear at first sight that expressions like (2.3) are well-defined since $u_n = U_V(t)u_n^0 \in L^\infty(\mathbb{R}_t; L^2(\mathbb{R}^2_z))$ is not square-integrable in the variable t. However, the symbol a here is compactly supported with respect to t. A classical decomposition of $\mathrm{Op}_1(a)$ with respect to the time variable (as an operator with compactly supported Schwartz kernel plus an operator with smooth Schwartz kernel having fast decay away from the diagonal) proves that this operator actually maps continuously $L^\infty(\mathbb{R}_t; L^2(\mathbb{R}^2_z))$ to $L^2_{\mathrm{comp}}(\mathbb{R}_t; L^2(\mathbb{R}^2_z))$, which clarifies the meaning of (2.3). See e.g. [30, Section 18.1]. Also, we shall see that if $u_n^0 \rightharpoonup 0$, then the limit object of (2.3) as $n \to +\infty$ only depends on the principal symbol of the operator $\mathrm{Op}_1(a)$ (i.e. on a_{hom}): hence, according to pseudodifferential calculus, we shall equivalently study instead $\langle \chi_T u_n, \mathrm{Op}_1(a)\chi_T u_n\rangle_{L^2(\mathbb{R}^2_z\times\mathbb{R}_t)}$ for some $\chi_T \in C_c^\infty(\mathbb{R})$ equal to one in the neighborhood of the time support of a.

Usual estimates on pseudodifferential operators now imply that W_{u_n} forms a bounded sequence in $\mathcal{S}_0'$. The main goal of this article is to understand properties of weak limits of (W_{u_n}) that are valid for any sequence of initial conditions (u_n^0).

The problem also has a semiclassical variant. In this version, one considers $a \in C_c^\infty(T^*\mathbb{R}^2 \times T^*\mathbb{R})$, a real parameter $h > 0$, and one defines the *semiclassical Wigner distributions* at scale h by

$$W^h_{u_n}(a) := \langle u_n, \mathrm{Op}_1(a(z,h\xi,t,h^2H))u_n\rangle_{L^2(\mathbb{R}^2_z\times\mathbb{R}_t)}, \tag{2.5}$$

where $\mathrm{Op}_1(a(z,h\xi,t,h^2H)) = a(z,hD_z,t,h^2D_t) = \mathrm{Op}_h(a(z,\xi,t,hH))$, see (2.4). Note that this scaling relation is the natural one for solutions of (1.1), and its interest will be made clear below. Again $W^h_{u_n}$ is well defined, and forms a bounded sequence in $\mathcal{D}'(T^*\mathbb{R}^2 \times T^*\mathbb{R})$ if h stays bounded. This formulation is most meaningful if the parameter $h = h_n$ is chosen in relation with the typical scale of oscillation of our sequence of initial conditions (u_n^0).

Definition 2.3 Given a bounded sequence (w_n) in $L^2(\mathbb{D})$, we shall say that it is (h_n)-oscillating from above (resp. (h_n)-oscillating from below) if the sequence (w_n) extended by zero outside of $\mathbb{D}$ satisfies:

$$\lim_{R\to\infty}\limsup_{n\to\infty}\int_{|\xi|\geq R/h_n}|\widehat{w_n}(\xi)|^2\,d\xi=0,$$
$$\left(\text{resp.}\quad \lim_{\epsilon\to 0}\limsup_{n\to\infty}\int_{|\xi|\leq \epsilon/h_n}|\widehat{w_n}(\xi)|^2\,d\xi=0\right),$$

where $\widehat{w_n}$ is the Fourier transform of w_n on $\mathbb{R}^2$.

The property of being (h_n)-oscillating from above is only relevant if $h_n\to 0$; if u_n^0 is (h_n)-oscillating for (h_n) bounded away from 0, the (extended) sequence (u_n^0) is compact in L^2 and the accumulation points of $(W_{u_n}^{h_n})$ are just the Wigner measures W_u^h, where $h>0$ is an accumulation point of h_n and u is an accumulation point of u_n in L^2. Therefore, we shall always assume that $h_n\to 0$. Note that one can always find (h_n) tending to zero such that (u_n^0) is h_n-oscillating from above (to see that, note that for fixed n one may choose h_n such that $\int_{|\xi|\geq 1/h_n}|\widehat{u_n^0}(\xi)|^2 d\xi\leq n^{-1}$). However, the choice of the sequence h_n is by no means unique (h_n-oscillating sequences are also h'_n-oscillating as soon as $h'_n\leq h_n$), although in many cases there is a natural scale h_n given by the problem under consideration.

One can find (h'_n) such that (u_n^0) is h'_n-oscillating from below if and only if the functions (u_n^0) (extended by the value 0 outside $\mathbb{D}$) converge to 0 weakly in $L^2(\mathbb{R}^2)$. It is not always possible to find a common (h_n) such that (u_n^0) is h_n-oscillating both from above and below (see [24] for an example of a sequence with this behavior). However, when it is the case, the semiclassical Wigner distributions contain more information that the microlocal ones (see Sect. 2.6). On the other hand, if no h_n exists such that (u_n^0) is h_n-oscillating from above and below, the accumulation points of $W_{u_n}^{h_n}$ may fail to capture completely the asymptotic phase-space distribution of the sequence (u_n), either because some mass will escape to $|\xi|=\infty$ or because the fraction of the mass going to infinity at a rate slower that h_n^{-1} will give a contribution concentrated on $\xi=0$. In those cases, the microlocal formulation is still able to describe the asymptotic distribution of the sequence on the reduced phase-space $\mathbb{R}_z^2\times\mathbb{R}_t\times\mathbb{S}^2_{\xi,H}$.

This is one of the motivations that has led us to study both points of view, semiclassical and microlocal.

2.2 The billiard flow

Microlocal and semiclassical analysis provide a connection between the Schrödinger equation and the billiard on the underlying phase space. In this section we first clarify what we mean by "billiard flow" in the disk. The phase space associated with the billiard flow on the disk can be defined as a quotient of $\overline{\mathbb{D}}\times\mathbb{R}^2$ (position $\times$ frequency). We first define the symmetry with respect to the line tangent to the circle $\partial\mathbb{D}$ at $z\in\partial\mathbb{D}$ by

$$\sigma_z(\xi) = \xi - 2(z \cdot \xi)z, \quad \text{and we denote } \sigma(z,\xi) = (z, \sigma_z(\xi)), \quad \text{for } z \in \partial\mathbb{D}.$$

Then, we work on the quotient space

$$\mathbb{W} = \overline{\mathbb{D}} \times \mathbb{R}^2 / \sim \quad \text{where } (z,\xi) \sim \sigma(z,\xi) \quad \text{for } |z| = 1.$$

We denote by π the canonical projection $\overline{\mathbb{D}} \times \mathbb{R}^2 \to \mathbb{W}$ which maps a point (z, ξ) to its equivalence class modulo $\sim$. Note that π is one-one on $\mathbb{D} \times \mathbb{R}^2$, so that $\mathbb{D} \times \mathbb{R}^2$ may be seen as a subset of $\mathbb{W}$.

A function $a \in C^0(\mathbb{W})$ can be identified with the function $\tilde{a} = a \circ \pi \in C^0(\overline{\mathbb{D}} \times \mathbb{R}^2)$ satisfying $\tilde{a}(z,\xi) = \tilde{a} \circ \sigma(z,\xi)$ for $(z,\xi) \in \partial\mathbb{D} \times \mathbb{R}^2$.

The billiard flow $(\phi^\tau)_{\tau\in\mathbb{R}}$ on $\mathbb{W}$ is the (uniquely defined) action of $\mathbb{R}$ on $\mathbb{W}$ such that the map $(\tau, z, \xi) \mapsto \phi^\tau(z,\xi)$ is continuous on $\mathbb{R} \times \mathbb{W}$, satisfies $\phi^{\tau+\tau'} = \phi^\tau \circ \phi^{\tau'}$, and such that

$$\phi^\tau(z,\xi) = (z + \tau\xi, \xi)$$

whenever $z \in \mathbb{D}$ and $z + \tau\xi \in \mathbb{D}$.

In order to understand how the completely integrable nature of the flow ϕ^τ influences the structure of Wigner measures, we need to introduce adapted coordinates. We denote by

$$\Phi : (s, \theta, E, J) \mapsto (x, y, \xi_x, \xi_y), \tag{2.6}$$

the "action-angle" coordinates for the billiard flow (see also Sect. 3.1), defined by:

$$\begin{cases} x = \frac{J}{E}\cos\theta - s\sin\theta, \\ y = \frac{J}{E}\sin\theta + s\cos\theta, \\ \xi_x = -E\sin\theta, \\ \xi_y = E\cos\theta. \end{cases}$$

These coordinates are illustrated in Fig. 1. The inverse map is given by the formulas

$$\begin{cases} E = \sqrt{\xi_x^2 + \xi_y^2}, \quad \text{(velocity)} \\ J = x\xi_y - y\xi_x, \quad \text{(angular momentum)} \\ \theta = -\arctan\left(\frac{\xi_x}{\xi_y}\right), \text{(angle of } \xi \text{ with the vertical)} \\ s = -x\sin\theta + y\cos\theta, \quad \text{(abscissa of } (x,y) \text{ along the line} \\ \qquad \left(\frac{J}{E}\cos\theta, \frac{J}{E}\sin\theta\right) + \mathbb{R}\xi). \end{cases}$$

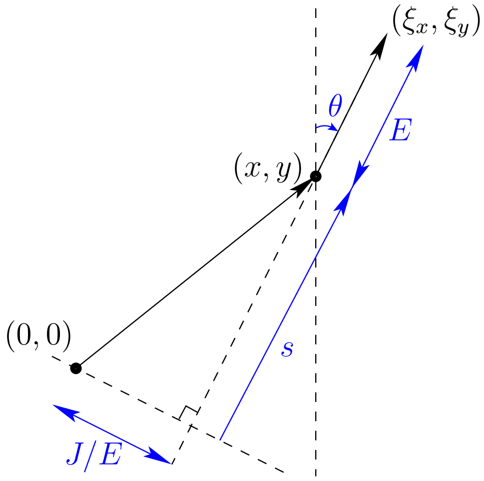


Fig. 1 Action-angle coordinates for the billiard flow on the disk. On the picture, we have $J > 0$ and $\theta \in (-\frac{\pi}{2}, 0)$

In other words, we have:

$$\begin{cases} E = |\xi|, \\ J = z \cdot \xi^{\perp}, \\ s = z \cdot \frac{\xi}{|\xi|}, \end{cases}$$

where $\xi^{\perp} := (\xi_y, -\xi_x)$, and

$$\begin{cases} \xi = (\xi_x, \xi_y) = E(-\sin(\theta), \cos(\theta)), \\ z = (x, y) = s(-\sin(\theta), \cos(\theta)) + \frac{J}{E}(\cos(\theta), \sin(\theta)) \\ \quad = \left(z \cdot \frac{\xi}{|\xi|}\right) \frac{\xi}{|\xi|} + \left(z \cdot \frac{\xi^{\perp}}{|\xi|}\right) \frac{\xi^{\perp}}{|\xi|}. \end{cases}$$

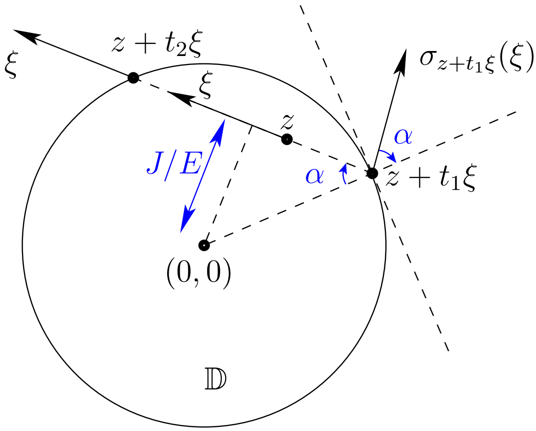
Note that the velocity E and the angular momentum J are preserved both by the free transport flow in $\mathbb{R}^2 \times \mathbb{R}^2$ and the symmetry σ. Hence, they are also preserved along ϕ^{τ}; the variables s and θ play the role of "angle" coordinates. We call $\alpha = -\arcsin\left(\frac{J}{E}\right)$ the angle that a billiard trajectory makes with the normal to the circle, when it hits the boundary (see Fig. 2). The quantity α is preserved by the billiard flow.

Let us denote $T_{(E,J)}$ the level sets of the pair (E, J), namely

$$T_{(E,J)} = \{(z, \xi) \in \overline{\mathbb{D}} \times \mathbb{R}^2 \;:\; (|\xi|, z \cdot \xi^{\perp}) = (E, J)\}. \tag{2.7}$$

For $E \neq 0$ let us denote $\lambda_{E,J}$ the probability measure on $T_{(E,J)}$ that is both invariant under the billiard flow and invariant under rotations. In the coordinates (s, θ, E, J), we have

Fig. 2 Angle α. On the picture, we have $J > 0$ and $\alpha \in (-\frac{\pi}{2}, 0)$



$$\lambda_{E,J}(ds, d\theta) = c(E, J)dsd\theta, \quad c(E, J) = \left(\int_{T(E,J)} dsd\theta\right)^{-1} > 0.$$

Note that for $E \neq 0$ and $\alpha \in \pi\mathbb{Q}$ the billiard flow is periodic on $T_{(E,J)}$ whereas $\alpha \notin \pi\mathbb{Q}$ corresponds to trajectories that hit the boundary on a dense set. More precisely, if $\alpha \notin \pi\mathbb{Q}$ then the billiard flow restricted to $T_{(E,J)}$ has a unique invariant probability measure, namely $\lambda_{E,J}$.

2.3 Standard facts about Wigner measures

We start formulating the question and results in a semiclassical framework: we have a parameter h_n going to 0, meant to represent the typical scale of oscillation of our sequence of initial conditions (u_n^0).

We simplify the notation by writing $h = h_n$, $u_h^0 = u_n^0$. We will always assume that the functions u_h^0 are normalized in $L^2(\mathbb{D})$. We define $u_h(z, t) = U_V(t)u_h^0(z)$ (the reader should be aware that u_h satisfies the classical Schrödinger equation (1.1); the index h only reminds its oscillation scale). Since this is a function on $\mathbb{D} \times \mathbb{R}$ it is natural to do a frequency analysis both in z and t. Recall that we keep the notation u_h after the extension by zero outside $\mathbb{D}$. The *semiclassical Wigner distribution* associated to u_h (at scale h) is a distribution on the cotangent bundle $T^*\mathbb{R}^2 \times T^*\mathbb{R} = \mathbb{R}_z^2 \times \mathbb{R}_\xi^2 \times \mathbb{R}_t \times \mathbb{R}_H$, defined by

$$W_{u_h}^h : a \mapsto \langle u_h, \mathrm{Op}_1(a(z, h\xi, t, h^2 H))u_h \rangle_{L^2(\mathbb{R}^2 \times \mathbb{R})}, \quad \text{for all } a \in C_c^\infty(T^*\mathbb{R}^2 \times T^*\mathbb{R}). \tag{2.8}$$

The scaling $\mathrm{Op}_1(a(z, h\xi, t, h^2 H))$ is performed in order to capture all the information when u_h is h-oscillating from above (otherwise, the discussion below remains entirely valid but part of the information about $u_h(z,t)$ is lost when studying $W^h_{u_h}(a)$). When no confusion arises, we shall denote W_h for $W^h_{u_h}$.

By standard estimates on the norm of $\mathrm{Op}_1(a)$, it follows that W_h belongs to $\mathcal{D}'(T^*\mathbb{R}^2 \times T^*\mathbb{R})$, and is uniformly bounded in that space as $h \to 0^+$. Thus, one can extract subsequences that converge in the weak-$*$ topology of $\mathcal{D}'(T^*\mathbb{R}^2 \times T^*\mathbb{R})$. In other words, after possibly extracting a subsequence, we have

$$W_h(a) \underset{h\to 0}{\longrightarrow} \mu_{sc}(a) \tag{2.9}$$

for all $a \in C_c^\infty(T^*\mathbb{R}^2 \times T^*\mathbb{R})$ (the fact that we may extract a common subsequence for all functions a follows from a diagonal extraction argument, see e.g. [23]).

In this paper such a measure μ_{sc} will be called a *semiclassical Wigner measure*, or in short *semiclassical measure*, associated with the initial conditions (u_h^0) and the scale h.

Remark 2.4 Fix $R > 0$; Lemma 8.2 below tells us that in order to compute the restriction of μ_{sc} to the set $\{|H| < R\}$ we may assume that $u_h^0 \in H_0^1(\mathbb{D})$ and $\|\nabla u_h^0\|_{L^2(\mathbb{D})} = O_R(h^{-1})$. In that case Proposition 8.1 says that the boundary data $h\partial_n(U_V(t)u_h^0)$ form a bounded sequence in $L^2_{\mathrm{loc}}(\mathbb{R} \times \partial\mathbb{D})$. We can work under these assumptions when necessary. Since R is arbitrary, this constitutes no loss of generality.

It follows from standard properties of pseudodifferential operators that the limit μ_{sc} in (2.9) has the following properties:

- μ_{sc} is a nonnegative measure, of the form $\mu_{sc}(dz, d\xi, dt, dH) = \mu_{sc}(dz, d\xi, t, dH)dt$ where $t \mapsto \mu_{sc}(t) \in L^\infty(\mathbb{R}_t; \mathcal{M}_+(T^*\mathbb{R}^2 \times \mathbb{R}_H))$. Moreover, for a.e. $t \in \mathbb{R}$, $\mu_{sc}(t)$ is supported in $\{|\xi|^2 = 2H\} \cap \left(\overline{\mathbb{D}} \times \mathbb{R}^2 \times \mathbb{R}_H\right)$. See [26,42] for a proof of nonnegativity; the time regularity and the localization of the support are shown in Proposition 9.1.
- From the normalization of u_h^0 in L^2, we have for a.e. t:

$$\int_{\overline{\mathbb{D}}\times\mathbb{R}^2\times\mathbb{R}} \mu_{sc}(dz, d\xi, t, dH) \le 1,$$

  the inequality coming from the fact that $\overline{\mathbb{D}} \times \mathbb{R}^2 \times \mathbb{R}$ is not compact, and that there may be an escape of mass to infinity. However, if u_h^0 is h-oscillating from above, escape of mass does not occur and we have $\int_{\overline{\mathbb{D}}\times\mathbb{R}^2\times\mathbb{R}} \mu_{sc}(dz, d\xi, t, dH) = 1$.

- The standard quantization enjoys the following property:

$$\left[-\frac{ih}{2}\Delta, \mathrm{Op}_h(a)\right] = \mathrm{Op}_h\left(\xi \cdot \partial_z a - \frac{ih}{2}\Delta_z a\right), \tag{2.10}$$

where Δ is the Laplacian on $\mathbb{R}^2$. From this identity together with (2.2), one can show that

$$\int_{\overline{\mathbb{D}}\times\mathbb{R}^2\times\mathbb{R}} \xi \cdot \partial_z a \ \mu_{sc}(dz, d\xi, t, dH) = 0 \tag{2.11}$$

for a.e. t and for every smooth a such that $a(z, \xi, t, H) = a(z, \sigma_z(\xi), t, H)$ for $|z| = 1$. Equivalently,

$$\int_{\overline{\mathbb{D}}\times\mathbb{R}^2\times\mathbb{R}} a \circ \phi^\tau \circ \pi(z, \xi, t, H)\mu_{sc}(dz, d\xi, t, dH)$$
$$= \int_{\overline{\mathbb{D}}\times\mathbb{R}^2\times\mathbb{R}} a \circ \pi(z, \xi, t, H)\mu_{sc}(dz, d\xi, t, dH)$$

for every $a \in C^0(\mathbb{W})$, $\tau \in \mathbb{R}$—where ϕ^τ is the billiard flow in the disk and $\pi : \overline{\mathbb{D}} \times \mathbb{R}^2 \to \mathbb{W}$ is the canonical projection, defined in Sect. 2.2. In other words, $\pi_*\mu$ is an invariant measure of the billiard flow.
We refer to Sect. 2.8 for a more general version of (2.11) (formulated initially in [25], see also [56]) involving a measure associated to boundary traces.

2.4 The structure theorem: semiclassical formulation

Now comes our central result, giving the structure of semiclassical measures arising as weak-$*$ limits of sequences (W_h) associated to solutions of (1.1). As a by-product it clarifies the dependence of $\mu_{sc}(t, \cdot)$ on the time parameter t. It was already noted in [45] that the dependence of $\mu_{sc}(t, \cdot)$ on the sequence of initial conditions is a subtle issue.

The statement of Theorem 2.5 is technical and needs introducing some notation. The notation (s, θ, E, J), α was introduced in Sect. 2.2. We first restrict our attention to the case where the initial conditions (u_h^0) are h-oscillating from below, or equivalently μ_{sc} does not charge $\{\xi = 0\}$ (otherwise, the restriction of μ_{sc} to $\{\xi = 0\}$ will be better understood at the end of Sect. 2.6); hence, we may restrict our discussion to $E \neq 0$. For each $\alpha_0 \in \pi\mathbb{Q} \cap (-\pi/2, \pi/2)$ we will introduce a flow $(\phi_{\alpha_0}^\tau)$ on the billiard phase space $\mathbb{W}$, all of whose orbits are periodic (Definition 3.4). It coincides with the billiard flow on the set

$$\mathcal{I}_{\alpha_0} = \{(s, \theta, E, J) \in \Phi^{-1}(\overline{\mathbb{D}} \times \mathbb{R}^2), \quad J = -E \sin\alpha_0\} = \{\alpha = \alpha_0\},$$

which is the union of all the lagrangian manifolds $T_{(E,J)}$ with $J = -E \sin \alpha_0$. If a is a function on $\mathbb{W}$, we shall denote by $\langle a \rangle_{\alpha_0}$ its average along the orbits of $\phi^\tau_{\alpha_0}$ (actually, $\langle a \rangle_{\alpha_0}$ is well defined even if a is not symmetric with respect to the boundary, since the set of hitting times of the boundary has measure 0). In the coordinates (s, θ, E, J), this is a function whose restriction to $\mathcal{I}_{\alpha_0}$ does not depend on s.

We will denote by

$$m_a^{\alpha_0}(s, E, t, H)$$

the operator on $L^2_{\text{loc}}(\mathbb{R}_\theta)$ acting by multiplication by the function

$$a\left(\Phi(s, \theta, E, -E \sin \alpha_0), t, H\right).$$

Remark that $m^{\alpha_0}_{\langle a \rangle_{\alpha_0}}$ does not depend on the variable s. For our potential V, the function $\langle V \rangle_{\alpha_0} \circ \Phi$ depends only on θ (and, of course, on t if V is time-dependent).

Given $\omega \in \mathbb{T} := \mathbb{R}/2\pi\mathbb{Z}$, we denote by $U_{\alpha_0,\omega}(t)$ the unitary propagator (starting at time 0) of the equation

$$-\cos^2 \alpha_0\ D_t v(t, \theta) + \left(-\frac{1}{2}\partial_\theta^2 + \cos^2 \alpha_0\ \langle V \rangle_{\alpha_0} \circ \Phi(t, \theta)\right) v(t, \theta) = 0$$

acting on the Hilbert space

$$\mathcal{H}_\omega = \{v \in L^2_{\text{loc}}(\mathbb{R}) : v(\theta + 2\pi) = v(\theta)e^{i\omega},\ \text{for a.e. } \theta \in \mathbb{R}\}, \tag{2.12}$$

i.e. with Floquet-periodic condition. In the statements below, each $\mathcal{H}_\omega$ is identified with $L^2(0, 2\pi)$ by taking restriction of functions to $(0, 2\pi)$. Note that, by definition of Φ, the functions $\langle V \rangle_{\alpha_0} \circ \Phi$, $\langle a \rangle_{\alpha_0} \circ \Phi$ are 2π-periodic in the variable θ, so that they indeed act on $\mathcal{H}_\omega$ by multiplication.

Theorem 2.5 *Let (u_h^0) be a family of initial data, assumed to be h-oscillating from below. Let μ_{sc} be a semiclassical measure, associated with the initial conditions (u_h^0) and the scale h. Then μ_{sc} can be decomposed into a countable sum of non-negative measures*:

$$\mu_{sc} = \nu_{Leb} + \sum_{\alpha_0 \in \pi\mathbb{Q} \cap [-\pi/2, \pi/2]} \nu_{\alpha_0},$$

satisfying:

(i) *Each of the measures in the decomposition above is carried by the set $\{H = \frac{E^2}{2}\}$ and is invariant under the billiard flow.*

(ii) *The measure* ν_{Leb} *is constant in* t *and is of the form* $\nu_{Leb} = \int_{E>0,|J|\le E} \lambda_{E,J} d\nu'(E,J)dt$ *for some nonnegative measure* ν' *on* $\mathbb{R}^2$. *In other words* ν_{Leb} *is a combination of Lebesgue measures on the invariant "tori"* $T_{(E,J)}$.

(iii) *For every* $\alpha_0 \in \pi\mathbb{Q} \cap (-\pi/2, \pi/2)$, *the measure* ν_{α_0} *is carried by the set* $\mathcal{I}_{\alpha_0} \cap \{H = E^2/2\}$ *and is characterized by*

$$\int_{\mathcal{I}_{\alpha_0}} a\, d\nu_{\alpha_0} = \int_{\mathcal{I}_{\alpha_0}} \mathrm{Tr}_{L^2(0,2\pi)} \left(m^{\alpha_0}_{\langle a \rangle_{\alpha_0}} \sigma_{\alpha_0} \right) d\ell_{\alpha_0} dt,$$
$$\text{for all } \; a \in C_c^\infty(T^*\mathbb{R}^2 \times T^*\mathbb{R}), \tag{2.13}$$

where $\ell_{\alpha_0}(d\omega, dE, dH)$ *is a nonnegative measure on* $\mathbb{T}_\omega \times \mathbb{R}_E \times \mathbb{R}_H$, *and*

$$\sigma_{\alpha_0} : \mathbb{T}_\omega \times \mathbb{R}_E \times \mathbb{R}_H \times \mathbb{R}_t \to \mathcal{L}^1_+ \left(L^2(0, 2\pi) \right),$$

is integrable with respect to ℓ_{α_0}, *continuous with respect to* t *and takes values in the set of nonnegative trace-class operators on* $L^2(0, 2\pi)$, *and, for* ℓ_{α_0}-*almost every* (ω, E, H), *satisfies* $\mathrm{Tr}(\sigma_{\alpha_0}(\omega, E, H, t)) = 1$.
In addition, for ℓ_{α_0}-*almost every* (ω, E, H), *we have*:

$$\sigma_{\alpha_0}(\omega, E, H, t) = U_{\alpha_0,\omega}(t)\sigma_{\alpha_0}(\omega, E, H, 0)U^*_{\alpha_0,\omega}(t). \tag{2.14}$$

Finally, ℓ_{α_0} *and* $\sigma_{\alpha_0}(\cdot, 0)$ *only depend on the sequence of initial conditions* (u^0_h).

(iv) *For* $\alpha_0 = \pm\frac{\pi}{2}$, ν_{α_0} *is a measure that does not depend on* t, *carried by* $\{H = |\xi|^2/2\} \cap T^*\partial\mathbb{D}$ *and is invariant under rotations around the origin.*

Note that if we do not identify $\mathcal{H}_\omega$ with $L^2(0, 2\pi)$, then for each (ω, E, H, t), the operator $\sigma_{\alpha_0}(\omega, E, H, t)$ is in $\mathcal{L}^1_+(\mathcal{H}_\omega)$, so that σ_{α_0} is actually a section of a vector bundle over $\mathbb{T}_\omega$. We would also like to stress the fact that the family of operators $\sigma_{\alpha_0}(\omega, E, H, 0)$ (which, recall, only depend on the sequence of initial data) are in some sense, more precise objects than the Wigner measure associated to the sequence of initial data. It may happen that two sequences of initial data have the same Wigner measure but give rise to different families of operators $\sigma_{\alpha_0}(\omega, E, H, 0)$. We refer the reader to reference [45] for examples of this type of behavior when the disk is replaced by the flat torus.

Remark 2.6 The conclusion of the results above (as well as their counterparts in the next section) also holds for semiclassical measures associated to sequences of approximate solutions of the Schrödinger equation, i.e. satisfying

$$\left(D_t + \frac{1}{2}\Delta - V(t,z)\right) u_h(z,t) = o_{L^2_{\mathrm{loc}}(\mathbb{D}\times\mathbb{R})}(1).$$

Moreover, as in the torus [1,6], all conclusions of Theorem 2.5 hold as well for solutions of

$$\left(D_t + \frac{1}{2}\Delta\right) u_h(z,t) = O_{L^2_{\mathrm{loc}}(\mathbb{D}\times\mathbb{R})}(1),$$

except for the continuity in time of σ_{α_0} and the propagation law (2.14). This is thus also the case for Corollary 1.1 (whose proof does not use these two properties). This includes for instance the case of "operator potentials" $V \in L^2_{\mathrm{loc}}(\mathbb{R}; \mathcal{L}(L^2(\mathbb{D})))$ (see also [14] for related results).

2.5 The structure theorem: microlocal formulation

We now give the microlocal version of Theorem 2.5. The main difference is that we now use the class of test functions $\mathcal{S}_0$ defined in Sect. 2.1.

Let (u_n^0) be a sequence of initial conditions, normalized in $L^2(\mathbb{D})$. Denote by $u_n(\cdot,t) := U_V(t)u_n^0$ the associated solution of (1.1), and recall that we also write u_n for its extension by zero to the whole $\mathbb{R}^2$. All over the paper we let $\chi \in C_c^\infty(\mathbb{R})$ be a nonnegative cut-off function that is identically equal to one near the origin. Let $R > 0$. For $a \in \mathcal{S}_0$, we define

$$\left\langle W_{n,R}^\infty, a\right\rangle := \left\langle u_n, \mathrm{Op}_1\left(\left(1-\chi\left(\frac{|\xi|^2+|H|}{R^2}\right)\right) a(z,\xi,t,H)\right) u_n\right\rangle_{L^2(\mathbb{R}^2\times\mathbb{R})},$$

and

$$\left\langle W_{c,n,R}, a\right\rangle := \left\langle u_n, \mathrm{Op}_1\left(\chi\left(\frac{|\xi|^2+|H|}{R^2}\right) a(z,\xi,t,H)\right) u_n\right\rangle_{L^2(\mathbb{R}^2\times\mathbb{R})}. \tag{2.15}$$

The Calderón–Vaillancourt theorem [18] ensures that both $W_{n,R}^\infty$ and $W_{c,n,R}$ are bounded in $\mathcal{S}_0'$. After possibly extracting subsequences, we have the existence of a limit: for every $a \in \mathcal{S}_0$,

$$\left\langle \mu^\infty, a\right\rangle := \lim_{R\to\infty}\lim_{n\to+\infty}\left\langle W_{n,R}^\infty, a\right\rangle, \tag{2.16}$$

and

$$\left\langle \mu_c, a\right\rangle := \lim_{R\to\infty}\lim_{n\to+\infty}\left\langle W_{c,n,R}, a\right\rangle. \tag{2.17}$$

As a consequence, after extraction, the subsequence W_n converges weakly-$*$ to a limit $\mu_{ml} \in \mathcal{S}_0'$, which can be decomposed into

$$\mu_{ml} = \mu^\infty + \mu_c.$$

The two limit objects μ_c and μ^∞ enjoy the following standard properties:

- The distribution μ_c vanishes if and only if the family (u_n^0) converges weakly to 0 in $L^2(\mathbb{D})$; in other words μ_c reflects the "compact part" of the sequence (u_n^0), hence the subscript $_c$ in the notation.
- The distribution μ^∞ is nonnegative, 0-homogeneous and supported at infinity in the variable (ξ, H) (*i.e.*, it vanishes when paired with a compactly supported function). As a consequence, μ^∞ may be identified with a nonnegative Radon measure on $\mathbb{R}^2_z \times \mathbb{R}_t \times \mathbb{S}^2_{\xi,H}$. Actually, μ^∞ is the microlocal defect measure of [23,58] (with the appropriate class of symbols $\mathcal{S}_0$).
- In addition, μ^∞ is of the form $\mu^\infty(dz, d\xi, dt, dH) = \mu^\infty(dz, d\xi, t, dH)\, dt$ where $t \mapsto \mu^\infty(t) \in L^\infty(\mathbb{R}_t; \mathcal{M}_+(\mathbb{R}^2_z \times \mathbb{S}^2_{\xi,H}))$. Moreover, for a.e. $t \in \mathbb{R}$, $\mu^\infty(t)$ is supported in $\{|\xi|^2 = 2H\} \cap (\overline{\mathbb{D}} \times \mathbb{S}^2_{\xi,H})$.
- The projection of the distribution $\mu_{ml} = \mu_c + \mu^\infty$ on the (z, t)-variables is the Radon measure ν defined in the introduction (Sect. 1). From the normalization of u_n^0 in L^2, we have for a.e. t:

$$\int_{\overline{\mathbb{D}} \times \mathbb{S}^2_{\xi,H}} \mu^\infty(dz, d\xi, t, dH) \leq 1,$$

  if $u_n^0 \rightharpoonup 0$ in $L^2(\mathbb{D})$, then we have $\int_{\overline{\mathbb{D}} \times \mathbb{S}^2_{\xi,H}} \mu^\infty(dz, d\xi, t, dH) = 1$.
- The measure μ^∞ satisfies the invariance property:

$$\left\langle \mu^\infty, \frac{\xi}{\sqrt{2H}}.\partial_z a \right\rangle = 0, \tag{2.18}$$

  for a satisfying the symmetry condition $a(z, \xi, t, H) = a(z, \sigma_z(\xi), t, H)$ for $|z| = 1$. In other words, $\pi_* \mu^\infty$ is invariant by the billiard flow.

These properties are well-known and won't be proved in detail here (the fact that it is carried on $\{H = \frac{|\xi|^2}{2}\}$ follows from Appendix A and the proof of invariance is essentially contained in [25] or [56]).

Let us now discuss separately the finer properties of μ^∞ (high frequencies) and of μ_c (low frequencies).

We first describe μ^∞ and state the analogue of Theorem 2.5 in the microlocal setting. As previously we call $T_{(E,J)}$ the level sets of (E, J) and $\mathcal{I}_{\alpha_0} = \{J = -\sin \alpha_0 E\}$. The only difference with the semiclassical formalism is that the test functions are homogeneous as in Definition 2.1. Recall that, if we identify homogeneous functions with functions on the 2-dimensional sphere $\mathbb{S}^2_{\xi,H} = \{(\xi, H), |\xi|^2 + 2|H| = 2\}$, the measure μ^∞ may be seen as a measure on $\mathbb{R}^2_z \times \mathbb{R}_t \times \mathbb{S}^2_{\xi,H}$ (supported by $\overline{\mathbb{D}} \times \mathbb{R}_t \times \mathbb{S}^2_{\xi,H}$). The microlocal version of Theorem 2.5 reads as follows:

Theorem 2.7 *Let (u_n^0) be normalized in $L^2(\mathbb{D})$, and such that (2.16) holds. Then the measure μ^∞ can be decomposed as a countable sum of nonnegative measures on $\mathbb{R}^2 \times \mathbb{R}_t \times \mathbb{S}^2$:*

$$\mu^\infty = \mu_{Leb} + \sum_{\alpha_0 \in \pi\mathbb{Q}\cap[-\pi/2,\pi/2]} \mu_{\alpha_0},$$

satisfying:

(i) *Each of the measures in the above decomposition is carried by $\overline{\mathbb{D}} \times \mathbb{R}_t \times \mathbb{S}^2 \cap \{|\xi|^2 = 2H\}$ and by the cone $\{|J| \leq E\}$, and is invariant under the billiard flow.*

(ii) *The measure μ_{Leb} does not depend on t and is of the form $\mu_{Leb} = \int_{E>0,|J|\leq E} \lambda_{E,J} d\mu'(E, J)dt$ for some nonnegative measure μ', defined on the set of pairs (E, J) modulo homotheties.*

(iii) *For every $\alpha_0 \in \pi\mathbb{Q} \cap (-\pi/2, \pi/2)$, the measure μ_{α_0} is carried by the set $\mathcal{I}_{\alpha_0} \cap \{H = E^2/2\}$ and is defined by:*

$$\int_{\mathcal{I}_{\alpha_0}} a d\mu_{\alpha_0} = \int_{\mathcal{I}_{\alpha_0}} \mathrm{Tr}_{L^2(0,2\pi)} \left(m^{\alpha_0}_{\langle a\rangle_{\alpha_0}} \sigma_{\alpha_0} \right) d\ell_{\alpha_0} dt, \quad \textit{for all } \; a \in \mathcal{S}_0,$$

where $\ell_{\alpha_0}(d\omega, dE, dH)$ is a non-negative measure on $\mathbb{T}_\omega \times \{E^2+2|H| = 2\}$ carried by $\{H = E^2/2\}$ and

$$\sigma_{\alpha_0} : \mathbb{T}_\omega \times \{E^2 + 2|H| = 2\} \times \mathbb{R}_t \to \mathcal{L}^1_+ \left(L^2(0, 2\pi)\right),$$

is integrable with respect to ℓ_{α_0}, continuous in t and takes values in the set of nonnegative trace-class operators on $L^2(0, 2\pi)$, and, for ℓ_{α_0}-almost every (ω, E, H), satisfies $\mathrm{Tr}(\sigma_{\alpha_0}(\omega, E, H, t)) = 1$.
Moreover, for ℓ_{α_0}-almost every (ω, E, H), we have

$$\sigma_{\alpha_0}(\omega, E, H, t) = U_{\alpha_0,\omega}(t)\sigma_{\alpha_0}(\omega, E, H, 0)U^*_{\alpha_0,\omega}(t).$$

Finally, ℓ_{α_0} and $\sigma_{\alpha_0}(\cdot, 0)$ only depend on the sequence of initial conditions (u_n^0).

(iv) *For $\alpha_0 = \pm\frac{\pi}{2}$, μ_{α_0} does not depend on t, it is a measure carried by the set $\mathcal{I}_{\pm\frac{\pi}{2}} \cap \{H = E^2/2\}$ (which consists of vectors tangent to $\partial\mathbb{D}$) and is invariant under rotations around the origin.*

In point (iii), the measure ℓ_{α_0} arises as an element of the dual of the space of continuous homogeneous functions (satisfying $a(\omega, \lambda E, \lambda^2 H) = a(\omega, E, H)$). This space has been identified with the space of continuous functions on $\mathbb{R}/2\pi\mathbb{Z} \times \{(E, H), E^2 + 2|H| = 2\}$.

To conclude the description of μ_{ml}, it now remains to describe more precisely μ_c. After extraction, we can always assume that the sequence of initial conditions (u_n^0) has a weak limit u^0 in $L^2(\mathbb{D})$.

Theorem 2.8 *Assume that (u_n^0) has a weak limit u^0 in $L^2(\mathbb{D})$, and set $u(x, t) = [U_V(t)u^0](x)$. Then, for all $a \in \mathcal{S}_0$, we have*

$$\langle \mu_c, a \rangle = \left\langle u, \mathrm{Op}_1\left(a(z, \xi, t, H)\right) u \right\rangle_{L^2(\mathbb{R}^2 \times \mathbb{R})}. \tag{2.19}$$

As a consequence, the projection of μ_c on $\mathbb{D} \times \mathbb{R}_t$ is a nonnegative Radon measure, which is absolutely continuous, and continuous with respect to t.

We refer to Remark 2.2 for the meaning of (2.19). The proof of this result is given in Sect. 6.1.

2.6 Link between microlocal and semiclassical Wigner measures

Let us clarify the link between the two approaches in the context of the present article (see also [22,25] for a related discussion).

2.6.1 *Sequences h_n-oscillating from above and below*

As was said, if (u_n^0) is h_n-oscillating from above and below, the semiclassical Wigner measures convey more information than the microlocal ones. In fact, if (u_n^0) is h_n-oscillating from above and below (with $h_n \to 0$), we have for $a \in \mathcal{S}_0$, and χ a smooth cut-off function that equals 1 in a neighborhood of the origin,

$$\begin{aligned}
\lim_{n\to\infty} W_{u_n}(a) &= \lim_{n\to\infty} \left\langle u_n, \mathrm{Op}_1\left(a(z, \xi, t, H)\left(\chi\left(\frac{h_n^2(|\xi|^2 + |H|)}{R}\right)\right.\right.\right. \\
&\quad \left.\left.\left. - \chi\left(\frac{h_n^2(|\xi|^2 + |H|)}{\epsilon}\right)\right)\right)u_n\right\rangle_{L^2(\mathbb{R}^2\times\mathbb{R})} + o(1)_{\epsilon\to 0, R\to+\infty} \\
&= \lim_{n\to\infty} \left\langle u_n, \mathrm{Op}_1\left(a_{\mathrm{hom}}(z, h_n\xi, t, h_n^2 H)\left(\chi\left(\frac{h_n^2(|\xi|^2 + |H|)}{R}\right)\right.\right.\right. \\
&\quad \left.\left.\left. - \chi\left(\frac{h_n^2(|\xi|^2 + |H|)}{\epsilon}\right)\right)\right)u_n\right\rangle_{L^2(\mathbb{R}^2\times\mathbb{R})} + o(1)_{\epsilon\to 0, R\to+\infty}
\end{aligned}$$

$$= \lim_{n\to\infty} W_{u_n}^{h_n}\left(a_{\text{hom}}\left(\chi\left(\frac{(|\xi|^2+|H|)}{R}\right)-\chi\left(\frac{(|\xi|^2+|H|)}{\epsilon}\right)\right)\right) + o(1)_{\epsilon\to 0, R\to+\infty}$$

From this identity, one sees that if W_{u_n} converges weakly to μ_{ml} and $W_{u_n}^{h_n}$ converges weakly to μ_{sc}, and if (u_n^0) is h_n-oscillating from above and below, we have

$$\mu_{ml}(a) = \mu_{sc}(a_{\text{hom}}).$$

The right-hand side is well-defined since μ_{sc} is a nonnegative measure which is bounded on sets of the form $\overline{\mathbb{D}} \times \mathbb{R}^2 \times [-T, T] \times \mathbb{R}$ for any $T > 0$.

2.6.2 Sequences not h_n-oscillating from below

On the other hand, if in Theorem 2.5 the sequence (u_n^0) is not h_n-oscillating from below, then μ_{sc} does charge the set $\{\xi = 0\}$, and we have for any compactly supported function a and any cut-off $\chi \in C_c^\infty((-1, 1))$ with $\chi(0) = 1$,

$$\begin{aligned}\mu_{sc}\rceil_{(\xi,H)=0}(a) &= \lim_{\epsilon\to 0}\int_{T^*(\mathbb{R}^2\times\mathbb{R})} a(z,\xi,t,H)\chi^2\left(\frac{|\xi|^2+|H|}{\epsilon}\right)d\mu_{sc}\\ &= \lim_{\epsilon\to 0}\int_{T^*(\mathbb{R}^2\times\mathbb{R})} a(z,\xi,t,H)\chi^2\left(\frac{3|H|}{\epsilon}\right)d\mu_{sc}\\ &= \lim_{\epsilon\to 0}\lim_{n\to+\infty} W_{U_V(t)v_{n,\epsilon}^0}(a),\end{aligned} \tag{2.20}$$

where $v_{n,\epsilon}^0 = \chi(\frac{3h_n^2|D_t|}{\epsilon})\chi_T u_n\rceil_{t=0}$, for some $\chi_T = \chi_T(t) \in C_c^\infty(\mathbb{R})$ equal to one in a neighborhood of the t-support of a. Equality of the first two lines comes from the fact that the measure μ_{sc} is suported by $\{|\xi|^2 = 2H\}$ (see Proposition 8.3), and equality of the last two lines is proved in Appendix A (see Lemma 8.2). We see that the microlocal Wigner measures $\mu_{ml,\epsilon}$ associated with $U_V(t)v_{n,\epsilon}^0$ encompass the description of $\mu_{sc}\rceil_{(\xi,H)=0}$ (on the time interval where $\chi_T = 1$): calling $\mu_{ml,0} = \lim_{\epsilon\to 0}\mu_{ml,\epsilon}$ (in $\mathcal{S}_0'$ and after extraction), we have

$$\mu_{sc}\rceil_{(\xi,H)=0}(a(z,\xi,t,H)) = \mu_{ml,0}(a(z,0,t,0)).$$

Obtained as a limit of $\mu_{ml,\epsilon}$, the measure $\mu_{ml,0}$ also possesses the structure described in Theorem 2.7.

2.6.3 *Sequences not h_n-oscillating from above*

Finally, if (u_n^0) is not h_n-oscillating from above, we see that (using cutoff functions χ and χ_T as above),

$$\lim_{R\to+\infty}\lim_{n\to+\infty} W^{h_n}_{(1-\chi)\left(\frac{h_n^2(|D_z|^2+|D_t|)}{R}\right)\chi_T u_n}(a) = 0$$

for compactly supported a, whereas the limit

$$\lim_{R\to+\infty}\lim_{n\to+\infty} W_{(1-\chi)\left(\frac{h_n^2(|D_z|^2+|D_t|)}{R}\right)\chi_T u_n}(a)$$

does not necessarily vanish for $a \in \mathcal{S}_0$. This last limit coincides with

$$\lim_{R\to+\infty}\lim_{n\to+\infty} W_{(1-\chi)\left(\frac{3h_n^2|D_t|}{R}\right)\chi_T u_n}(a) = \lim_{R\to+\infty}\lim_{n\to+\infty} W_{U_V(t)w^0_{n,R}}(a)$$

where $w^0_{n,R} = (1-\chi)(\frac{3h_n^2|D_t|}{R})\chi_T u_n\rceil_{t=0}$, and equality of the limits is proved in Appendix A (see Lemma 8.2).

We may thus conclude from the last two paragraphs that the frequencies of u_n^0 that are of order $\gg h_n^{-1}$ or $\ll h_n^{-1}$ are better captured by the microlocal approach.

2.7 Application to the regularity of limit measures

As will be explained at the end of Sect. 5.2, Theorems 2.5 or 2.7 imply Corollary 1.1. Here, we state a more precise version of this result (say, in the semiclassical setting), and first need for this an intermediate proposition. As in Theorem 2.5, let us first assume that we are considering a sequence of initial data (u_h^0) which is h-oscillating from below.

Proposition 2.9 *Suppose that μ_{sc} is a semiclassical measure associated to (u_h) solution of* (1.1)–(1.2). *Denote by $\bar{\mu}_{sc}(dE, dJ, t)$ the image of the measure $\mu_{sc}(dz, d\xi, t, dH)$ under the moment map*

$$M : (z = (x, y), \xi, H) \mapsto (E, J) = (|\xi|, x\xi_y - y\xi_x)$$

(*velocity and angular momentum*). *Then $\bar{\mu}_{sc}$ does not depend on t.*

This proposition is proved in Sect. 5.1.

The following theorem holds—and implies Corollary 1.1 for sequences that are h-oscillating from below.

Theorem 2.10 *Define by $\mu_{E,J}(t,\cdot)$ the disintegration of $\mu_{sc}(t,\cdot)$ with respect to the variables (E,J), carried on the 2-dimensional (Lagrangian) manifold $T_{(E,J)} = \{(z,\xi) \in \overline{\mathbb{D}} \times \mathbb{R}^2, (|\xi|, x\xi_y - y\xi_x) = (E,J)\}$, i.e.*

$$\int_{\mathbb{R}_H} \int_{\overline{\mathbb{D}}\times\mathbb{R}^2} f(z,\xi,t,H)\mu_{sc}(dz,d\xi,t,dH)$$
$$= \int_{\mathbb{R}^2} \left(\int_{T_{(E,J)}} f\left(z,\xi,t,\frac{E^2}{2}\right) \mu_{E,J}(t,dz,d\xi) \right) \bar{\mu}_{sc}(dE,dJ),$$

for every bounded measurable function f, for $t \in \mathbb{R}$.

Then for $\bar{\mu}_{sc}$-almost every (E,J) with $|J| \neq E$, the measure $\mu_{E,J}(t,\cdot)$ is absolutely continuous on $T_{(E,J)}$.

Note that $|J| = E$, with $E \neq 0$, means that $T_{(E,J)} \cap \left(\overline{\mathbb{D}} \times \mathbb{R}^2\right)$ is contained in the set $\{(z,\xi), |z| = 1, z \perp \xi\}$ of tangent rays to the boundary. The restriction of $\mu_{sc}(t)$ to that set may be considered trivial, since (2.11) implies that it is invariant under rotation.

Finally, for $J = E = 0$, we can use the last lines of Sect. 2.6, combined with Theorem 2.7: the measure μ_{sc} restricted to $\{\xi = 0\} = \overline{\mathbb{D}} \times \{0\}$ is the sum of an absolutely continuous measure carried by the interior $\mathbb{D}$ and a multiple of the Lebesgue measure on $\partial\mathbb{D}$.

Remark 2.11 The analogues of Proposition 2.9 and Theorem 2.10 hold as well in the microlocal setting, for the measure μ^∞. In particular, if $\bar{\mu}^\infty$ is the image of $\mu^\infty(t)$ under the map $(z,\xi,t,H) \mapsto (E,J)$, this measure is independent of t. Concerning the compact part μ_c, it is clear from Theorem 2.8 that its projection on $\overline{\mathbb{D}} \times \mathbb{R}_t$ is absolutely continuous, and its density is $|u(x,t)|^2$.

Remark 2.12 Proposition 2.9 (and Remark 2.11) allows us to complete the proof of the necessity of the assumption $\Omega \cap \partial\mathbb{D} \neq \emptyset$ in Theorem 1.2 when V does not identically vanish (see the discussion in Sect. 1.6.2). Taking for instance as initial data $u_n^0 := \psi_{n,0}^\pm / \|\psi_{n,0}^\pm\|_{L^2(\mathbb{D})}$ (see (1.7)) with and $n \to \infty$, then one has $|u_n^0|^2 dx \rightharpoonup (2\pi)^{-1}\delta_{\partial\mathbb{D}}$ (see Sect. 1.3); more precisely, the Wigner measures associated with the initial data $u_n^0 := \psi_{n,0}^\pm / \|\psi_{n,0}^\pm\|_{L^2(\mathbb{D})}$ concentrate on the set $\{|J| = E\}$. Combined with Proposition 2.9 (and Remark 2.11), this shows that $\bar{\mu}^{ml}$ is entirely carried by the set $\{|J| = E\}$, and thus μ^{ml} itself does not charge the interior of the disk, where $|J| < E$. This shows that (1.5) cannot hold if Ω does not touch the boundary.

2.8 Measures at the boundary

In this section, we define and compare different measures on $\partial\mathbb{D}$. Given any invariant measure for the billiard flow (for instance μ_{sc} or μ_{ml} obtained from sequences of solutions of (1.1)), we first define the associated "projected measure" on the boundary. Second, we define semiclassical and microlocal measures associated with the Neumann trace at the boundary of sequences of solutions of (1.1). We finally explain the links between these objects.

2.8.1 Projection on the boundary of an invariant measure

We recall the following standard construction from the theory of Poincaré sections in dynamical systems. We define the sets

$$S^{\pm} = \{(z,\xi), |z| = 1, \pm\xi \cdot z > 0\}, \quad S = S^{+} \cup S^{-}, \tag{2.21}$$

which represent the set of outward (S^+) or inward (S^-) pointing vectors, and the set of nontangential vectors (S).

When $(z,\xi) \in S^+$, we denote as above by $\alpha(z,\xi) = -\arcsin\left(\frac{J(z,\xi)}{|\xi|}\right)$ the angle of the vector ξ with the normal at z to the disk. The map

$$\begin{aligned} P : \{(z,\xi,\tau) \in S^+ \times \mathbb{R}, \tau \in [0, 2\cos\alpha(z,\xi)]\} &\to \overline{\mathbb{D}} \times \mathbb{R}^2 \\ (z,\xi,\tau) &\mapsto \left(z + \frac{\tau}{|\xi|}\sigma_z(\xi), \sigma_z(\xi)\right) \end{aligned}$$

is a measurable bijection onto its image $S \cup (\mathbb{D} \times \mathbb{R}^2)$, and $\pi \circ P$ is a measurable bijection onto its image (recall that π is the projection from $\overline{\mathbb{D}} \times \mathbb{R}^2$ to $\mathbb{W}$). If μ is a nonnegative measure on $S \cup (\mathbb{D} \times \mathbb{R}^2)$ which does not charge S, and such that $\pi_*\mu$ is invariant under the billiard flow, then $P_*^{-1}\mu$ must be of the form

$$P_*^{-1}\mu = \mu^S \otimes d\tau$$

where μ^S is a measure on S^+ which is invariant under the first return map

$$(z,\xi) \mapsto \left(z + \frac{2\cos\alpha(z,\xi)}{|\xi|}\sigma_z(\xi), \sigma_z(\xi)\right).$$

This implies that

$$\begin{aligned} \int_{\overline{\mathbb{D}} \times \mathbb{R}^2} \xi.\partial_z a \, d\mu &= \int_{\{(z,\xi,\tau) \in S^+ \times \mathbb{R}, \tau \in [0, 2\cos\alpha(z,\xi)]\}} |\xi| \partial_\tau (a \circ P) d\mu^S \otimes d\tau \\ &= \int_{S^+} |\xi| \left(a\left(z + \frac{2\cos\alpha(z,\xi)}{|\xi|}\sigma_z(\xi), \sigma_z(\xi)\right) - a(z, \sigma_z(\xi)) \right) \mu^S(dz, d\xi) \\ &= \int_{S^+} |\xi| \left(a(z,\xi) - a(z,\sigma_z(\xi))\right) \mu^S(dz, d\xi). \end{aligned} \tag{2.22}$$

Note that the total mass of μ is $\int d\mu = \int_{S^+} 2\cos\alpha(z,\xi)\mu^S(dx, d\xi)$.

2.8.2 Semiclassical measure associated to Neumann trace

Let (u_h^0) be a family of initial conditions, normalized in $L^2(\mathbb{D})$. When we look at the semiclassical Wigner distributions (2.5), where we use *compactly supported* symbols, Remark 2.4 and Lemma 8.2 show that we may truncate (u_h^0) in frequency and assume, without changing the limit as $h \to 0$, that $u_h^0 \in H_0^1(\mathbb{D})$, $\|\nabla u_h^0\|_{L^2(\mathbb{D})} = O(h^{-1})$. Proposition 8.1 then entails that the boundary data $h\partial_n(U_V(t)u_h^0)$ form a bounded sequence in $L^2_{\mathrm{loc}}(\mathbb{R} \times \partial\mathbb{D})$.

Now, let $\mu_{sc}^\partial \in \mathcal{M}_+(T^*\partial\mathbb{D} \times T^*\mathbb{R})$ be a semiclassical measure associated with the boundary data $h\partial_n u_h(t)$ defined by quantizing test functions on $T^*\partial\mathbb{D} \times T^*\mathbb{R}$ with the same scaling $(hj, h^2 H)$ in the cotangent variables as in (2.8). Then μ_{sc}^∂ is carried by the set (see [25])

$$\overline{\mathscr{H}} = \{(u, j, t, H) \in T^*\partial\mathbb{D} \times T^*\mathbb{R}, |j| \le \sqrt{2H}\}.$$

If μ_{sc} and μ_{sc}^∂ are obtained through the same sequence of initial data, then we have the relation (see again [25])

$$\begin{aligned}\int_{\overline{\mathbb{D}}\times\mathbb{R}^2\times\mathbb{R}\times\mathbb{R}} &\xi \cdot \partial_z a \;\; \mu_{sc}(dz, d\xi, t, dH)dt \\ &= \int_{\mathscr{H}} \frac{a(u, \xi^+(j,H)) - a(u, \xi^-(j,H))}{2\sqrt{2H - j^2}} \mu_{sc}^\partial(du, dj, dt, dH),\end{aligned} \tag{2.23}$$

valid for any smooth function a. For $(u, j) \in T^*\partial\mathbb{D}$ with $|j| \le \sqrt{2H}$, the vectors $\xi^\pm(j, H)$ are the two vectors (pointing outwards and inwards, and coinciding if $|j| = \sqrt{2H}$) in $T_u^*\mathbb{R}^2$ of norm $= \sqrt{2H}$, whose projection to $T_u^*\partial\mathbb{D}$ is j. Note that the expression under the integral on the right hand side of (2.23) has a well-defined finite limit as $|j| \to \sqrt{2H}$. Let us point out three consequences of Identity (2.23):

- First, the measure μ_{sc} does not charge the set S defined in (2.21) (otherwise the left-hand side of (2.23) would define a distribution of order 1 which is not a measure). Note that (2.23) is stronger than (2.11).
- Second, let $\mu_{sc}^S(t)$ be the measure associated to $\mu_{sc}(t)$ as in Sect. 2.8.1. Comparing (2.23) with (2.22), we see that for any a vanishing in a neighborhood of $\overline{S^-}$, we have

$$\int_{\{(u,j)\in T^*\partial\mathbb{D}, |j|<\sqrt{2H}\}\times\mathbb{R}\times\mathbb{R}} a(u, \xi^+(j, H), t, H)\mu_{sc}^{\partial}(du, dj, dt, dH)$$
$$= \int_{S^+\times\mathbb{R}\times\mathbb{R}} 2|\xi|^2 \cos\alpha(z,\xi)a(z,\xi)\mu_{sc}^{S}(dz, d\xi, t, dH)dt.$$

- Third, (2.23) implies

$$\int_{T^*\partial\mathbb{D}\times\mathbb{R}\times\mathbb{R}} |\xi|^2 a(z, \xi, t, H)\mu_{sc}(dz, d\xi, t, dH)dt$$
$$= \int_{\{(u,j)\in T^*\partial\mathbb{D}, |j|=\sqrt{2H}\}\times\mathbb{R}\times\mathbb{R}} a(u, j, t, H)\mu_{sc}^{\partial}(du, dj, dt, dH).$$

  This identity can be obtained by replacing a by $\delta\chi(\frac{1-|z|}{\delta})\chi(\frac{z\cdot\xi}{\delta})a$ in (2.23) where $\chi \in C_c^\infty(\mathbb{R})$ satisfies $\chi(0) = \chi'(0) = 1$, and then letting δ tend to zero. In particular, note that $\mu_{sc}^{\partial}\rceil_{H=0}$ vanishes, since $H = 0$ corresponds to $\xi = 0$ on the left-hand side.

Identities (2.11) and (2.23) are essentially proved in [25] (see also [56]) for general domains and for time-independent solutions of (1.1); we do not reproduce the proofs here.

2.8.3 Microlocal measure associated to the Neumann trace

The sequences considered here $u_n = U_V(t)u_n^0$ are bounded in $L^\infty(\mathbb{R}; L^2(\mathbb{D}))$. Since normal traces are not convenient to work with at this level of regularity, the definition of associated microlocal measures needs a little care.

For this, let us define $\psi \in C^\infty(\mathbb{R})$, such that $\psi = 0$ on $(-\infty, 1]$ and $\psi = 1$ on $[2, +\infty)$ and the operator $A(D_t) = \mathrm{Op}_1(\frac{\psi(H)}{\sqrt{2H}})$. We have the following regularity result.

Lemma 2.13 *For all $\varphi \in C^\infty(\mathbb{R}_t \times \overline{\mathbb{D}}_z)$ with compact support in the first variable $t \in \mathbb{R}$, there exists a constant $C = C(\varphi, \psi) > 0$ such that for any $u^0 \in L^2(\mathbb{D})$, the associated solution $u(t) = U_V(t)u^0$ of (1.1)–(1.2) satisfies*

$$\|A(D_t)\varphi u\|_{L^2(\mathbb{R};H^1(\mathbb{D}))} \le C\|u^0\|_{L^2(\mathbb{D})}.$$

This Lemma is proved at the end of Appendix A. We now define, for any $g \in C_c^\infty(\mathbb{R})$, the sequence $\tilde{u}_n = A(D_t)g(t)u_n$, solution of

$$\left(D_t + \frac{1}{2}\Delta\right)\tilde{u}_n = A(D_t)\left(g(t)V(t,z) + ig'(t)\right)u_n. \tag{2.24}$$

As a consequence of Lemma 2.13, we have $\|\tilde{u}_n\|_{L^2(\mathbb{R};H^1(\mathbb{D}))} \leq C\|u_n^0\|_{L^2(\mathbb{D})}$ together with

$$\|A(D_t)\left(g(t)V(t,z)u_n + ig'(t)\right)u_n\|_{L^2(\mathbb{R};H^1(\mathbb{D}))} \leq C\|u_n^0\|_{L^2(\mathbb{D})}.$$

This, together with Eq. (2.24) and basic energy estimates (see (8.3)) then implies that $\|\tilde{u}_n\|_{L^\infty((-T,T);H^1(\mathbb{D}))} \leq C_T\|u_n^0\|_{L^2(\mathbb{D})}$ for any $T > 0$, and that $A(D_t)g(t)\partial_n u_n = \partial_n \tilde{u}_n$ is bounded in $L^2(\mathbb{R} \times \partial\mathbb{D})$ by $\|u_n^0\|_{L^2(\mathbb{D})}$, according to the hidden regularity of Proposition 8.1. Hence, if we take g to be constant equal to 1 on the support of a, after extraction of subsequences, the following limit exists

$$\langle \mu_{ml}^\partial, a\rangle = \lim_{R\to\infty}\lim_{n\to+\infty}\left\langle \partial_n\tilde{u}_n, \mathrm{Op}_1\left(\left(1-\chi\left(\frac{|H|}{R^2}\right)\right)a(u,j,t,H)\right)\partial_n\tilde{u}_n\right\rangle_{L^2(\partial\mathbb{D}\times\mathbb{R})},$$

for symbols $a \in C^\infty(T^*(\partial\mathbb{D}\times\mathbb{R}))$, compactly supported in the variables z, t, such that

$$a(u,j,t,H) = a(u,\lambda j,t,\lambda^2 H), \quad \text{for } |H| > R_0 \quad \text{and } \lambda \geq 1.$$

Then μ_{ml}^∂ is carried by the set $\overline{\mathscr{H}}$. If moreover μ_{ml} and μ_{ml}^∂ are obtained through the same sequence of initial data, then we have the relation (see again [25])

$$\begin{aligned}\int_{\overline{\mathbb{D}}\times\mathbb{R}_t\times\mathbb{S}^2_{\xi,H}} \frac{\xi}{\sqrt{2H}}\cdot\partial_z a\ \mu^\infty(dz,d\xi,t,dH)dt &= \int_{|j|\leq\sqrt{2H}} \frac{1}{2}\left(\frac{2H}{2H-j^2}\right)^{\frac{1}{2}} \\ &\times\left(a(u,\xi^+(j,H)) - a(u,\xi^-(j,H))\right)\mu_{ml}^\partial(du,dj,dt,dH)\end{aligned} \tag{2.25}$$

valid for any $a \in \mathcal{S}_0$. The vectors $\xi^\pm(j,H)$ are the two vectors (pointing outwards and inwards) in $T_u^*\mathbb{R}^2$ of norm $=\sqrt{2H}$, whose projection to $T_u^*\partial\mathbb{D}$ is j. As above, this implies that μ^∞ does not charge the set S; we then denote by $\mu_{ml}^S(t)$ the measure associated to $\mu^\infty(t)$ as in Sect. 2.8.1. Comparing with (2.22), we see that for any $a \in \mathcal{S}_0$, we have

$$\begin{aligned}&\int_{(u,j)\in T^*\partial\mathbb{D},|j|<\sqrt{2H}} a(u,\xi^+(j,H),t,H)\mu_{ml}^\partial(du,dj,dt,dH) \\ &\quad= \int_{S^+} 2\cos\alpha(z,\xi)a(z,\xi)\mu_{ml}^S(dz,d\xi,t,dH)dt.\end{aligned} \tag{2.26}$$

Moreover, (2.25) implies

$$\int_{(u,\xi)\in T^*\partial\mathbb{D},|\xi|=\sqrt{2H}} a(z,\xi,t,H)\mu^{\infty}(dz,d\xi,t,dH)dt$$
$$= \int_{|j|=\sqrt{2H}} a(u,j,t,H)\mu^{\partial}_{ml}(du,dj,dt,dH). \tag{2.27}$$

These links between the different measures shall be particularly useful when proving the boundary observability result of Theorem 1.3.

2.9 Plan of the proofs

Section 3 first deals with the understanding of action-angle coordinates and the appropriate decomposition of measures that are invariant by the billiard flow. Section 3.1 discusses in more detail the coordinates described in the introduction, in which the dynamics of the billiard can be integrated, and introduces the Fourier Integral Operator corresponding to this change of coordinates. Section 3.2 reduces the study of invariant measures on the disk to their restriction to all invariant tori of the dynamics (more precisely, their restriction to the level sets $\mathcal{I}_\alpha$, which are unions of invariant tori).

Sections 4 and 5 are devoted to the proof of Theorem 2.5 (semiclassical version of the result). In Sect. 4, we perform the second microlocalization on a level set $\mathcal{I}_\alpha$: we start by introducing the adapted class of symbols in Sect. 4.1 and the appropriate coordinates (which are a modification of the action-angle coordinates) in Sect. 4.2. This allows us to construct the two different second-microlocal measures in Sect. 4.3. We then prove their structure properties in Sects. 4.4 and 4.5. To complete the analysis, we prove that they obey invariance laws in Sect. 4.4 and 4.6 respectively. Section 5 then concludes the proof of Theorem 2.5.

In Sect. 6 we explain how to adapt the proof to obtain the microlocal version, Theorem 2.7.

The observability inequalities of Theorems 1.2 and 1.3 are then derived in Sect. 7.

Appendices A and B collect generalities on solutions of Schrödinger equations: Appendix A is concerned with the localization in frequency of oscillating sequences of solutions, and Appendix B states and proves the L^∞ regularity in time of Wigner measures.

Finally, Appendices C, D and E are devoted to the technical calculations needed to change coordinates from polar to action-angle ones.

3 Action-angle coordinates and decomposition of invariant measures

3.1 Action-angle coordinates and their quantization

Recall that the change of coordinates Φ, mapping action-angle coordinates to cartesian ones, is introduced in Sect. 2.2 (see (2.6)). The map

$$\begin{aligned}\Phi : \{(s,\theta,E,J) \ : \ E>0, \theta\in\mathbb{R}/2\pi\mathbb{Z}, s\in\mathbb{R}, J\in\mathbb{R}\} \\ \longrightarrow \{(z,\xi)\in\mathbb{R}^2\times\mathbb{R}^2 \ : \ \xi\neq 0\}\end{aligned}$$

is a diffeomorphism satisfying, in particular,

$$\begin{aligned}&\Phi^{-1}\left(\mathbb{D}\times(\mathbb{R}^2\backslash\{0\})\right)\\ &\quad=\left\{(s,\theta,E,J) \ : \ (\theta,E)\in\mathbb{R}/2\pi\mathbb{Z}\times(0,\infty), (J/E)^2+s^2<1\right\}.\end{aligned}$$

Write for $\theta\in\mathbb{T}=\mathbb{R}/2\pi\mathbb{Z}$,

$$\omega(\theta) := (-\sin\theta, \cos\theta)\,;$$

the transformation Φ admits the generating function

$$S(z,s,\theta,E) = E\omega(\theta)\cdot z - Es,$$

meaning that

$$\begin{aligned}\text{Graph}\,\Phi &= \{(s,\theta,E,J,z,\xi) \ : \ (z,\xi)=\Phi(s,\theta,E,J)\}\\ &= \left\{(s,\theta,E,J,z,\xi) \ : \ \frac{\partial S}{\partial E}=0, \xi=\frac{\partial S}{\partial z}, J=-\frac{\partial S}{\partial\theta}, E=-\frac{\partial S}{\partial s}\right\}.\end{aligned}$$

The existence of such a generating function implies that the diffeomorphism Φ preserves the symplectic form (see for instance [61, Theorem 2.7]), i.e.

$$d\xi_x\wedge dx + d\xi_y\wedge dy = dE\wedge ds + dJ\wedge d\theta.$$

Using this generating function we define a unitary operator that quantizes the canonical transformation Φ. The operator

$$\mathscr{U}f(s,\theta) = (2\pi)^{-3/2}\int_0^\infty\int_{\mathbb{R}^2} e^{-iS(z,s,\theta,E)}f(z)\sqrt{E}dzdE, \tag{3.1}$$

mapping functions on $\mathbb{R}^2_z$ to functions on $\mathbb{R}_s\times\mathbb{R}_\theta$ that are 2π-periodic in the variable θ, is in fact a classical Fourier Integral Operator associated with Φ^{-1} (the choice of the term $\sqrt{E}$ in this expression is for $\mathscr{U}$ to be unitary, see Lemma 3.1 below). Noting that S is linear in E, the change of variables $E \leftarrow E/h$ allows to recast $\mathscr{U}$ in a semiclassical scaling as

$$\mathscr{U} f(s,\theta) = (2\pi h)^{-3/2} \int_0^\infty \int_{\mathbb{R}^2} e^{-i\frac{E\omega(\theta)\cdot z - Es}{h}} f(z)\sqrt{E}\, dz\, dE, \quad \text{for all } h > 0. \tag{3.2}$$

Note also from (3.1) that $\mathscr{U} f$ can be rewritten as

$$\mathscr{U} f(s,\theta) = \int_0^\infty e^{iEs} \widehat{f}(E\omega(\theta)) \sqrt{E} \frac{dE}{(2\pi)^{3/2}},$$

where $\widehat{f}(\xi) = \int_{\mathbb{R}^2} e^{-iz\cdot\xi} f(z) dz$ stands for the usual Fourier transform of f.

Therefore, the Fourier transform with respect to s of $\mathscr{U} f(s,\theta)$ is merely:

$$(2\pi)^{-1/2} \widehat{f}(E\omega(\theta)) \mathbb{1}_{[0,\infty)}(E) \sqrt{E}.$$

From this it is clear that for any symbol $\phi : \mathbb{R} \to \mathbb{R}$ one has:

$$\phi(D_s) \mathscr{U} f = \mathscr{U} \phi(|D_z|) f,$$

and, by Plancherel's theorem,

$$\begin{aligned}\int_0^{2\pi}\int_0^\infty \mathscr{U} f(s,\theta)\overline{\mathscr{U} g(s,\theta)} ds\, d\theta &= \int_0^{2\pi}\int_0^\infty \widehat{f}(E\omega(\theta))\, \overline{\widehat{g}(E\omega(\theta))} E \frac{ds\, d\theta}{(2\pi)^2} \\ &= \langle g, f\rangle_{L^2(\mathbb{R}^2)}.\end{aligned}$$

In particular, the following Lemma has been proved:

Lemma 3.1 (i) *The operator $\mathscr{U}$ is unitary from $L^2(\mathbb{R}^2)$ to $L^2(\mathbb{R}\times\mathbb{R}/2\pi\mathbb{Z})$:* $\mathscr{U}^*\mathscr{U} = I$.

(ii) *For $f \in C_c^\infty(\mathbb{R}^2)$, we have $\partial_s^2 \mathscr{U} f = \mathscr{U}\Delta f$.*

As a consequence,

$$-\mathscr{U}\Delta\mathscr{U}^* = -\partial_s^2.$$

Note that the adjoint operator $\mathscr{U}^*$ is described explicitly for $g \in C_c^\infty(\mathbb{R}_s \times (\mathbb{R}/2\pi\mathbb{Z})_\theta)$ by

$$\mathscr{U}^* g(z) = (2\pi)^{-3/2} \int_0^\infty \int_{\mathbb{R}_s\times(\mathbb{R}/2\pi\mathbb{Z})_\theta} e^{+iS(z,s,\theta,E)} g(s,\theta)\, ds\, d\theta \sqrt{E}\, dE,$$

for $z \in \mathbb{R}^2$, and shall be also used in the semiclassical scaling as

$$\mathscr{U}^* g(z) = (2\pi h)^{-3/2} \int_0^\infty \int_{\mathbb{R}_s \times (\mathbb{R}/2\pi\mathbb{Z})_\theta} e^{+i \frac{S(z,s,\theta,E)}{h}} g(s,\theta) ds d\theta \sqrt{E} dE,$$
$$z \in \mathbb{R}^2, \ h > 0. \tag{3.3}$$

Notation. We denote by $P_0(z,\xi) = \frac{|\xi|^2}{2}$ the hamiltonian generating the geodesic flow in $\mathbb{R}^2 \times \mathbb{R}^2$; and $P_1(z,\xi) = x\xi_y - y\xi_x$ the hamiltonian generating the (unit speed) rotation. We denote by $X_{P_0} = \xi \cdot \partial_z$ and $X_{P_1} = z^\perp \cdot \partial_z + \xi^\perp \cdot \partial_\xi$ the corresponding hamiltonian vector fields on $T^*\mathbb{R}^2$. We denote by $G^\tau(z,\xi) = (z + \tau\xi, \xi)$ the geodesic flow (generated by P_0) and R^τ the flow generated by P_1 (rotation of angle τ of both z and ξ). Note that R^τ is given explicitely by $R^\tau(z,\xi) = (R(\tau)z, R(\tau)\xi)$, where $R(\tau)$ is the rotation matrix of angle τ.

In the new coordinates, these hamiltonians and vector fields are slightly simpler since $P_0 \circ \Phi = \frac{E^2}{2}$, $P_1 \circ \Phi = J$ together with $X_{P_0 \circ \Phi} = E\partial_s$ and $X_{P_1 \circ \Phi} = \partial_\theta$. In these coordinates, the flows G^τ and R^τ thus admit the expressions

$$(s, \theta, E, J) \mapsto (s + \tau E, \theta, J, E), \quad \tau \in \mathbb{R},$$

and

$$(s, \theta, E, J) \mapsto (s, \theta + \tau, J, E), \quad \tau \in \mathbb{R},$$

respectively. Very often, we shall (with a slight abuse of notation) use the letter J to mean the function P_1, and E for the function $\sqrt{2P_0}$.

3.2 Decomposition of an invariant measure of the billiard

This section aims at describing properties shared by all measures μ invariant by the billiard flow (even if they are not necessarily linked with solutions to a partial differential equation). It collects a few facts that will be useful in the next sections when studying measures arising from solutions of the Schrödinger equation (1.1).

Let $(z,\xi) \in \overline{\mathbb{D}} \times \mathbb{R}^2$. There exist $t_1 \leq 0, t_2 \geq 0$ such that $|z + t_1\xi| = |z + t_2\xi| = 1$. Note that if $(z,\xi) \in \mathbb{D} \times \mathbb{R}^2$, then t_1 and t_2 are unique and $t_1 > 0, t_2 < 0$ (see Fig. 2).

Recall that $\alpha \in [-\pi/2, \pi/2]$ (defined in Sect. 2.2) is the oriented angle between $-(z+t_1\xi)$ and ξ (that is, the angle between the velocity ξ and the inner normal to the disk, at the point where the oriented straight line $\{z + t\xi, t \in \mathbb{R}\}$ first hits the disk). See Fig. 2. One has the expression

$$\alpha = -\arcsin\left(\frac{x\xi_y - y\xi_x}{|\xi|}\right).$$

Our work is based on the following partition of phase space:

$$\overline{\mathbb{D}} \times (\mathbb{R}^2\backslash\{0\}) = \alpha^{-1}(\pi\mathbb{Q} \cap [-\pi/2, \pi/2]) \sqcup \alpha^{-1}(\mathbb{R}\backslash\pi\mathbb{Q}),$$

from which the following lemma follows.

Lemma 3.2 *Let μ be any finite, nonnegative Radon measure*[1] *on $\overline{\mathbb{D}} \times \mathbb{R}^2$. Then μ decomposes as a sum of nonnegative measures:*

$$\mu = \mu\rceil_{\alpha\notin\pi\mathbb{Q}} + \sum_{r\in\mathbb{Q}\cap[-1/2,1/2]} \mu\rceil_{\alpha=r\pi} + \mu\rceil_{\xi=0}. \tag{3.4}$$

Note that the functions P_0, P_1, and thus also α, are preserved by the symmetry σ, and hence are well-defined on the billiard phase space $\mathbb{W}$. Thus the previous lemma applies as well to measures on $\mathbb{W}$.

In what follows, we shall call *nonnegative invariant measure* a nonnegative Radon measure on $\mathbb{W}$ which is invariant under the billiard flow. We shall extend this terminology to measures μ defined a priori on $\overline{\mathbb{D}} \times \mathbb{R}^2$, to mean that $\pi_*\mu$ (the image of μ under the projection π) is invariant under the billiard flow ϕ^τ on $\mathbb{W}$.

Lemma 3.3 *Let μ be a nonnegative invariant measure on $\mathbb{W}$. Then every term in the decomposition (3.4) is a nonnegative invariant measure, and $\mu\rceil_{\alpha\notin\pi\mathbb{Q}}$ is invariant under the rotation flow (R^τ), as well as $\mu\rceil_{\alpha=\pm\pi/2}$.*

The rotation flow (R^τ) is well defined on $\mathbb{W}$, so the last sentence makes sense. The assertion for $\alpha = \pm\pi/2$ comes from the fact that the rotation flow coincides with the billiard flow (up to time change) on the set $\{\alpha = \pm\pi/2\}$. The assertion for $\alpha \notin \pi\mathbb{Q}$ is a standard fact. It comes from the remark that, for any given value $\alpha_0 \notin \pi\mathbb{Q}$, we can find $T = T(\alpha_0) > 0$ such that ϕ^T coincides with an irrational rotation on the set $\{\alpha = \alpha_0\}$. Thus, the measures $\mu\rceil_{\alpha\notin\pi\mathbb{Q}}$ and $\mu\rceil_{\alpha=\pm\pi/2}$ are of the form described in Theorem 2.5 (ii) and (iv), and there is nothing more to say.

Now consider a term $\mu\rceil_{\alpha=r_0\pi}$, where $r_0 \in \mathbb{Q}\cap(-1/2, 1/2)$ is fixed. Let us denote $\alpha_0 = \pi r_0$. Introduce the vector field on $T^*\mathbb{R}^2$:

$$(\alpha_0 - \alpha)X_{P_1} + \frac{\cos\alpha}{E}X_{P_0}.$$

On the set $\mathcal{I}_{\alpha_0} = \{J = -E\sin\alpha_0\}$ it coincides with X_{P_0} up to a constant factor. We shall denote by $\phi^\tau_{\alpha_0}$ the flow on $\mathbb{W}$ generated by $(\alpha_0-\alpha)X_{P_1} + \frac{\cos\alpha}{E}X_{P_0}$ with

[1] We denote by $\mathcal{M}_+(\overline{\mathbb{D}} \times \mathbb{R}^2)$ the set of all such measures.

reflection on the boundary of the disk. More precisely, we have the following definition.

Definition 3.4 For $\alpha \in \pi\mathbb{Q} \cap (-\pi/2, \pi/2)$, we denote by $\phi^{\tau}_{\alpha_0}$ the unique continuous flow defined on $\mathbb{W}$ such that

$$\phi^{\tau}_{\alpha_0}(z, \xi) = R^{(\alpha_0-\alpha)\tau}\left(z + \tau\frac{\cos\alpha}{|\xi|}\xi, \xi\right)$$

whenever $z \in \mathbb{D}$ and $z + \tau\frac{\cos\alpha}{|\xi|}\xi \in \mathbb{D}$, with $\alpha = -\arcsin\frac{P_1(z,\xi)}{|\xi|}$.

In the coordinates (s, θ, E, J), this flow simply reads

$$\phi^{\tau}_{\alpha_0} \circ \Phi(s, \theta, E, J) = \Phi(s + \tau\cos\alpha, \theta + (\alpha_0 - \alpha)\tau, E, J),$$
$$\alpha = -\arcsin(J/E),$$

with reflection on the boundary of the disk.

All its orbits are periodic: actually, we determined the coefficients $(\alpha_0 - \alpha)$ and $\frac{\cos\alpha}{E}$ precisely for that purpose, see Fig. 3. Some trajectories of the flow are represented on Fig. 4.

Since $\phi^{\tau}_{\alpha_0}$ coincides with the billiard flow on the set $\{\alpha = \alpha_0\}$, we have now proved the following lemma.

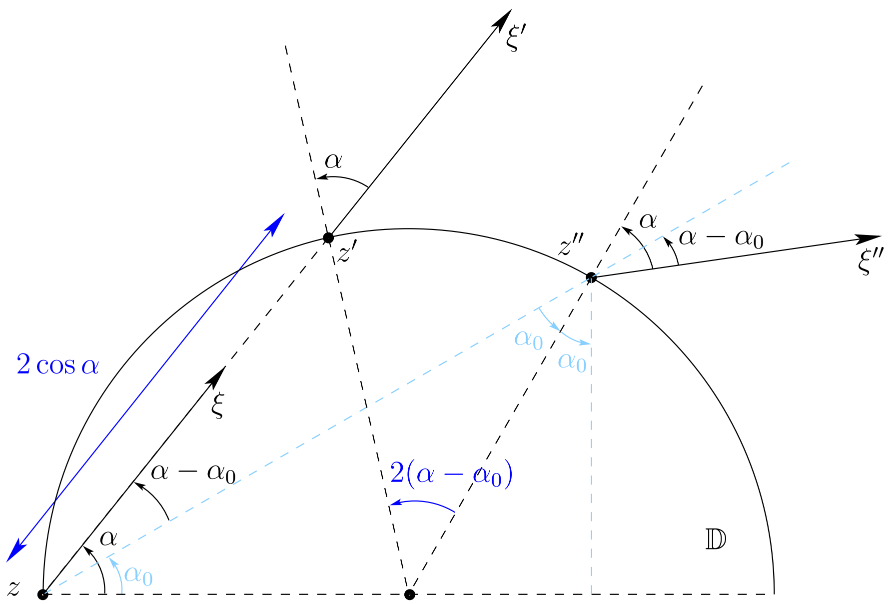


Fig. 3 Construction of the flow $\phi^{\tau}_{\alpha_0}$ with $\alpha_0 = \pi/6$. On the figure, $(z', \xi') = (z + 2\frac{\cos\alpha}{|\xi|}\xi, \xi)$ and $(z'', \xi'') = R^{2(\alpha_0-\alpha)}(z', \xi') = \phi^{2}_{\alpha_0}(z, \xi)$

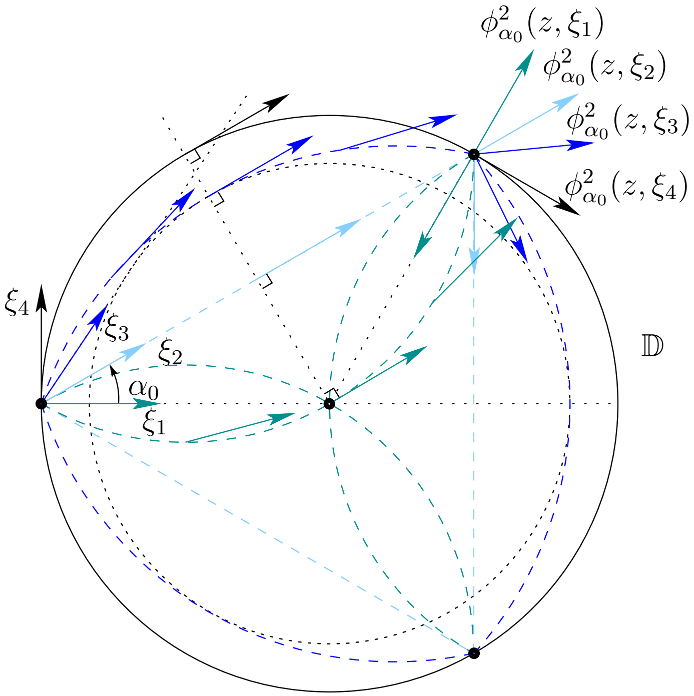


Fig. 4 Approximate representation of some trajectories of the flow $\phi^\tau_{\alpha_0}$ with $\alpha_0 = \pi/6$, issued from (z, ξ_j) with $z = (-1, 0)$ and ξ_j, $j \in \{1, 2, 3, 4\}$ such that $\alpha(z, \xi_1) = 0$, $\alpha(z, \xi_2) = \alpha_0$, $\alpha(z, \xi_3) \in (\alpha_0, \pi/2)$ and $\alpha(z, \xi_4) = \pi/2$

Lemma 3.5 *Let μ be a nonnegative invariant measure on $\mathbb{W}$. Then, for all $a \in C^0(\mathbb{W})$ and all $\tau \in \mathbb{R}$, we have*

$$\int a \circ \phi^\tau_{\alpha_0} d\mu\rceil_{\alpha=\alpha_0} = \int a d\mu\rceil_{\alpha=\alpha_0}.$$

Equivalently, we have for all $a \in C^0(\mathbb{W})$

$$\int a d\mu\rceil_{\alpha=\alpha_0} = \int \langle a\rangle_{\alpha_0} d\mu\rceil_{\alpha=\alpha_0},$$

where

$$\langle a\rangle_{\alpha_0} = \lim_{T\to+\infty} \frac{1}{T}\int_0^T a \circ \phi^\tau_{\alpha_0} dt.$$

Remark that $\langle a\rangle_{\alpha_0}$ is well defined even if a is a bounded measurable function on $\mathbb{D}$ (the times τ where the trajectories of $\phi^\tau_{\alpha_0}$ hit the boundary form a set of measure 0 in $\mathbb{R}$).

4 Second microlocalization on a rational angle

This section and the next one are devoted to proving the semiclassical version of our result, Theorem 2.5.

Let $\left(u_h^0\right)$ be a bounded family in $L^2(\mathbb{D})$. Denote by $u_h(z,t) = U_V(t)u_h^0(z)$ the corresponding solutions to (1.1). After extracting a subsequence, we suppose that its Wigner distributions W_h (defined by (2.5)) converge to a semiclassical measure μ_{sc} in the weak-$*$ topology of $\mathcal{D}'(\mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}_t \times \mathbb{R}_H)$. The measure $\mu_{sc} \in L^\infty(\mathbb{R}_t; \mathcal{M}_+(\mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}_H))$ is for a.e. $t \in \mathbb{R}$ supported by $\overline{\mathbb{D}} \times \mathbb{R}^2 \times \mathbb{R}_H \cap \{H = \frac{E^2}{2}\}$.

From now on, we skip the index sc to lighten the notation since there is no possible confusion here (only semiclassical measures are considered until Sect. 6).

The aim of this section is to understand the term $\mu\rceil_{\mathcal{I}_{\alpha_0}}$, where $\mathcal{I}_{\alpha_0} = \{\alpha = \alpha_0\}$ and $\alpha_0 \in \pi\mathbb{Q}$. In view of Lemma 3.5, it suffices to characterize the action of $\mu\rceil_{\mathcal{I}_{\alpha_0}}$ on test functions that are $(\phi_{\alpha_0}^\tau)$-invariant.

4.1 Classes of test functions

Here is a list of properties that we may want to impose on our symbols in the course of our proof. We express these properties both in the "old" coordinates (x, y, ξ_x, ξ_y) and in the "new" ones (s, θ, E, J).

Definition 4.1 Let a be a smooth function of $(x, y, \xi_x, \xi_y, t, H)$, supported away from $\{\xi = 0\}$. Then $a \circ \Phi$ is a smooth function of (s, θ, E, J, t, H) supported away from $\{E = 0\}$. We shall say that a satisfies Assumption

(A) if the symbol a is compactly supported w.r.t. ξ, t and H. This is equivalent to $a \circ \Phi$ being compactly supported w.r.t. s, E, J, t and H. Note also that $a \circ \Phi$ is 2π-periodic w.r.t. θ.

(B) if for $|z| = 1$, we have $a\,(z, \xi) = a \circ \sigma\,(z, \xi)$ where σ is the orthogonal symmetry with respect to the boundary of the disk at z. In the coordinates of Sect. 3.1, this reads (forgetting to write the (t, H)-dependence of a)

$$a \circ \Phi\,(\cos\alpha, \theta, E, J) = a \circ \Phi\,(-\cos\alpha, \theta + \pi + 2\alpha, E, J)$$

for all θ, E, J and for $\alpha = -\arcsin\left(\frac{J}{E}\right)$. **Terminology.** In what follows, we shall say that a is a smooth function on $\mathbb{W}$ if a is a smooth functon on $\overline{\mathbb{D}} \times \mathbb{R}^2$ that satisfies (B).

(C) if a satisfies (B), and, in addition, if $a \circ \pi \circ \phi^\tau$ defines a smooth function on $\mathbb{W}$ for all τ. This is equivalent to requiring that

$$\partial_s^k(a \circ \Phi)\,(\cos\alpha, \theta, E, J) = \partial_s^k(a \circ \Phi)\,(-\cos\alpha, \theta + \pi + 2\alpha, E, J)$$

for all k, for all θ, E, J and for $\alpha = -\arcsin(\frac{J}{E})$. In other words, all the derivatives of $a \circ \Phi$ w.r.t. s satisfy the symmetry condition (B).

(D) if the function a satisfies (C), and in addition, a is $\phi^{\tau}_{\alpha_0}$-invariant, which reads

$$\left((\alpha_0 - \alpha) X_{P_1} + \frac{\cos\alpha}{E} X_{P_0}\right) a = 0,$$

or, in the new coordinates,

$$[(\alpha_0 - \alpha)\partial_\theta + \cos\alpha\,\partial_s] a \circ \Phi = 0.$$

Furthermore, to fix ideas, let us assume that the support of a with respect to t is contained in $(-1, 1)$. This implies that

$$\begin{aligned} W_h(a) &= \langle g(t)u_h, \mathrm{Op}_1(a(z, h\xi, t, h^2 H)g(t)u_h\rangle_{L^2(\mathbb{R}^2\times\mathbb{R})} + O(h^\infty) \\ &= \langle g(t)u_h, \mathrm{Op}_h(a(z, \xi, t, hH)g(t)u_h\rangle_{L^2(\mathbb{R}^2\times\mathbb{R})} + O(h^\infty), \quad (4.1) \end{aligned}$$

for any smooth cut-off function g supported in $(-2, 2)$ and taking the value 1 in a neighborhood of $[-1, 1]$. In other words, we need only consider the restriction of $u_h(z, t)$ to $t \in (-2, 2)$.

4.2 Coordinates adapted to the second microlocalization on $\mathcal{I}_{\alpha_0}$

We wish to study the concentration of W_h around the set $\{J = -E\sin\alpha_0\}$. If the limit measure $(\Phi^{-1})_*\mu$ is supported on the set $\{E = \sqrt{2H}\}$ this is equivalent to studying the concentration of W_h around $\{J = -\sqrt{2H}\sin\alpha_0\}$. Since this assumption is satisfied for sequences (u_h) satisfying (1.8), we shall study the concentration of W_h around this set. The reason why it is more convenient to localize in H than in E is that D_t is tangential to the boundary of $\mathbb{R}_t \times \mathbb{D}_z$ (and hence commutes with the equation, at least when V is time-independent), whereas $|D_z|$, or Δ (the Laplacian on $\mathbb{R}^2$), are not. Note that in the associated stationary problem studied in [3], the measure is concentrated on $\{E = E_0\}$ for a fixed positive constant E_0, so that we do not need the introduction of the FIO $\mathscr{V}$, which simplifies the analysis.

Thus we make the (symplectic) change of variables

$$\left(s, \theta, E, J' - \sqrt{2H}\sin\alpha_0, t' + \frac{\theta\sin\alpha_0}{\sqrt{2H}}, H\right) = (s, \theta, E, J, t, H)$$

which sends $\{J' = 0\}$ to $\{J = -\sqrt{2H}\sin\alpha_0\}$ and leaves untouched the variables (s, E).

Consider the corresponding Fourier Integral Operator (which leaves untouched the variables (s, E), omitted here from the notation): for $f \in C_c^\infty(\mathbb{T}_\theta \times \mathbb{R}_t)$, we set

$$\mathscr{V} f(\theta, H) = (2\pi)^{-1/2} e^{i\sqrt{2H}\sin\alpha_0\theta/h} \int f(\theta, ht) e^{-iHt/h} dt.$$

Note that $h\mathscr{V} f(\theta, h^2 H)$ is in fact h-independent.

Lemma 4.2 *If $b \in C_c^\infty(\mathbb{R}_\theta \times \mathbb{R}_J \times \mathbb{R}_t \times \mathbb{R}_H)$, we have*

$$\mathscr{V}\,\mathrm{Op}_1(b(\theta, hJ, t, h^2 H))\mathscr{V}^* = \mathrm{Op}_h(\tilde{b}(\theta, J', H, ht)) + O(h), \tag{4.2}$$

where $\tilde{b}(\theta, J', H, ht) = b(\theta, J' - \sqrt{2H}\sin\alpha_0, -ht, H)$, and

$$\mathscr{V}\mathscr{V}^* = I. \tag{4.3}$$

Remark that Eq. 4.2 applies to functions depending on the variables (θ, H), their "dual variables" being now (J', t).

Proof First notice that we have

$$\mathscr{V}^* g(\theta, t) = (2\pi)^{-1/2} h^{-1} \int g(\theta, H) e^{iHt/h^2} e^{-i\sqrt{2H}\sin\alpha_0\theta/h} dH.$$

Second, we may now compute $A := \mathscr{V}\,\mathrm{Op}_1(b(\theta, hJ, t, h^2 H))\mathscr{V}^* e^{it_0 H/h} e^{iJ_0\theta/h}\rceil_{H=H_0,\theta=\theta_0}$. We have the exact formula

$$A = (2\pi h)^{-1} e^{i\sqrt{2H_0}\sin\alpha_0\theta_0/h} \int b(\theta_0, J_0 - \sqrt{2H}\sin\alpha_0, ht, H)$$
$$\times e^{iHt/h} e^{iHt_0/h} e^{iJ_0\theta_0/h} e^{-i\sqrt{2H}\sin\alpha_0\theta_0/h} e^{-iH_0 t/h} dH dt.$$

Note that this expression is exact and thus does not involve the derivatives of b w.r.t. θ or J. Taking $b = 1$ in this expression gives the exact formula $A = e^{it_0 H_0/h} e^{iJ_0\theta_0/h}$, which proves (4.3).

We carry on the computations with a general b. After a change of variables, A is now equal to

$$
\begin{aligned}
A &= (2\pi h)^{-1} e^{i\sqrt{2H_0}\sin\alpha_0\theta_0/h} e^{iJ_0\theta_0/h} \int b(\theta_0, J_0 - \sqrt{2H}\sin\alpha_0, ht, H) \\
&\qquad \times e^{iH(t+t_0)/h} e^{-i\sqrt{2H}\sin\alpha_0\theta_0/h} e^{-iH_0t/h} dH dt \\
&= (2\pi h)^{-1} e^{iH_0t_0/h} e^{i\sqrt{2H_0}\sin\alpha_0\theta_0/h} e^{iJ_0\theta_0/h} \\
&\qquad \times \int b\left(\theta_0, J_0 - \sqrt{2(H+H_0)}\sin\alpha_0, h(t-t_0), H_0 + H\right) \\
&\qquad \times e^{iHt/h} e^{-i\sqrt{2(H+H_0)}\sin\alpha_0\theta_0/h} dH dt.
\end{aligned}
$$

Standard application of the method of stationary phase shows that this expression is of order $O(h^\infty)$ if H_0 is outside the support of b. Besides, the phase has a single nondegenerate critical point at $(t, H) = (\frac{\sin\alpha_0\theta_0}{\sqrt{2H_0}}, 0)$, so that *uniformly in* $t_0 \in \mathbb{R}$ the method of stationary phase yields

$$
\begin{aligned}
A &= (2\pi h)^{-1} e^{iH_0t_0/h} e^{i\sqrt{2H_0}\sin\alpha_0\theta_0/h} e^{iJ_0\theta_0/h} \\
&\quad \times \left((2\pi h) e^{-i\sqrt{2H_0}\sin\alpha_0\theta_0/h} b\left(\theta_0, J_0 - \sqrt{2H_0}\sin\alpha_0, h\left(-t_0 + \frac{\sin\alpha_0\theta_0}{\sqrt{2H}}\right), H_0\right)\right) \\
&\quad + O(h).
\end{aligned}
$$

This is

$$
A = e^{iH_0t_0/h} e^{iJ_0\theta_0/h} b\left(\theta_0, J_0 - \sqrt{2H_0}\sin\alpha_0, -ht_0, H_0\right) + O(h),
$$

where $O(h)$ is uniform if θ_0 stays in a fixed compact set. This concludes the proof of (4.2). □

Recalling the definition of $W_h(a)$ in (2.5), we thus have

$$
\begin{aligned}
W_h(a) &= \langle \mathscr{V}\mathscr{U} u_h, (\mathscr{V}\mathscr{U}\,\mathrm{Op}_h(a(z,\xi,t,hH)\mathscr{U}^*\mathscr{V}^*)\mathscr{V}\mathscr{U} u_h\rangle_{L^2(\mathbb{R}_s\times\mathbb{T}_\theta\times\mathbb{R}_H)} \\
&= \langle \mathscr{V}\mathscr{U} u_h, \mathrm{Op}_h(\tilde{b}(s,\theta,J',E,H,ht))\mathscr{V}\mathscr{U} u_h\rangle_{L^2(\mathbb{R}_s\times\mathbb{T}_\theta\times\mathbb{R}_H)} + O(h)
\end{aligned}
$$

where

$$
\tilde{b}(s,\theta,J',E,H,ht)) = a\circ\Phi(s,\theta,E,J' - \sqrt{2H}\sin\alpha_0, -ht, H), \quad (4.4)
$$

and $\mathbb{T}_\theta = \mathbb{R}/2\pi\mathbb{Z}$ is the circle in which the variable θ takes values. By (4.1), u_h may actually be replaced by $g(t)u_h$ (being compactly supported in $t \in (-2, 2)$) in this formula, so that it is safe to apply $\mathscr{V}$ to $\mathscr{U} u_h$.

To work in our new coordinates, we now define

$$
\langle w_h, b\rangle = \langle \mathscr{V}\mathscr{U} u_h, \mathrm{Op}_h(b(s,\theta,E,J',H,ht))\mathscr{V}\mathscr{U} u_h\rangle_{L^2(\mathbb{R}_s\times\mathbb{T}_\theta\times\mathbb{R}_H)} \quad (4.5)
$$

for symbols b that satisfy

(A) the symbol b is compactly supported w.r.t. s, E, J', t and H, and 2π-periodic w.r.t. θ.

We then recover $W_h(a) = \langle w_h, \tilde{b}\rangle + O(h)$ with $\tilde{b}$ and a linked by (4.4).

Remark 4.3 Note that the bracket (4.5) can also be written as follows

$$\langle w_h, b\rangle = \langle \mathscr{V}\mathscr{U} u_h, \mathrm{Op}_h(\chi_0(\theta) b(s, \theta, E, J', H, ht))\mathscr{V}\mathscr{U} u_h\rangle_{L^2(\mathbb{R}_s\times\mathbb{R}_\theta\times\mathbb{R}_H)} \tag{4.6}$$

for any $\chi_0 \in C_c^\infty(\mathbb{R})$ satisfying $\sum_{k\in\mathbb{Z}} \chi_0(\theta + 2\pi k) \equiv 1$ on $\mathbb{R}$. Indeed, we have

$$\mathrm{Op}_h(\chi_0(\theta)b) = \chi_0(\theta)\,\mathrm{Op}_h(b) \tag{4.7}$$

(because Op denotes the standard quantization) and we write for any 2π-periodic function $f \in L^1_{\mathrm{loc}}(\mathbb{R})$, $\int_{\mathbb{R}} \chi_0(\theta) f(\theta) d\theta = \int_{\mathbb{T}} f(\theta) d\theta$. Because of (4.7), we may also take $\chi_0 = \mathbb{1}_{(0,2\pi)}$ when needed.

4.3 Second microlocalization

We now introduce two auxiliary distributions which describe more precisely how w_h concentrates on the set

$$\left\{(s, \theta, E, J, H, t) \in \Phi^{-1}(\overline{\mathbb{D}} \times (\mathbb{R}^2\backslash\{0\})) \times \mathbb{R}^2,\ \text{such that } -\frac{J}{\sqrt{2H}} = \sin\alpha_0\right\}$$

(whose intersection with $\{E = \sqrt{2H}\}$ is equal to $\mathcal{I}_{\alpha_0} \cap \{E = \sqrt{2H}\}$).

For this, we define an appropriate class of symbols depending on an additional variable η, which later in the calculations will be identified with $\frac{J'}{h} = \frac{J+\sqrt{2H}\sin\alpha_0}{h}$.

Definition 4.4 • We denote by $\mathcal{S}$ the class of smooth functions $b(s, \theta, E, J', \eta, H, t)$ on $\mathbb{R}^7$, supported away from $\{E = 0\}$ and that satisfy condition (A) in the variables (s, θ, E, J', H, t), and, in addition,

(E) b is homogeneous of degree zero at infinity in $\eta \in \mathbb{R}$. That is, there exist $R_0 > 0$ and $b_{\mathrm{hom}} \in C^\infty(\mathbb{R}^4 \times \{-1, +1\} \times \mathbb{R}^2)$ such that

$$b(s, \theta, E, J', \eta, H, t) = b_{\mathrm{hom}}\left(s, \theta, E, J', \frac{\eta}{|\eta|}, H, t\right),$$
$$\text{for } |\eta| > R_0 \text{ and } (s, \theta, E, J', H, t) \in \mathbb{R}^6.$$

• We denote by $\mathcal{S}^\sigma$ those symbols $b \in \mathcal{S}$ that satisfy conditions (B) and (C) (for all η, H, t, omitted here from the notation):

(B) $b(\cos\alpha, \theta, E, J') = b(-\cos\alpha, \theta+\pi+2\alpha, E, J')$ for all θ, E, J', and for $\alpha = -\arcsin\left(\frac{J'-\sqrt{2H}\sin\alpha_0}{E}\right)$.

(C) $\partial_s^k b(\cos\alpha, \theta, E, J') = \partial_s^k b(-\cos\alpha, \theta+\pi+2\alpha, E, J')$ for all k, for all θ, E, J', and for $\alpha = -\arcsin\left(\frac{J'-\sqrt{2H}\sin\alpha_0}{E}\right)$.

- We denote by $\mathcal{S}^{\sigma}_{\alpha_0}$ those symbols $b \in \mathcal{S}^{\sigma}$ satisfying the invariance condition (D) (for all η, H, t, omitted here from the notation):

(D) $[(\alpha_0 - \alpha)\partial_\theta + \cos\alpha\,\partial_s]b(s, \theta, E, J') = 0$ for all s, θ, E, J', and for $\alpha = -\arcsin\left(\frac{J'-\sqrt{2H}\sin\alpha_0}{E}\right)$.

Let $\chi \in C_c^\infty(\mathbb{R})$ be a nonnegative cut-off function that is identically equal to one near the origin and let $R > 0$. For $b \in \mathcal{S}$, we define

$$\left\langle w^{\alpha_0}_{h,R}, b\right\rangle := \left\langle \mathscr{V}\mathscr{U}u_h, \mathrm{Op}_h\left(\left(1-\chi\left(\frac{J'}{Rh}\right)\right)\chi_0(\theta)b\left(s, \theta, E, J', \frac{J'}{h}, H, ht\right)\right) \mathscr{V}\mathscr{U}u_h\right\rangle_{L^2(\mathbb{R}_s\times\mathbb{R}_\theta\times\mathbb{R}_H)},$$

and

$$\left\langle w_{\alpha_0,h,R}, b\right\rangle := \left\langle \mathscr{V}\mathscr{U}u_h, \mathrm{Op}_h\left(\chi\left(\frac{J'}{Rh}\right)\chi_0(\theta)b\left(s, \theta, E, J', \frac{J'}{h}, H, ht\right)\right) \mathscr{V}\mathscr{U}u_h\right\rangle_{L^2(\mathbb{R}_s\times\mathbb{R}_\theta\times\mathbb{R}_H)}, \tag{4.8}$$

where χ_0 is as in Remark 4.3.

The Calderón–Vaillancourt theorem [18] ensures that both $w^{\alpha_0}_{h,R}$ and $w_{\alpha_0,h,R}$ are bounded in $\mathcal{S}'$. After possibly extracting subsequences, we have the existence of a limit: for every $b \in \mathcal{S}$,

$$\left\langle \mu^{\alpha_0}, b\right\rangle := \lim_{R\to\infty}\lim_{h\to 0^+}\left\langle w^{\alpha_0}_{h,R}, b\right\rangle,$$

and

$$\left\langle \mu_{\alpha_0}, b\right\rangle := \lim_{R\to\infty}\lim_{h\to 0^+}\left\langle w_{\alpha_0,h,R}, b\right\rangle. \tag{4.9}$$

Positivity properties are described in the next proposition.

Proposition 4.5 (i) *The distribution μ^{α_0} is a nonnegative Radon measure being 0-homogeneous and supported at infinity in the variable η (i.e., it vanishes when paired with a compactly supported function). As a consequence, μ^{α_0} may be identified with a nonnegative measure on $\mathbb{R}^4 \times \{-1, +1\} \times \mathbb{R}_t \times \mathbb{R}_H$.*

(ii) *The projection of* μ_{α_0} *on* $\mathbb{R}^4_{s,\theta,E,J'} \times \mathbb{R}^2_{H,t}$ (*that is*, $\int_{\mathbb{R}} \mu_{\alpha_0}(d\eta)$) *is a non-negative measure*, *carried on* $\{J' = 0\}$.

Moreover, both μ_{α_0} and μ^{α_0} are carried by the set $\{E = \sqrt{2H}\}$, as can be seen from Proposition 8.3. Note also that the argument of Proposition 9.1 proves that μ^{α_0} enjoys L^∞ regularity in the time variable.

Proposition 4.5 (i) is proved at the beginning of Sect. 4.4, whereas (ii) shall be a consequence of Sect. 4.5.

Remark 4.6 If $a = a(z, \xi, t, H)$ is a continuous function on $\overline{\mathbb{D}} \times \mathbb{R}^2 \times \mathbb{R}^2$, let us define

$$m^{\alpha_0}(a) := \mu^{\alpha_0}(b \mathbb{1}_{J'=0}),$$
$$m_{\alpha_0}(a) := \mu_{\alpha_0}(b).$$

where $b(s, \theta, E, J', H, t) = a \circ \Phi(s, \theta, E, J' - \sqrt{2H} \sin \alpha_0, -t, H)$ (a function that does not depend on the additional variable η). Then we have

$$\mu \rceil_{\mathcal{I}_{\alpha_0}} = m^{\alpha_0} + m_{\alpha_0}. \tag{4.10}$$

Thus, understanding $\mu \rceil_{\mathcal{I}_{\alpha_0}}$ amounts to understanding both m^{α_0} and m_{α_0}, which we shall do by understanding the structure of μ^{α_0} and μ_{α_0}. Note that, as a consequence of Proposition 4.5, the distributions m_{α_0}, m^{α_0} are positive.

Let us continue describing properties of the distributions μ_{α_0} and μ^{α_0}. The following proposition states that they are both invariant under the billiard flow (in the coordinates (s, θ, E, J')).

Proposition 4.7 *The distributions* μ_{α_0} *and* μ^{α_0} *enjoy the following property*:

$$\langle \mu_{\alpha_0}, E\, \partial_s b \rangle = 0, \quad \langle \mu^{\alpha_0}, E\, \partial_s b \rangle = 0$$

for every $b \in \mathcal{S}^\sigma$.

The proof of this result uses as a "black-box" the technical calculations developed in Appendices C and D. The main point of these calculations is to understand how an operator of the form

$$\mathscr{U}^* \operatorname{Op}_h (P(s, \theta, E, J, t, hH)) \mathscr{U}$$

preserves or modifies the Dirichlet boundary condition, according to the properties of P (the technical difficulty is that our new coordinates (s, θ, E, J), well-adapted to the dynamics, are not adapted to express the Dirichlet boundary condition). More precisely, in the appendices we perform the following technical constructions:

(1) In Appendices C and D the operator $\mathscr{U}^* \mathrm{Op}_h(P(s, \theta, E, J, t, hH))\mathscr{U}$ is expressed (modulo a small remainder) as a pseudodifferential operator $\mathcal{A}_{t,h^2 D_t}(P)$ on $\mathbb{R}^2 \times \mathbb{R}$ (defined in Definition 11.2) in polar coordinates

$$z = (-r \sin u, r \cos u).$$

Note that the polar coordinates are the ones adapted to our boundary problem, since the boundary is given by the equation $r = 1$. Thus we have

$$\lim \langle u_h, \mathscr{U}^* \mathrm{Op}_h \left(P(s, \theta, E, J, t, hH)\right) \mathscr{U} u_h\rangle = \lim \langle u_h, \mathcal{A}_{t,h^2 D_t}(P) u_h\rangle.$$

(2) In Appendix D (see Definition 11.3), we then introduce a pseudodifferential operator $\widetilde{\mathcal{A}}_{t,h^2 D_t}(P)$, having the property that the symbols of $\mathcal{A}_{t,h^2 D_t}(P)$ and $\widetilde{\mathcal{A}}_{t,h^2 D_t}(P)$ coincide on $\{|\xi|^2 = 2H\}$. More precisely, we are able to prove (Lemma 11.5)

$$\lim_{h\to 0} \langle u_h, \mathcal{A}_{t,h^2 D_t}(P) u_h\rangle = \lim_{h\to 0} \langle u_h, \widetilde{\mathcal{A}}_{t,h^2 D_t}(P) u_h\rangle$$

if (u_h) is a solution to the Eq. (1.1).

(3) The explicit expression of $\widetilde{\mathcal{A}}_{t,h^2 D_t}(P)$ reads

$$A(r, u, h\sqrt{2D_t}, hD_u, t) + B(r, u, h\sqrt{2D_t}, hD_u, t) \circ hD_r \tag{4.11}$$

modulo terms of order $O(h)$, where $z = (-r \sin u, r \cos u)$ is the decomposition in polar coordinates. The functions A, B, C, D are expressed explicitly in terms of P in Definition 10.6. If P satisfies the symmetry condition (B), then $B \equiv 0$ for $r = 1$.

(4) Finally, we show in Proposition 12.2 that

$$\lim_{h\to 0} \langle u_h, \widetilde{\mathcal{A}}_{t,h^2 D_t}(E\, \partial_s P) u_h\rangle = \lim_{h\to 0} \left\langle u_h, \left[-\frac{ih\Delta}{2}, \widetilde{\mathcal{A}}_{t,h^2 D_t}(P)\right] u_h \right\rangle$$

where Δ is the laplacian on $\mathbb{R}^2$.

With this in hand, we may now prove the proposition.

Proof of Proposition 4.7 We prove the statement for μ_{α_0}, and, to this aim, we consider the function P (depending on both parameters R and h) defined by

$$
\begin{aligned}
&P(s,\theta,E,J,t,H)\\
&\quad = b\left(s,\theta,E,J+\sqrt{2H}\sin\alpha_0,\frac{J+\sqrt{2H}\sin\alpha_0}{h},H,-t\right)\\
&\qquad \times\chi\left(\frac{J+\sqrt{2H}\sin\alpha_0}{Rh}\right).
\end{aligned}
\tag{4.12}
$$

To prove the result for μ^{α_0}, the argument would be the same with the function

$$
\begin{aligned}
\tilde{P}(s,\theta,E,J,t,H)=&\,b\left(s,\theta,E,J+\sqrt{2H}\sin\alpha_0,\frac{J+\sqrt{2H}\sin\alpha_0}{h},H,-t\right)\\
&\times(1-\chi)\left(\frac{J+\sqrt{2H}\sin\alpha_0}{Rh}\right).
\end{aligned}
\tag{4.13}
$$

With b and P related by (4.12), and writing in the next few lines $\lim_{R,h}$ for $\lim_{R\to\infty}\lim_{h\to 0^+}$, we now compute

$$
\begin{aligned}
\langle \mu_{\alpha_0}, E\,\partial_s b\rangle &= \lim_{R,h}\left\langle w_{\alpha_0,h,R}, E\,\partial_s b\right\rangle = \lim_{R,h}\Big\langle \mathscr{V}\mathscr{U}u_h,\\
&\quad \mathrm{Op}_h\left(E\,\partial_s P(s,\theta,E,J'-\sqrt{2H}\sin\alpha_0,-ht,H)\right)\mathscr{V}\mathscr{U}u_h\Big\rangle_{L^2(\mathbb{R}_s\times\mathbb{T}_\theta\times\mathbb{R}_H)}\\
&= \lim_{R,h}\left\langle \mathscr{U}u_h, \mathrm{Op}_h\left(E\,\partial_s P(s,\theta,E,J,t,hH)\right)\mathscr{U}u_h\right\rangle_{L^2(\mathbb{R}_s\times\mathbb{T}_\theta\times\mathbb{R}_t)},
\end{aligned}
$$

where the last line is a consequence of Lemma 4.2. Using now the remarks preceding the present proof (proved in the appendices), we obtain (with Δ being the Laplacian on $\mathbb{R}^2$)

$$
\begin{aligned}
\langle \mu_{\alpha_0}, E\,\partial_s b\rangle &= \lim_{R,h}\left\langle u_h, \widetilde{\mathcal{A}}_{t,h^2D_t}(E\,\partial_s P)u_h\right\rangle\\
&= \lim_{R,h}\left\langle u_h, \left[-\frac{ih}{2}\Delta, \widetilde{\mathcal{A}}_{t,h^2D_t}(P)\right]u_h\right\rangle\\
&= \lim_{R,h}\left\langle u_h, \left[-\frac{ih}{2}\Delta + ihV - h\partial_t, \widetilde{\mathcal{A}}_{t,h^2D_t}(P)\right]u_h\right\rangle \qquad (4.14)\\
&= \lim_{R,h}\left\langle \frac{ih}{2}\frac{\partial u_h}{\partial n}\otimes\delta_{\partial\mathbb{D}}, \widetilde{\mathcal{A}}_{t,h^2D_t}(P)u_h\right\rangle\\
&\quad - \lim_{R,h}\left\langle u_h, \widetilde{\mathcal{A}}_{t,h^2D_t}(P)\frac{ih}{2}\frac{\partial u_h}{\partial n}\otimes\delta_{\partial\mathbb{D}}\right\rangle. \qquad (4.15)
\end{aligned}
$$

The last line (4.15) comes from the fact that u_h, extended to $\mathbb{R}^2$ by the value 0 outside $\mathbb{D}$, satisfies

$$\left(-\frac{ih}{2}\Delta + ihV - h\partial_t\right) u_h = \frac{ih}{2}\frac{\partial u_h}{\partial n} \otimes \delta_{\partial\mathbb{D}} \tag{4.16}$$

where Δ is the laplacian on $\mathbb{R}^2$. The previous line (4.14) holds simply because the additional terms hV and $h\partial_t$ give a vanishing contribution as $h \to 0$ (if P depends on t, we can write $[h\partial_t, \widetilde{\mathcal{A}}_{t,h^2D_t}(P)] = h\widetilde{\mathcal{A}}_{t,h^2D_t}(\partial_t P)$).

We use now the explicit expression (4.11) of $\widetilde{\mathcal{A}}_{t,h^2D_t}(P)$, modulo terms that vanish at the limit. Using the fact that u_h satisfies Dirichlet boundary conditions, and the fact that B vanishes for $r = 1$ if P satisfies the symmetry condition (B), we see that the last line (4.15) vanishes.

Note that only the limit $h \to 0$ was actually used, so that the result holds even before taking the limit $R \to +\infty$. □

Note that Appendices C and D provide with identities modulo $O(h^2)$. Here we only used these results with remainders of order $O(h)$; the full results of Appendices C and D shall be used in the next sections.

Remark 4.8 More generally, let $b(s, \theta, E, J, \eta, H, t)$ be a smooth function on $\mathbb{R} \times \mathbb{R}/2\pi\mathbb{Z} \times \mathbb{R}^5$ with bounded derivatives, and compactly supported w.r.t. s, E, J, H, t. Let $\mathbf{P}(s, \theta, E, J, t, H) = b(s, \theta, E, J + \sqrt{2H}\sin\alpha_0, \frac{J+\sqrt{2H}\sin\alpha_0}{h}, H, -t)$. Then, the same proof yields,without using the symmetry condition (B), the formula

$$\begin{aligned}
&\lim_h \left\langle \mathscr{V}\mathscr{U}u_h, \mathrm{Op}_h\left(E\,\partial_s b(s,\theta,E,J',\frac{J'}{h},H,ht)\right)\mathscr{V}\mathscr{U}u_h\right\rangle_{L^2(\mathbb{R}_s\times\mathbb{T}_\theta\times\mathbb{R}_H)}\\
&\quad= \lim_h \left\langle \mathscr{U}u_h, \mathrm{Op}_h\left(E\,\partial_s\mathbf{P}(s,\theta,E,J,t,hH)\right)\mathscr{U}u_h\right\rangle_{L^2(\mathbb{R}_s\times\mathbb{T}_\theta\times\mathbb{R}_t)}\\
&\quad= -\lim_h \left\langle h\frac{\partial u_h}{\partial n}, \mathbf{B}(1,u,h\sqrt{2D_t},hD_u,t)h\frac{\partial u_h}{\partial n}\right\rangle_{L^2(\partial\mathbb{D}\times\mathbb{R})}
\end{aligned} \tag{4.17}$$

where $\mathbf{B}$ is the function associated to $\mathbf{P}$ by the formulas of Definition 10.6. Again, if $\mathbf{P}$ satisfies (B), the operator $\mathbf{B}(1, u, h\sqrt{2D_t}, hD_u, t)$ vanishes.

This formula, relating the semiclassical measures of boundary data to the semiclassical measures of interior data, is analogous to formula (2.23) but is expressed in a different set of coordinates.

Applying (4.17) to $\frac{s}{E}\mathbf{P}$ instead of $\mathbf{P}$ (that is, $\frac{s}{E}b$ instead of b) has the following consequence that will be used later on:

$$\lim_h \left\langle \mathscr{U} u_h, \mathrm{Op}_h \left((\mathbf{P} + s\partial_s \mathbf{P})(s, \theta, E, J, t, hH) \right) \mathscr{U} u_h \right\rangle_{L^2(\mathbb{R}_s \times \mathbb{T}_\theta \times \mathbb{R}_t)}$$
$$= -\lim_h \left\langle h \frac{\partial u_h}{\partial n}, (E^{-2}\mathbf{P})^\sigma (1, u, h\sqrt{2D_t}, hD_u, t) h \frac{\partial u_h}{\partial n} \right\rangle_{L^2(\partial \mathbb{D} \times \mathbb{R})} \tag{4.18}$$

with the notation (10.5).

The following result states that both μ^{α_0} and μ_{α_0} have some extra regularity (for two different reasons).

Theorem 4.9 (i) *The measure μ^{α_0} satisfies the invariance property*:

$$\langle \mu^{\alpha_0}, \partial_\theta b \rangle = 0, \quad \textit{for every } \ b \ \textit{ in } \ \mathcal{S}^\sigma_{\alpha_0}. \tag{4.19}$$

(ii) *The distribution μ_{α_0} is concentrated on $\{J' = 0\}$ and its projection onto the variables (s, θ) is a nonnegative absolutely continuous measure. More precisely, we can write*

$$\int b(s, \theta, E, J, H, t) \mu_{\alpha_0}(ds, d\theta, dJ, d\eta, dE, dH, dt)$$
$$= \int \left(\int b(s, \theta, E, 0, H, t) d\nu_{E,H,t}(s, \theta) \right) \tilde{\ell}_{\alpha_0}(dE, dH, dt)$$

where $\tilde{\ell}_{\alpha_0}$ is a positive measure on $\mathbb{R}_E \times \mathbb{R}_H \times \mathbb{R}_t$, and for almost all E, H, t, $\nu_{E,H,t}$ is a positive measure on $\mathbb{R}_s \times \mathbb{R}/2\pi\mathbb{Z}$ which is absolutely continuous.

Section 4.4 is devoted to the study of the properties of μ^{α_0} and gives the proofs of Proposition 4.5 (i) and Theorem 4.9 (i). The study of the structure of μ_{α_0} is performed in Sect. 4.5 using the notion of second-microlocal measures. This structure will imply (2.13) in Theorem 2.5 (iii). In particular, we prove at the end of Sect. 4.5 that it yields Theorem 4.9 (ii).

4.4 Structure and propagation of μ^{α_0}

In this section, we prove Proposition 4.5 (i) and the invariance property given by Theorem 4.9 (i).

The positivity of μ^{α_0} can be deduced following the lines of [21] Section 2.1, or those of the proof of Theorem 1 in [23]; or also Corollary 27 in [6]. The argument will not be reproduced here. Given $b \in \mathcal{S}$ there exists $R_0 > 0$ and $b_{\mathrm{hom}} \in C_c^\infty \left(\mathbb{R}^4 \times \{-1, +1\} \times \mathbb{R}^2 \right)$ such that

$$b\left(s, \theta, E, J', \eta, H, t\right) = b_{\mathrm{hom}} \left(s, \theta, E, J', \frac{\eta}{|\eta|}, H, t \right), \quad \text{for} \quad |\eta| \geq R_0.$$

Clearly, for R large enough, the value $\langle w_{h,R}^{\alpha_0}, b\rangle$ only depends on b_{hom}. Therefore, the limiting distribution μ^{α_0} can be viewed as an element of the dual space of $C_c^\infty(\mathbb{R}^4 \times \{-1,+1\} \times \mathbb{R}^2)$. Its positivity implies that it is a measure, which proves Proposition 4.5 (i).

We now prove the invariance property of Theorem 4.9 (i). Let $b \in \mathcal{S}_{\alpha_0}^\sigma$, and define $\tilde{P}$ as in formula (4.13). Because of property (D) in the definition of the class $\mathcal{S}_{\alpha_0}^\sigma$, we have:

$$\partial_\theta \tilde{P}\left(s, \theta, \sqrt{2H}, J, t, H\right) = -\frac{\cos\alpha}{\alpha_0 - \alpha}\partial_s \tilde{P}\left(s, \theta, \sqrt{2H}, J, H, t\right) \tag{4.20}$$

where $\alpha = -\arcsin(\frac{J}{\sqrt{2H}})$. The crucial point in what follows is that

$$\left|\frac{\cos\alpha}{\alpha_0 - \alpha}\right| \le \frac{C}{hR} \tag{4.21}$$

on the support of $\tilde{P}(s, \theta, \sqrt{2H}, J, t, H)$.

Recall that by definition

$$\langle \mu^{\alpha_0}, \partial_\theta b\rangle = \lim_{R\to\infty}\lim_{h\to 0^+}\left\langle w_{h,R}^{\alpha_0}, \partial_\theta b\right\rangle.$$

Let us first fix R and study the limit $h \to 0$. Arguing as in the proof of Proposition 4.7, we have

$$\begin{aligned}
\lim_{h\to 0^+}\left\langle w_{h,R}^{\alpha_0}, \partial_\theta b\right\rangle &= \lim\left\langle \mathscr{U} u_h, \mathrm{Op}_h\left(\partial_\theta \tilde{P}(s, \theta, E, J, t, hH)\right)\mathscr{U} u_h\right\rangle_{L^2(\mathbb{R}_s\times\mathbb{T}_\theta\times\mathbb{R}_H)}\\
&= \lim\left\langle u_h, \mathcal{A}_{t,h^2D_t}\left(\partial_\theta \tilde{P}\right)u_h\right\rangle\\
&= \lim\left\langle u_h, \widetilde{\mathcal{A}}_{t,h^2D_t}\left(\partial_\theta \tilde{P}\right)u_h\right\rangle\\
&= \lim\left\langle u_h, \widetilde{\mathcal{A}}_{t,h^2D_t}\left(-\frac{\cos\alpha}{(\alpha_0-\alpha)}\partial_s \tilde{P}\right)u_h\right\rangle
\end{aligned}$$

after having used (4.20). Now using (4.21), we obtain

$$\begin{aligned}
&\lim_{h\to 0^+}\left\langle w_{h,R}^{\alpha_0}, \partial_\theta b\right\rangle\\
&=\lim\left\langle u_h, \left[-\frac{ih}{2}\Delta + ihV - h\partial_t, \widetilde{\mathcal{A}}_{t,h^2D_t}\left(-\frac{\cos\alpha}{E(\alpha_0-\alpha)}\tilde{P}\right)\right]u_h\right\rangle + O(R^{-1})
\end{aligned}$$

$$
\begin{aligned}
&= \lim \left\langle \frac{ih}{2}\frac{\partial u_h}{\partial n} \otimes \delta_{\partial \mathbb{D}}, \widetilde{\mathcal{A}}_{t,h^2 D_t}\left(-\frac{\cos\alpha}{E(\alpha_0-\alpha)}\tilde{P}\right) u_h \right\rangle \\
&\quad - \lim \left\langle u_h, \widetilde{\mathcal{A}}_{t,h^2 D_t}\left(-\frac{\cos\alpha}{E(\alpha_0-\alpha)}\tilde{P}\right) \frac{ih}{2}\frac{\partial u_h}{\partial n} \otimes \delta_{\partial \mathbb{D}} \right\rangle + O(R^{-1})
\end{aligned}
\tag{4.22}
$$

where we used again (4.16).

But $\widetilde{\mathcal{A}}_{t,h^2 D_t}(-\frac{\cos\alpha}{E(\alpha_0-\alpha)}\tilde{P})$ equals (modulo terms which only add an error $O(R^{-1})$ to the whole calculation)

$$
\begin{aligned}
&-\tilde{A}(r,u,h\sqrt{2D_t},hD_u,t)\frac{\cos\alpha(hD_u,h^2D_t)}{h\sqrt{2D_t}(\alpha_0-\alpha(hD_u,h^2D_t))} \\
&-\tilde{B}(r,u,h\sqrt{2D_t},hD_u,t)\frac{\cos\alpha(hD_u,h^2D_t)}{h\sqrt{2D_t}(\alpha_0-\alpha(hD_u,h^2D_t))} \circ hD_r
\end{aligned}
$$

where $z = (-r\sin u, r\cos u)$ is the decomposition in polar coordinates, and $\tilde{A}$, $\tilde{B}$ are the functions associated to $\tilde{P}$ by the formulas of Definition 10.6.

If $\tilde{P}$ satisfies (B) then $\tilde{B} \equiv 0$ for $r = 1$. Since u_h satisfies Dirichlet boundary conditions, we see that the last terms in (4.22) vanish.

To conclude the proof of Theorem 4.9 (i), we take $R \to +\infty$ after taking $h \to 0$, so that the terms estimated as $O(R^{-1})$ vanish.

4.5 Second microlocal structure of μ_{α_0}

If $\mathcal{H}$ is a Hilbert space, we shall denote by $\mathcal{L}(\mathcal{H})$, $\mathcal{K}(\mathcal{H})$ and $\mathcal{L}^1(\mathcal{H})$ the spaces of bounded, compact and trace class operators on $\mathcal{H}$. It is well known that $\mathcal{L}^1(\mathcal{H})$ is the dual of $\mathcal{K}(\mathcal{H})$ (see e.g. [53, Section VI.6]). A measure on a polish space T, taking values in $\mathcal{L}^1(H)$, is defined as a bounded linear functional ρ from $C_c(T)$ to $\mathcal{L}^1(H)$; ρ is said to be nonnegative if, for every nonnegative $b \in C_c(T)$, $\rho(b)$ is a nonnegative hermitian operator. The set of such measures is denoted by $\mathcal{M}_+(T;\mathcal{L}^1(H))$; they can be identified in a natural way to nonnegative linear functionals on $C_c(T;\mathcal{K}(H))$. Background and further details on operator-valued measures may be found for instance in [23].

For each $\omega \in \mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$, we recall that $\mathcal{H}_\omega$, defined in (2.12), is the space of functions f on $\mathbb{R}$ satisfying $f(\theta+2\pi) = f(\theta)e^{i\omega}$ and that are square-integrable on $(0, 2\pi)$.

We shall denote by $\mathcal{K}^{2\pi}$ the space of operators on $L^2(\mathbb{R})$ whose kernel K satisfies $K(\theta+2\pi,\theta'+2\pi) = K(\theta,\theta')$ and that define compact operators on each $\mathcal{H}_\omega$. Each Hilbert space $\mathcal{H}_\omega$ is isometric to $L^2(0,2\pi)$ (just by restricting functions to $(0,2\pi)$), and in this identification the kernel of K acting on $\mathcal{H}_\omega$ is given by

$$K_\omega(\theta,\theta') := \mathbb{1}_{(0,2\pi)}(\theta)\mathbb{1}_{(0,2\pi)}(\theta')\sum_{n\in\mathbb{Z}} K(\theta,\theta'+2\pi n)e^{in\omega}. \tag{4.23}$$

The idea of the Floquet-Bloch theory (see e.g. [52, Section XIII.16]) is that it is completely equivalent to know $K(\theta,\theta')$ and to know $K_\omega(\theta,\theta')$ for almost all ω, by decomposing

$$L^2(\mathbb{R}) = \int_{(0,2\pi)}^{\oplus} \mathcal{H}_\omega d\omega.$$

In particular, we recover K from $(K_\omega)_{\omega\in\mathbb{T}}$ by

$$K(\theta,\theta') = \int_{\mathbb{T}} K_\omega(\theta,\theta')d\omega.$$

Besides, the operator with kernel K is nonnegative (resp. bounded) if and only if K_ω is nonnegative (resp. bounded) for a.e. ω.

Examples of operators in $\mathcal{K}^{2\pi}$ are, given a symbol $b\in\mathcal{S}$ and parameters R, h, t, H, s, E, furnished by

$$K_{b,h,R}(s,E,H,t) = b(s,\theta,E,hD_\theta,D_\theta,H,t)\chi(D_\theta/R). \tag{4.24}$$

Note that, as $h\to 0$, we have $K_{b,h,R}(s,E,H,t) = K_{b,0,R}(s,E,H,t) + O_R(h)$.

If b satisfies the symmetry condition (B), note that the operator $K_{b,0,R}(s,E,H,t)$ has the property

$$K(\cos\alpha,E,H,t) = R^{-1}_{\pi+2\alpha}\circ K(-\cos\alpha,E,H,t)\circ R_{\pi+2\alpha}, \tag{4.25}$$

where R is a translation operator on $L^2(\mathbb{R}_\theta)$: $R_\alpha f(\theta) = f(\theta-\alpha)$ and where $\alpha = \arcsin(\frac{\sqrt{2H}\sin\alpha_0}{E})$. In particular,

$$K(\cos\alpha_0,\sqrt{2H},H,t) = R^{-1}_{\pi+2\alpha_0}\circ K(-\cos\alpha_0,\sqrt{2H},H,t)\circ R_{\pi+2\alpha_0}. \tag{4.26}$$

Remark 4.10 The fact that the orbits of the billiard flow are periodic on $\mathcal{I}_{\alpha_0}$ ($\alpha_0\in\pi\mathbb{Q}$) is reflected in the fact that the function $s\mapsto K(s,\sqrt{2H},H,t)$ is periodic, if K satisfies (4.26).

For $K \in C_c^\infty\left(\mathbb{T}_\omega \times \mathbb{R}_s \times \mathbb{R}_E \times \mathbb{R}_H \times \mathbb{R}_t; \mathcal{K}\left(L^2(0, 2\pi)\right)\right)$, let us define:

$$\left\langle n_h^{\alpha_0}, K\right\rangle = (2\pi h)^{-2} \int_{\omega=0}^{2\pi} \sum_{H, \sqrt{2H}\sin\alpha_0/h \equiv \omega(2\pi)} \frac{h}{\sin\alpha_0}\sqrt{\frac{H}{2}}$$
$$\int_{s,s',E,H',t} \left\langle \chi_0 \mathscr{V}\mathscr{U} u_h(s', H'), K\left(\omega, s, E, H', ht\right) \chi_0 \mathscr{V}\mathscr{U} u_h(s, H)\right\rangle_{L^2(0,2\pi)}$$
$$\times e^{iE(s'-s)/h} e^{it(H'-H)/h} ds\, dE\, ds'\, dH'\, dt\, d\omega \tag{4.27}$$

where χ_0 is $\mathbb{1}_{(0,2\pi)}$ as in Remark 4.3. This can be rewritten as

$$\langle \chi_0 \mathscr{V}\mathscr{U} u_h, \mathscr{K} \chi_0 \mathscr{V}\mathscr{U} u_h\rangle_{L^2(\mathbb{R}_s \times \mathbb{R}_H, L^2(0,2\pi))},$$

where $\mathscr{K}$ is the pseudodifferential operator with operator-valued symbol:

$$k(s, E, H', ht) = \int_{\omega=0}^{2\pi} \sum_{H, \sqrt{2H}\sin\alpha_0/h \equiv \omega(2\pi)} \frac{h}{\sin\alpha_0}\sqrt{\frac{H}{2}} K\left(\omega, s, E, H', ht\right) d\omega. \tag{4.28}$$

Remark 4.11 As noted earlier, it is equivalent (by the relation (4.23)) to consider a family $K(\omega)$ of kernels on $(0, 2\pi)^2$ and a kernel K on $\mathbb{R}^2$ satisfying $K(\theta, \theta') = K(\theta + 2\pi, \theta' + 2\pi)$. With this identification in mind, formula (4.27) amounts to

$$\begin{aligned}(2\pi h)^{-2} &\int_{s,s',E,H,H',t} \left\langle \chi_0 \mathscr{V}\mathscr{U} u_h(s', H'), k\left(s, E, H', ht\right) \mathscr{V}\mathscr{U} u_h(s, H)\right\rangle_{L^2(\mathbb{R})} \\ &\times e^{iE(s'-s)/h} e^{it(H'-H)/h} ds dE ds' dH dH' dt \\ &= \left\langle \chi_0 \mathscr{V}\mathscr{U} u_h, k\left(s, hD_s, h^2 D_t, t\right) \mathscr{V}\mathscr{U} u_h\right\rangle_{L^2(\mathbb{R}_s \times \mathbb{R}_H, L^2(\mathbb{R}_\theta))}.\end{aligned} \tag{4.29}$$

The motivation for rewriting (4.29) in the apparently more complicated form (4.27) is that it will be more convenient to use the compact operators $K(\omega)$ on each $\mathcal{H}_\omega$ than the non-compact operator K on $L^2(\mathbb{R})$.

The relevance of definition (4.27) for us is that we have the relation

$$\begin{aligned}\langle w_{\alpha_0,h,R}, b\rangle &= \left\langle n_h^{\alpha_0}, K_{b,h,R}\right\rangle \\ &= \left\langle n_h^{\alpha_0}, K_{b,0,R}\right\rangle + O_R(h),\end{aligned} \tag{4.30}$$

where $K_{b,h,R}$ was defined in (4.24).

Proposition 4.12 *Suppose (u_h^0) is bounded in $L^2(\mathbb{D})$. Then, modulo taking subsequences, the following convergence takes place:*

$$\lim_{h\to 0^+} \langle n_h^{\alpha_0}, K\rangle = \int_0^{2\pi} \int_{\mathbb{R}_s\times\mathbb{R}_E\times\mathbb{R}_H\times\mathbb{R}_t} \operatorname{Tr}\left\{K(\omega, s, E, H, t)\,\rho_{\alpha_0}(d\omega, ds, dE, dH, dt)\right\}, \tag{4.31}$$

for every $K \in C_c^\infty(\mathbb{T}_\omega\times\mathbb{R}_s\times\mathbb{R}_E\times\mathbb{R}_H\times\mathbb{R}_t; \mathcal{K}(L^2(0,2\pi)))$. In other words, ρ_{α_0} is the limit of $n_h^{\alpha_0}$ in the weak-$*$ topology of

$$\mathcal{D}'\left(\mathbb{T}_\omega\times\mathbb{R}_s\times\mathbb{R}_E\times\mathbb{R}_H\times\mathbb{R}_t, \mathcal{L}^1\left(L^2(0,2\pi)\right)\right).$$

The distribution ρ_{α_0} is a nonnegative, $\mathcal{L}^1(L^2(0,2\pi))$-valued measure on $\mathbb{R}/2\pi\mathbb{Z}\times\mathbb{R}_s\times\mathbb{R}_E\times\mathbb{R}_H\times\mathbb{R}_t$.

In addition, ρ_{α_0} is supported in $\{s\in[-\cos\alpha_0, \cos\alpha_0], E=\sqrt{2H}\}$.

Proof Note that $\chi_0\mathscr{V}\mathscr{U}u_h(s,H)$ is bounded in $L^2(\mathbb{R}_s\times\mathbb{R}_H, L^2(0,2\pi))$. The Calderón–Vaillancourt theorem [18] gives that the operators $\mathcal{K}$ with symbols of the form (4.28) are uniformly bounded with respect to h. Therefore, the linear map

$$L_h : K \mapsto \langle n_h^{\alpha_0}, K\rangle$$

is uniformly bounded as $h\to 0$. As a consequence, for any K, up to extraction of a subsequence, it has a limit $l(K)$.

Considering a countable dense subset of $C_c^\infty(\mathbb{T}_\omega\times\mathbb{R}_s\times\mathbb{R}_E\times\mathbb{R}_H\times\mathbb{R}_t; \mathcal{K}(L^2(0,2\pi)))$, and using a diagonal extraction process, one finds a sequence (h_n) tending to 0 as n goes to $+\infty$ such that for any $K\in C_c^\infty(\mathbb{R}/2\pi\mathbb{Z}\times\mathbb{R}_s\times\mathbb{R}_E\times\mathbb{R}_H\times\mathbb{R}_t; \mathcal{K}(L^2(0,2\pi)))$, the sequence $L_{h_n}(K)$ has a limit as n goes to $+\infty$.

The limit is a linear form on $C_c^\infty(\mathbb{T}_\omega\times\mathbb{R}_s\times\mathbb{R}_E\times\mathbb{R}_H\times\mathbb{R}_t; \mathcal{K}(L^2(0,2\pi)))$, characterized by an element ρ_{α_0} of the dual space $\mathcal{D}'(\mathbb{T}_\omega\times\mathbb{R}_s\times\mathbb{R}_E\times\mathbb{R}_H\times\mathbb{R}_t, \mathcal{L}^1(L^2(0,2\pi)))$.

The positivity of the limit is classical. Note that it is immediately seen in the expression (4.29). □

Comparing with (4.30), we obtain the following result.

Corollary 4.13 *For every $b\in\mathcal{S}$, we have*

$$\int b(s,\theta,E,J,\eta,H,t)\mu_{\alpha_0}(ds,d\theta,dJ,d\eta,dE,dH,dt)$$
$$= \operatorname{Tr}_{L^2(0,2\pi)}\int K_{b,0,\infty}(s,E,H,t)_\omega\ \rho_{\alpha_0}(d\omega,ds,dE,dH,dt).$$

Remember that $K_{b,0,R}(s,E,H,t)=b(s,\theta,0,D_\theta,H,t)\chi(D_\theta/R)$, so that $K_{b,0,\infty}(s,E,H,t)=b(s,\theta,0,D_\theta,H,t)$.

Corollary 4.14 *If b does not depend on η then the above identity can be rewritten as*:

$$\int b(s,\theta,E,J,H,t)\mu_{\alpha_0}(ds,d\theta,dJ,d\eta,dE,dH,dt)$$
$$=\mathrm{Tr}_{L^2(0,2\pi)}\int m_b(s,E,H,t)\ \rho_{\alpha_0}(d\omega,ds,dE,dH,dt),$$

where $m_b(s,E,H,t)$ is the multiplication operator by $b(s,\theta,E,0,H,t)$ acting on $L^2(0,2\pi)$.

Note that $\int b(s,\theta,E,J,H,t)\mu_{\alpha_0}(ds,d\theta,dJ,d\eta,dE,dH,dt)\geq 0$ if b does not depend on η and $b\geq 0$. Thus the projection on μ_{α_0} on the variables (s,θ,E,J,H,t) defines a nonnegative measure.

We finish this section by explaining why this implies Theorem 4.9 (ii). The fact that μ_{α_0} is carried by the set $\{J'=0\}$ is obvious from the last line of Corollary 4.14.

If $b\in\mathcal{S}^\sigma$ does not depend on η, Proposition 4.7 implies that

$$\int b(s,\theta,E,J,H,t)\mu_{\alpha_0}(ds,d\theta,dJ,d\eta,dE,dH,dt)$$
$$=\int \langle b\rangle_{\alpha_0}(\theta,E,J,H,t)\mu_{\alpha_0}(ds,d\theta,dJ,d\eta,dE,dH,dt).$$

We know from Sect. 2.8.2 that $\mu=\mu_{sc}$ does not charge the set S. Since $\mu_{\alpha_0}\leq\mu$ by (4.10), the measure μ_{α_0} does not charge the set $\{s=\pm\cos\alpha_0\}$, and the previous equality actually holds for all $b\in\mathcal{S}$. Thus for all $b\in\mathcal{S}$ we get the formula

$$\int b(s,\theta,E,J,H,t)\mu_{\alpha_0}(ds,d\theta,dJ,d\eta,dE,dH,dt)$$
$$=\mathrm{Tr}_{L^2(0,2\pi)}\int m_{\langle b\rangle_{\alpha_0}}(E,0,H,t)\,\rho_{\alpha_0}(d\omega,ds,dE,dH,dt)$$
$$=\mathrm{Tr}_{L^2(0,2\pi)}\int m_{\langle b\rangle_{\alpha_0}}(E,0,H,t)\,\tilde\rho_{\alpha_0}(dE,dH,dt),$$

where $\tilde\rho_{\alpha_0}$ is the measure ρ_{α_0} integrated with respect to ω and s.

The Radon–Nikodym theorem [23, Appendix] implies that the operator valued measure $\tilde\rho_{\alpha_0}$ can also be written as $\tilde\rho_{\alpha_0}=\tilde\sigma_{\alpha_0}\tilde\ell_{\alpha_0}$ where $\tilde\ell_{\alpha_0}=\mathrm{Tr}(\tilde\rho_{\alpha_0})$ is a nonnegative scalar measure on $\mathbb{R}_E\times\mathbb{R}_H\times\mathbb{R}_t$, and

$$\tilde\sigma_{\alpha_0}:\mathbb{R}_E\times\mathbb{R}_H\times\mathbb{R}_t\to\mathcal{L}^1_+\left(L^2(0,2\pi)\right),$$

is an integrable function with respect to $\tilde{\ell}_{\alpha_0}$, taking values in the set of non-negative trace-class operators on $L^2(0, 2\pi)$. Note that $\mathrm{Tr}(\tilde{\sigma}_{\alpha_0}) = 1$. For all $b \in \mathcal{S}$ we finally get the formula

$$\begin{aligned}&\int b(s, \theta, E, J, H, t)\mu_{\alpha_0}(ds, d\theta, dJ, d\eta, dE, dH, dt)\\&\quad = \int \Big(\mathrm{Tr}_{L^2(0,2\pi)} m_{\langle b\rangle_{\alpha_0}}(E, 0, H, t)\ \tilde{\sigma}_{\alpha_0}(E, H, t)\Big)\,\tilde{\ell}_{\alpha_0}(dE, dH, dt).\end{aligned}$$

Let us fix (E, H, t) and define

$$\begin{aligned}&\int b(s, \theta, E, 0, H, t) d\nu_{E,H,t}(s, \theta)\\&\quad := \mathrm{Tr}_{L^2(0,2\pi)} m_{\langle b\rangle_{\alpha_0}}(E, 0, H, t)\ \tilde{\sigma}_{\alpha_0}(E, H, t).\end{aligned}$$

This formula, defined a priori for continuous b, extends to b measurable with respect to (s, θ). If $b(\cdot, \cdot, E, 0, H, t)$ vanishes for Lebesgue-almost every (s, θ), the multiplication operator $m_{\langle b\rangle_{\alpha_0}}(E, 0, H, t)$ vanishes on $L^2(0, 2\pi)$, and

$$\mathrm{Tr}_{L^2(0,2\pi)} m_{\langle b\rangle_{\alpha_0}}(E, 0, H, t)\ \tilde{\sigma}_{\alpha_0}(E, H, t) = 0,$$

which proves that the measure $\nu_{E,H,t}$ is absolutely continuous as announced in Theorem 4.9 (ii).

4.6 Propagation law for ρ_{α_0}

We now show that the operator-valued measure ρ_{α_0} constructed in the previous section possesses some invariance properties. Below, the notation $\langle V\rangle_{\alpha_0}$ stands short for the function $\langle V\rangle_{\alpha_0} \circ \Phi(s, \theta, E, -E \sin\alpha_0, t)$, a function that actually does not depend on s and is 2π-periodic in θ.

Proposition 4.15 (i) *If K satisfies (4.26), we have*

$$\mathrm{Tr}_{L^2(0,2\pi)} \int E\, \partial_s K\,(\omega, s, E, H, t)\, \rho_{\alpha_0}\,(d\omega, ds, dE, dH, dt) = 0. \quad (4.32)$$

(ii) *If in addition $K(s, \sqrt{2H}, H, t)$ does not depend on s, we have*

$$\begin{aligned}&\int \mathrm{Tr}\left(-\cos^2\alpha_0\,\partial_t K + i\left[-\frac{\partial_\theta^2}{2} + \cos^2\alpha_0 \langle V\rangle_{\alpha_0}, K\right]_\omega\right)(\omega, s, E, H, t)\\&\quad \rho_{\alpha_0}\,(d\omega, ds, dE, dH, dt) = 0, \qquad (4.33)\end{aligned}$$

where $[-\frac{\partial_\theta^2}{2} + \cos^2\alpha_0\langle V\rangle_{\alpha_0}, K]_\omega$ *means that we are considering* ∂_θ^2 *acting on* $\mathcal{H}_\omega$ *(in other words,* $L^2(0, 2\pi)$ *with Floquet-periodic boundary condition* $(f(\theta + 2\pi) = f(\theta)e^{i\omega})$*.*

The proof of this key proposition is postponed to the end of this Section. Let us first draw some of its consequences in view of Theorem 2.5.

Remark 4.16 Proposition 4.15 (ii) implies the following. Take $K = a(\omega, E, H, t)Id_{L^2(0,2\pi)}$ with a a scalar continous function (independent on s), then

$$\mathrm{Tr}\left(\int \partial_t a\,(\omega, E, H, t)\,\rho_{\alpha_0}\,(d\omega, ds, dE, dH, dt)\right) = 0.$$

Therefore, the image of ρ_{α_0} by the projection on $\mathbb{R}_s$,

$$\overline{\rho}_{\alpha_0}(d\omega, dE, dH, dt) := \int \rho_{\alpha_0}(d\omega, ds, dE, dH, dt)$$

is such that $\mathrm{Tr}(\overline{\rho}_{\alpha_0})$ does not depend on t.

Remark 4.17 The Radon–Nikodym theorem [23, Appendix] now implies that the operator valued measure $\overline{\rho}_{\alpha_0}$ can also be written as $\overline{\rho}_{\alpha_0} = \sigma_{\alpha_0}\ell_{\alpha_0}$ where $\ell_{\alpha_0} = \mathrm{Tr}(\overline{\rho}_{\alpha_0})$ is a nonnegative scalar measure on $\mathbb{R}/2\pi\mathbb{Z} \times \mathbb{R}_E \times \mathbb{R}_H$, and

$$\sigma_{\alpha_0} : \mathbb{T}_\omega \times \mathbb{R}_E \times \mathbb{R}_H \times \mathbb{R}_t \to \mathcal{L}^1_+\left(L^2(0, 2\pi)\right),$$

is an integrable function with respect to ℓ_{α_0}, taking values in the set of non-negative trace-class operators on $L^2(0, 2\pi)$. Note that $\mathrm{Tr}(\sigma_{\alpha_0}) = 1$.

Corollary 4.18 *Let* $\overline{\rho}_{\alpha_0}$ *as in Remark 4.16 and let* ℓ_{α_0} *and* σ_{α_0} *as in Remark 4.17. Then for* ℓ_{α_0}*-almost every* (ω, E, H)*, we have*

$$-\cos^2\alpha_0\partial_t\sigma_{\alpha_0} + i\left[-\frac{\partial_\theta^2}{2} + \cos^2\alpha_0\langle V\rangle_{\alpha_0}, \sigma_{\alpha_0}\right]_\omega = 0$$

in $\mathcal{D}'(\mathbb{R}_t; \mathcal{L}^1_+(L^2(0, 2\pi)))$*.*

Therefore, for ℓ_{α_0}*-almost every* (ω, E, H)*,* σ_{α_0} *coincides with a continuous function in*

$$C^0\left(\mathbb{R}_t; \mathcal{L}^1_+\left(L^2(0, 2\pi)\right)\right)$$

and

$$\sigma_{\alpha_0}(\omega, E, H, t) = U_{\alpha_0,\omega}(t)\sigma_{\alpha_0}(\omega, E, H, 0)U^*_{\alpha_0,\omega}(t),$$

where $U_{\alpha_0,\omega}(t)$ *is the unitary propagator of the equation*

$$-\cos^2\alpha_0 D_t v(t,\theta) + \left(-\frac{1}{2}\partial_\theta^2 + \cos^2\alpha_0 \langle V\rangle_{\alpha_0} \circ \Phi\right) v(t,\theta) = 0.$$

Proof We first rewrite (4.33) for s-independent operators K as

$$\int \mathrm{Tr}\left\{\left(-\cos^2\alpha_0\,\partial_t K + i\left[-\frac{\partial_\theta^2}{2} + \cos^2\alpha_0\langle V\rangle_{\alpha_0}, K\right]_\omega\right)\sigma_{\alpha_0}\right\}$$
$$\ell_{\alpha_0}\,(d\omega, dE, dH)\,dt = 0.$$

Therefore, we have, for all such K, the identity

$$\int \mathrm{Tr}\left\{K\left(-\cos^2\alpha_0\,\partial_t\sigma_{\alpha_0} + i\left[-\frac{\partial_\theta^2}{2} + \cos^2\alpha_0\langle V\rangle_{\alpha_0}, \sigma_{\alpha_0}\right]_\omega\right)\right\}$$
$$\ell_{\alpha_0}\,(d\omega, dE, dH)\,dt = 0,$$

which concludes the proof of Corollary 4.18. □

To conclude this section, let us now prove its main result, namely Propos-tion 4.15.

Proof of Propostion 4.15 As was already mentioned, it is equivalent to consider a family of kernels depending on ω, $K(\omega, s, E, H, t)(\theta, \theta')$ defined for $(\theta,\theta') \in (0, 2\pi)^2$, and a kernel $K(s, E, H, t)(\theta, \theta')$ defined for $(\theta, \theta') \in \mathbb{R}^2$ and satisfying $K(s, E, H, t)(\theta, \theta') = K(s, E, H, t)(\theta + 2\pi, \theta' + 2\pi)$. The link between both representations is the formula

$$K(\omega, s, E, H, t)(\theta, \theta') = \sum_{n\in\mathbb{Z}} K(s, E, H, t)(\theta, \theta' + 2n\pi)e^{in\omega}.$$

By a density argument, it is enough to treat the case where $K(s, E, H, t)$ is smooth in (s, E, H, t) and is a pseudodifferential operator on $L^2(\mathbb{R})$. By this, we mean that there is a $b_0(s, \theta, E, \eta, H, t) \in C_c^\infty(\mathbb{R}\times\mathbb{R}/2\pi\mathbb{Z}\times\mathbb{R}^4)$ such that $K(s, E, H, t) = b_0(s, \theta, E, D_\theta, H, t)$. As ρ_{α_0} is supported by $\{E = \sqrt{2H}\}$, we may further assume that K satisfies (4.25) instead of (4.26).

If K satisfies (4.25), then we have $b_0(\cos\alpha, \theta, E, \eta, H, t) = b_0(-\cos\alpha, \theta + \pi + 2\alpha, E, \eta, H, t)$ for $\alpha = \arcsin(\frac{\sqrt{2H}\sin\alpha_0}{E})$. We can extend b_0 to a function $b(s, \theta, E, J', \eta, H, t) \in C_c^\infty(\mathbb{R}\times\mathbb{R}/2\pi\mathbb{Z}\times\mathbb{R}^5)$ such that, for $J' = 0$, we have $b(s, \theta, E, 0, \eta, H, t) = b_0(s, \theta, E, \eta, H, t)$, and such that b satisfies the symmetry condition (B) with $\sin\alpha = -\frac{J' - \sqrt{2H}\sin\alpha_0}{E}$. We are now back to our previous notation. The proof Proposition 4.15 (i) goes exactly along the lines of the proof of Proposition 4.7 (see Remark 4.8).

Let us now focus on the proof of (4.33).

If $K(s, \sqrt{2H}, H, t)$ does not depend on s, then $b_0(s, \sqrt{2H}, \theta, \eta, H, t)$ does not depend on s, and we can impose that the function b constructed above satisfy equation (D).

Letting $\eta = \frac{J'}{h}$, we note that, for η in the (compact) support of $b(s, \sqrt{2H}, \theta, J', \eta, H, t)$, we have

$$\alpha - \alpha_0 \sim \frac{-h\eta}{\sqrt{2H}\cos\alpha_0}(1 + O(h))$$

so that

$$\frac{-\eta\cos\alpha}{\sqrt{2H}(\alpha - \alpha_0)} \sim \frac{\cos^2\alpha_0}{h}(1 + O(h)). \tag{4.34}$$

We set

$$\mathcal{Q}_0 := \int \mathrm{Tr}\left(-\cos^2\alpha_0\,\partial_t K + i\left[-\frac{\partial_\theta^2}{2} + \cos^2\alpha_0\langle V\rangle_{\alpha_0}, K\right]_\omega\right)(\omega, s, E, H, t)$$
$$\rho_{\alpha_0}(d\omega, ds, dE, dH, dt),$$

so that proving (4.33) amounts to showing that $\mathcal{Q}_0 = 0$.

First note that

$$\begin{aligned}\mathcal{Q}_0 &= \int \mathrm{Tr}\left(-\cos^2\alpha_0\,\partial_t K + i\left[-\frac{\partial_\theta^2}{2} + \cos^2\alpha_0\langle V\rangle_{\alpha_0}, K\right]_\omega\right)\left(\omega, s, \sqrt{2H}, H, t\right)\\ &\quad \rho_{\alpha_0}(d\omega, ds, dE, dH, dt)\\ &= \int \mathrm{Tr}\left(-\cos^2\alpha_0\,\partial_t K + i\left[-\frac{\partial_\theta^2}{2} + \cos^2\alpha_0 V, K\right]_\omega\right)\left(\omega, s, \sqrt{2H}, H, t\right)\\ &\quad \rho_{\alpha_0}(d\omega, ds, dE, dH, dt),\end{aligned} \tag{4.35}$$

since ρ_{α_0} is carried by $E = \sqrt{2H}$. With a slight abuse of notation we denoted by $V = V(s, E, t)$ the operator of multiplication by $V \circ \Phi(s, \theta, E, -E\sin\alpha_0, t)$ acting on $L^2(0, 2\pi)$. Note that it does not depend on ω. It satisfies the condition (4.26) since the function $V \circ \Phi$ satisfies the symmetry condition (B) (since V is only a function of z in the old coordinates). In (4.35) we used the fact that $K(s, \sqrt{2H}, H, t)$ does not depend on s, and the result of Proposition 4.15 (i), to replace $\langle V\rangle_{\alpha_0}$ by V.

Now, by definition of ρ_{α_0}, we have

$$\begin{aligned}\mathcal{Q}_0 &= \lim\left\langle n_h^{\alpha_0}, -\cos^2\alpha_0\,\partial_t K + i\left[-\frac{\partial_\theta^2}{2}+\cos^2\alpha_0 V, K\right]_\omega\right\rangle\\ &= \lim\Big\langle \chi_0\mathscr{V}\mathscr{U}u_h,\\ &\quad \left(-\cos^2\alpha_0\,\partial_t K + i\left[-\frac{\partial_\theta^2}{2}+\cos^2\alpha_0 V, K\right]\right)\left(s, hD_s, H, h^2D_t\right)\mathscr{V}\mathscr{U}u_h\Big\rangle_{L^2(\mathbb{R}_s\times\mathbb{R}_H, L^2(\mathbb{R}_\theta))}.\end{aligned}$$

Using the fact that $K(s, E, H, t) = b_0(s, \theta, E, D_\theta, H, t)$ and the commutator calculus rule (2.10) for the standard quantization, we obtain

$$\begin{aligned}\mathcal{Q}_0 &= \lim_{h\to 0^+} i\left\langle \chi_0\mathscr{V}\mathscr{U}u_h, \left(\left[\cos^2\alpha_0 V, K\right]\right)\left(s, hD_s, H, h^2D_t\right)\mathscr{V}\mathscr{U}u_h\right\rangle_{L^2(\mathbb{R}_s\times\mathbb{R}_H, L^2(\mathbb{R}_\theta))}\\ &\quad+\left\langle \chi_0\mathscr{V}\mathscr{U}u_h, \mathrm{Op}_h\left(\left(\eta\partial_\theta - i\frac{\partial_\theta^2}{2}-\cos^2\alpha_0\partial_t\right)b\left(s,\theta,E,J',\frac{J'}{h},H,ht\right)\right)\mathscr{V}\mathscr{U}u_h\right\rangle_{L^2(\mathbb{R}_s\times\mathbb{R}_\theta\times\mathbb{R}_H)},\end{aligned}$$

using the notation $\eta = \frac{J'}{h} = \frac{J+\sqrt{2H}\sin\alpha_0}{h}$.

We now set

$$\begin{aligned}\mathcal{Q}_1 &:= \lim_{h\to 0^+}\left\langle \chi_0\mathscr{V}\mathscr{U}u_h, \mathrm{Op}_h\left(\left(\eta\partial_\theta - \cos^2\alpha_0\partial_t\right)b\left(s,\theta,E,J',\frac{J'}{h},H,ht\right)\right)\mathscr{V}\mathscr{U}u_h\right\rangle_{L^2(\mathbb{R}_s\times\mathbb{R}_\theta\times\mathbb{R}_H)}\\ &\quad+i\left\langle \chi_0\mathscr{V}\mathscr{U}u_h, \left(\left[\cos^2\alpha_0 V, K\right]\right)\left(s, hD_s, H, h^2D_t\right)\mathscr{V}\mathscr{U}u_h\right\rangle_{L^2(\mathbb{R}_s\times\mathbb{R}_H, L^2(\mathbb{R}_\theta))},\end{aligned}$$

so that we have

$$\mathcal{Q}_0 = \mathcal{Q}_1 + \lim_{h\to 0^+}\left\langle \chi_0\mathscr{V}\mathscr{U}u_h, \mathrm{Op}_h\left(-i\frac{\partial_\theta^2}{2}b\left(s,\theta,E,J',\frac{J'}{h},H,ht\right)\right)\mathscr{V}\mathscr{U}u_h\right\rangle_{L^2(\mathbb{R}_s\times\mathbb{R}_\theta\times\mathbb{R}_H)}. \tag{4.36}$$

Let us for the moment focus on the term $\mathcal{Q}_1$, involving only derivatives of order 1 of b. As in Remark 4.8, we let $\mathbf{P}(s, \theta, E, J, t, H) = b(s, \theta, E, J + \sqrt{2H}\sin\alpha_0, \frac{J+\sqrt{2H}\sin\alpha_0}{h}, H, -t)$. Since b is compactly supported in the fifth variable, this is also, modulo $O(h)$,

$$\mathbf{P}(s, \theta, E, J, t, H) = b\left(s, \theta, E, 0, \frac{J+\sqrt{2H}\sin\alpha_0}{h}, H, -t\right).$$

Still using the notation $\eta = \frac{J'}{h} = \frac{J+\sqrt{2H}\sin\alpha_0}{h}$, we have

$$\begin{aligned}
\mathcal{Q}_1 &= \lim \left\langle \mathscr{U} u_h, \mathrm{Op}_h \left((\eta\partial_\theta - \cos^2\alpha_0 \partial_t)\mathbf{P}(s,\theta,E,J,t,hH) \right) \mathscr{U} u_h \right\rangle_{L^2(\mathbb{R}_s\times\mathbb{T}_\theta\times\mathbb{R}_t)} \\
&\quad + i \left\langle \chi_0 \mathscr{V}\mathscr{U} u_h, \left(\left[\cos^2\alpha_0 V, K \right] \right) \left(s, hD_s, H, h^2 D_t\right) \mathscr{V}\mathscr{U} u_h \right\rangle_{L^2(\mathbb{R}_s\times\mathbb{R}_H, L^2(\mathbb{R}_\theta))} \\
&= \lim \left\langle u_h, \widetilde{\mathcal{A}}_{t,h^2D_t} \left((\eta\partial_\theta - \cos^2\alpha_0\partial_t)\mathbf{P} \right) u_h \right\rangle_{L^2(\mathbb{R}^2\times\mathbb{R}^2\times\mathbb{R}_t)} \\
&\quad + i \left\langle u_h, \left[\cos^2\alpha_0 V, \widetilde{\mathcal{A}}_{t,h^2D_t}(\mathbf{P}) \right] u_h \right\rangle_{L^2(\mathbb{R}^2\times\mathbb{R}^2\times\mathbb{R}_t)} \\
&= \lim \left\langle u_h, \widetilde{\mathcal{A}}_{t,h^2D_t} \left(-\frac{\eta\cos\alpha}{E(\alpha_0-\alpha)} (E\,\partial_s - h\partial_t)\mathbf{P} \right) u_h \right\rangle \\
&\quad + i \left\langle u_h, \left[\cos^2\alpha_0 V, \widetilde{\mathcal{A}}_{t,h^2D_t}(\mathbf{P}) \right] u_h \right\rangle.
\end{aligned}$$

Using that b (and thus also $\mathbf{P}$) satisfies equation (D), together with (4.34), we obtain

$$\begin{aligned}
\mathcal{Q}_1 = \lim \frac{\cos^2\alpha_0}{h} &\left\langle u_h, \widetilde{\mathcal{A}}_{t,h^2D_t} \left((E\,\partial_s - h\partial_t)\mathbf{P} \right) u_h \right\rangle \\
&+ i \left\langle u_h, \left[\cos^2\alpha_0 V, \widetilde{\mathcal{A}}_{t,h^2D_t}(\mathbf{P}) \right] u_h \right\rangle.
\end{aligned} \tag{4.37}$$

Finally, we use again the Schrödinger equation (4.16) satisfied by u_h extended to $\mathbb{R}^2$, and rewrite the last line as

$$\begin{aligned}
\mathcal{Q}_1 &= \lim \frac{\cos^2\alpha_0}{h} \left\langle u_h, \left[-\frac{ih}{2}\Delta + ihV - h\partial_t, \widetilde{\mathcal{A}}_{t,h^2D_t}(\mathbf{P}) \right] u_h \right\rangle \\
&= \lim -\frac{\cos^2\alpha_0}{h} \left\langle \frac{ih}{2}\frac{\partial u_h}{\partial n} \otimes \delta_{\partial\mathbb{D}}, \widetilde{\mathcal{A}}_{t,h^2D_t}(\mathbf{P})\, u_h \right\rangle \\
&\quad + \lim -\frac{\cos^2\alpha_0}{h} \left\langle u_h, \widetilde{\mathcal{A}}_{t,h^2D_t}(\mathbf{P}) \frac{ih}{2}\frac{\partial u_h}{\partial n} \otimes \delta_{\partial\mathbb{D}} \right\rangle.
\end{aligned} \tag{4.38}$$

Here we need the knowledge of $\widetilde{\mathcal{A}}_{t,h^2D_t}(\mathbf{P})$ modulo $O(h^2)$ (because of the factor $\frac{\cos^2\alpha_0}{h}$ that appears in the previous expression). Our calculations of Proposition 10.7 give us the expression

$$\begin{aligned}
\widetilde{\mathcal{A}}_{t,h^2D_t}(\mathbf{P}) &= \mathbf{A}(r,u,h\sqrt{2D_t},hD_u,t) + \mathbf{B}(r,u,h\sqrt{2D_t},hD_u,t) \circ hD_r \\
&\quad + ih\mathbf{C}(r,u,h\sqrt{2D_t},hD_u,t) + ih\mathbf{D}(r,u,h\sqrt{2D_t},hD_u,t) \circ hD_r
\end{aligned}$$

if $z = (-r\sin u, r\cos u)$ is the decomposition in polar coordinates and $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, $\mathbf{D}$ are the functions associated to $\mathbf{P}$ by the formulas of Definition 10.6.

The terms $\mathbf{A}$, $\mathbf{C}$ give a vanishing contribution in formula (4.38) because they are radial operators and u_h satisfies a Dirichlet boundary condition. The term $\mathbf{B}$ gives a vanishing condition if b (and hence $\mathbf{P}$) satisfy the symmetry condition (B): in that case we have $\mathbf{B}(1, u, h\sqrt{2D_t}, hD_u, t) = 0$. So there just remains to look at the term $\mathbf{D}(1, u, h\sqrt{2D_t}, hD_u, t)$.

Look at formula (10.10) defining the function $\mathbf{D}$. Remember that $\mathbf{P}(s, \theta, \sqrt{2H}, J, t, H)$ is supported where $J + \sqrt{2H}\sin\alpha_0 = O(h)$, so that we have $\partial_s \mathbf{P} = O(h)$; also note that, on the set $\{J = -E\sin\alpha_0\}$, the boundary equation $r = 1$ amounts to $s = \pm\cos\alpha_0$, $\cos\theta_1(r, J, E) = \pm\cos\alpha_0$, so that $s\cos\theta_1(r, J, E) = \cos^2\alpha_0$ in formulas (10.10) and the following lines. We see that the function $\mathbf{D}(1, u, \sqrt{2H}, J, t)$ coincides, modulo $O(h)$, with $\frac{1}{2H\cos^2\alpha_0}\mathbf{P}^\sigma(1, u, \sqrt{2H}, J, t)$, so that

$$\begin{aligned}\mathcal{Q}_1 &= -\lim\left\langle \frac{h}{2}\frac{\partial u_h}{\partial n} \otimes \delta_{\partial\mathbb{D}}, \mathbf{D}(1, u, h\sqrt{2D_t}, hD_u, t) \circ hD_r u_h \right\rangle \\ &\quad + h\lim\left\langle u_h, \mathbf{D}(1, u, h\sqrt{2D_t}, hD_u, t) \circ hD_r \frac{h}{2}\frac{\partial u_h}{\partial n} \otimes \delta_{\partial\mathbb{D}} \right\rangle \\ &= -\frac{1}{2\cos^2\alpha_0}\lim\left\langle h\frac{\partial u_h}{\partial n}, (E^{-2}\partial_2^2\mathbf{P}^\sigma)(1, u, h\sqrt{2D_t}, -h\sqrt{2D_t}\alpha_0, ht, h^2 D_t)h\frac{\partial u_h}{\partial n} \right\rangle.\end{aligned}$$

Hence, we obtain

$$\begin{aligned}\mathcal{Q}_1 &= -\lim\frac{\cos^2\alpha_0}{h} ih\left\langle \frac{h}{2}\frac{\partial u_h}{\partial n} \otimes \delta_{\partial\mathbb{D}}, \mathbf{D}(1, u, h\sqrt{2D_t}, hD_u, t) \circ hD_r u_h \right\rangle \\ &= -\frac{i}{2}\lim\left\langle h\frac{\partial u_h}{\partial n}, (E^{-2}\partial_2^2\mathbf{P})(1, u, h\sqrt{2D_t}, -h\sqrt{2D_t}\alpha_0, ht, h^2 D_t)h\frac{\partial u_h}{\partial n} \right\rangle.\end{aligned}$$

Using Remark 4.8, this limit expressed in terms of boundary data can also be expressed in terms of the interior, and we see that it equals

$$\begin{aligned}\mathcal{Q}_1 &= \frac{i}{2}\lim\left\langle \mathscr{U}u_h, \mathrm{Op}_h\left(\partial_2^2\mathbf{P}(s, \theta, E, J', \frac{J'}{h}, H, ht)\right)\mathscr{U}u_h \right\rangle_{L^2(\mathbb{R}_s\times\mathbb{T}_\theta\times\mathbb{R}_t)} \\ &= \frac{i}{2}\lim\left\langle \mathscr{V}\mathscr{U}u_h, \mathrm{Op}_h\left(\partial_2^2 b(s, \theta, E, J', \frac{J'}{h}, H, ht)\right)\mathscr{V}\mathscr{U}u_h \right\rangle_{L^2(\mathbb{R}_s\times\mathbb{T}_\theta\times\mathbb{R}_H)}.\end{aligned}$$

Finally coming back to (4.36), this yields $\mathcal{Q}_0 = 0$, that is, identity (4.33). This concludes the proof of Proposition 4.15. □

5 End of the semiclassical construction: proof of Theorem 2.5

In this section, we first prove Proposition 2.9, and then conclude the proof of Theorem 2.5.

5.1 A proof of Proposition 2.9

For a a smooth compactly supported function on $\mathbb{R}^4$, we show that

$$\lim_{h\to 0} \langle u_h, \mathrm{Op}_h(\partial_t a(|\xi|^2, x\xi_y - y\xi_x, t, hH)u_h\rangle_{L^2(\mathbb{R}^2\times\mathbb{R})} = 0.$$

According to Proposition 8.3, this limit is the same as

$$\lim_{h\to 0} \langle u_h, \mathrm{Op}_h(\partial_t a(2hH, x\xi_y - y\xi_x, t, hH)u_h\rangle_{L^2(\mathbb{R}^2\times\mathbb{R})},$$

which is

$$\begin{aligned}&\lim_{h\to 0} \langle u_h, [\partial_t, \mathrm{Op}_h(a(2hH, x\xi_y - y\xi_x, t, hH)]u_h\rangle_{L^2(\mathbb{R}^2\times\mathbb{R})}\\ &\quad = \lim_{h\to 0} \langle u_h, [\partial_t, a(2h^2 D_t, hD_u, t, h^2 D_t)]u_h\rangle,\end{aligned}$$

where $z = (-r\sin u, r\cos u)$ is the decomposition of $z = (x, y)$ into polar coordinates. Because of the equation satisfied by u_h, this is also (with Δ_D the Dirichlet laplacian)

$$\lim_{h\to 0} \left\langle u_h, \left[-i\frac{\Delta_D}{2} + iV, a(2h^2 D_t, hD_u, t, h^2 D_t)\right] u_h\right\rangle.$$

Note that $a(2h^2 D_t, hD_u, t, h^2 D_t)$ actually defines an operator on $L^2(\mathbb{D})$ as it is tangential to $\partial\mathbb{D}$, which is why the scalar product on $\mathbb{R}^2$ has been replaced by a scalar product on $\mathbb{D}$. This limit vanishes, because Δ_D commutes with $a(2h^2 D_t, hD_u, t, h^2 D_t)$ and because

$$\left[V, a(2h^2 D_t, hD_u, t, h^2 D_t)\right] = O(h).$$

This concludes the proof of Proposition 2.9.

5.2 End of the proofs of Theorems 2.5 and 2.10

There only remains to fit all the pieces together to conclude the proofs of Theorems 2.5 and 2.10.

The measures m_{α_0} ($\alpha_0 \in \pi\mathbb{Q}$) in the theorem are the ones defined in (4.10). The fact that m_{α_0} is of the announced form is the contents of Corollaries 4.14 and 4.18, the measure ℓ_{α_0} and the function σ_{α_0} of Theorem 2.5 (iii) being the ones appearing in Remark 4.17.

The object called ν_{Leb} in Theorem 2.5 (ii) is defined as

$$\nu_{Leb} = \mu_{sc}\rceil_{\alpha\notin\pi\mathbb{Q}} + \sum_{\alpha_0\in\pi\mathbb{Q}} m^{\alpha_0}$$

where m^{α_0} was defined in Remark 4.6. For $\alpha \notin \pi\mathbb{Q}$, we must have

$$\mu_{sc}\rceil_{\mathcal{I}_\alpha}(t) = \int c_1(t, E, J)\lambda_{E,J}\, d\nu_1(E, J)$$

for some nonnegative measure ν_1 (carried by $\{J = -\sin\alpha E\}$) and some measurable function $c_1(t, E, J)$. But, because the image of μ_{sc} under the map $M : (z, \xi) \mapsto (E, J)$ does not depend on t (Proposition 2.9, proved in the previous section), the function $c_1(t, E, J)$ actually does not depend on t.

The two invariance properties Proposition 4.7 (invariance w.r.t. s) and Theorem 4.9 (i) (invariance w.r.t. θ) also imply that m^{α_0} is of the form

$$m^{\alpha_0}(t) = \int c_2(t, E, J)\lambda_{E,J}\, d\nu_2(E, J)$$

for some nonnegative measure ν_2 (carried by $\{J = -\sin\alpha_0 E\}$).

We now prove that the function $c_2(t, E, J)$ actually does not depend on t. For this, we remark that the same proof as that of Proposition 2.9 above applies if we replace

$$\mathrm{Op}_h\left(\partial_t a(|\xi|^2, x\xi_y - y\xi_x, t, hH)\right)$$

in the first line by

$$\mathscr{U}^*\,\mathrm{Op}_h\left(\partial_t a(E^2, J, t, hH)(1-\chi)\left(\frac{J + \sqrt{2H}\sin\alpha_0}{hR}\right)\right)\mathscr{U}$$

in the limits $h \to 0$ followed by $R \to +\infty$.

Using the notation of Remark 4.6, this shows that the image of m^{α_0} under the map M is independent of t. Since we already know that $m^{\alpha_0}(t)$ is of the form $\int c_2(t, E, J)\lambda_{E,J} d\nu_2(E, J)$ for some nonnegative measure ν_2, we conclude that $c_2(t, E, J)$ actually does not depend on t.

The proof of Theorem 2.5 is now complete. Theorem 2.10 also follows if we remember Theorem 4.9.

6 The microlocal construction: sketch of the proof of Theorem 2.7

Herein we use the definitions and notation introduced in Sects. 2.1 and 2.5.

Let (u_n^0) be a sequence of initial data, normalized in L^2, and as above denote $u_n(z,t) = U_V(t)u_n^0(z)$ or in short $u_n = U_V u_n^0$.

6.1 Structure of μ_c, proof of Theorem 2.8

Let $a \in \mathcal{S}_0$. Recall that $\langle \mu_c, a\rangle$ is defined as the limit (after extraction of subsequences) as $n \to +\infty$ followed by $R \to +\infty$ of

$$\left\langle W_{c,n,R}, a\right\rangle := \left\langle u_n, \mathrm{Op}_1\left(\chi\left(\frac{|\xi|^2+|H|}{R^2}\right) a(z,\xi,t,H)\right) u_n\right\rangle_{L^2(\mathbb{R}^2\times\mathbb{R})}. \tag{6.1}$$

This is also

$$\left\langle W_{c,n,R}, a\right\rangle = \left\langle u_n^0, \int_{\mathbb{R}} U_V(t)^{-1}\mathbb{1}_{\mathbb{D}}\,\mathrm{Op}_1\left(\chi\left(\frac{|\xi|^2+|H|}{R^2}\right) a(z,\xi,t,H)\right) \mathbb{1}_{\mathbb{D}}[U_V(\cdot)u_n^0]dt\right\rangle_{L^2(\mathbb{D})}.$$

Note now that for any given R, the operator

$$f \in L^2(\mathbb{D}) \mapsto \int_{\mathbb{R}} U_V(t)^{-1}\mathbb{1}_{\mathbb{D}}\,\mathrm{Op}_1\left(\chi\left(\frac{|\xi|^2+|H|}{R^2}\right) a(z,\xi,t,H)\right) \mathbb{1}_{\mathbb{D}}[U_V(\cdot)f]dt$$

is compact from $L^2(\mathbb{D})$ to itself. As a consequence, calling u^0 a weak-$*$ limit of the sequence (u_n^0) in $L^2(\mathbb{D})$ and writing $u(t,x) = [U_V(t)u^0](x)$, we have for fixed $R > 0$ that the sequence $\left\langle W_{c,n,R}, a\right\rangle$ converges to

$$\left\langle u^0, \int_{\mathbb{R}} U_V(t)^{-1}\mathbb{1}_{\mathbb{D}}\,\mathrm{Op}_1\left(\chi\left(\frac{|\xi|^2+|H|}{R^2}\right) a(z,\xi,t,H)\right) \mathbb{1}_{\mathbb{D}}[U_V(\cdot)u^0]dt\right\rangle_{L^2(\mathbb{D})}.$$

Letting now $R \to +\infty$ we find the expression

$$\begin{aligned}\langle \mu_c, a\rangle &= \left\langle u^0, \int_{\mathbb{R}} U_V(t)^{-1}\mathbb{1}_{\mathbb{D}}\,\mathrm{Op}_1\left(a(z,\xi,t,H)\right) \mathbb{1}_{\mathbb{D}}[U_V(\cdot)u^0]dt\right\rangle_{L^2(\mathbb{D})}\\ &= \left\langle u, \mathrm{Op}_1\left(a(z,\xi,t,H)\right) u\right\rangle_{L^2(\mathbb{R}^2\times\mathbb{R})},\end{aligned}$$

which concludes the proof of Theorem 2.8.

6.2 Structure of μ^∞, proof of Theorem 2.7

Let $\tilde{\xi} = \frac{\xi}{\sqrt{2H}}$. On the support of μ^∞, $\tilde{\xi}$ has norm 1. To any pair $(z, \tilde{\xi}) \in \mathbb{D} \times \mathbb{S}^1$ we now associate $j = x\tilde{\xi}_y - y\tilde{\xi}_x$ and $\alpha = -\arcsin j$ which is the angle that the billiard ray issued from (z, η) makes with the inner normal when it bounces on the boundary of the disk. Exactly as in Lemma 3.2 we decompose μ^∞ as a sum of nonnegative measures:

$$\mu^\infty = \mu^\infty\rceil_{\alpha \notin \pi\mathbb{Q}} + \sum_{r \in \mathbb{Q} \cap [-1/2, 1/2]} \mu^\infty\rceil_{\alpha = r\pi}. \tag{6.2}$$

The invariance (2.18) implies that $\mu^\infty\rceil_{\alpha \notin \pi\mathbb{Q}}$ is of the form $\int_{E>0, |J| \leq E, \alpha \notin \pi\mathbb{Q}} \lambda_{E,J} d\bar{\mu}^\infty(E, J)$. Note that the measure $\bar{\mu}^\infty$ is in the dual of the space of continuous 0-homogeneous functions of (E, J); in other words we may see it as a measure on the set $\{E = 1, |J| \leq 1\}$ and consider only the measures $\lambda_{1,J}$. The fact that $\bar{\mu}^\infty$ does not depend on t is the microlocal version of Proposition 2.9.

We now fix $r_0 \in \mathbb{Q} \cap (-1/2, 1/2)$, write $\alpha_0 = r_0\pi$ and wish to study $\mu^\infty\rceil_{\alpha=\alpha_0}$. We define

$$\begin{aligned}
&\langle \mu^\infty_{\alpha_0}, a \rangle \\
&\quad := \lim_{R' \to +\infty} \lim_{R \to \infty} \lim_{n \to +\infty} \\
&\quad \left\langle u_n, \mathrm{Op}_1 \left(\left(1 - \chi\left(\frac{|\xi|^2 + H}{R^2} \right) \right) a(z, \xi, t, H) \chi\left(\frac{J + \sqrt{2H}\sin\alpha_0}{R'} \right) \right) u_n \right\rangle_{L^2(\mathbb{R}^2 \times \mathbb{R})} \\
&\quad = \lim_{R' \to +\infty} \lim_{R} \lim_{n} \\
&\quad \left\langle \mathscr{U} u_n, \mathrm{Op}_1 \left(\left(1 - \chi\left(\frac{H}{R^2} \right) \right) a \circ \Phi(s, \theta, \sqrt{2H}, J, t, H) \chi\left(\frac{J + \sqrt{2H}\sin\alpha_0}{R'} \right) \right) \mathscr{U} u_n \right\rangle_{L^2(\mathbb{R} \times \mathbb{T} \times \mathbb{R})}
\end{aligned}$$

and

$$\begin{aligned}
&\langle \mu^{\infty,\alpha_0}, a \rangle \\
&\quad := \lim_{R'} \lim_{R \to \infty} \lim_{n \to +\infty} \\
&\quad \left\langle u_n, \mathrm{Op}_1 \left(\left(1 - \chi\left(\frac{|\xi|^2 + H}{R^2} \right) \right) a(z, \xi, t, H) \left(1 - \chi\left(\frac{J + \sqrt{2H}\sin\alpha_0}{R'} \right) \right) \right) u_n \right\rangle_{L^2(\mathbb{R}^2 \times \mathbb{R})} \\
&\quad = \lim_{R'} \lim_{R} \lim_{n} \\
&\quad \left\langle \mathscr{U} u_n, \mathrm{Op}_1 \left(\left(1 - \chi\left(\frac{H}{R^2} \right) \right) a \circ \Phi(s, \theta, \sqrt{2H}, J, t, H) \left(1 - \chi\left(\frac{J + \sqrt{2H}\sin\alpha_0}{R'} \right) \right) \right) \mathscr{U} u_n \right\rangle.
\end{aligned}$$

The following results are proved in essentially the same way as in the semiclassical case. Again, we identify homogeneous functions in (E, H) with functions on the "sphere" $\mathbb{S}^2_{E,H} = \{(E, H), E^2 + 2|H| = 2\}$, which explains that our measures are also defined there.

Theorem 6.1 (i) μ^{∞,α_0} *is invariant under rotations* (*that is*, *under the flow of* P_1)*. Thus it is a multiple of the Lebesgue measure* $\lambda_{1,-\sin\alpha_0}$*. That is,* $\mu^{\infty,\alpha_0} = c(\alpha_0)\lambda_{1,-\sin\alpha_0}$ *with* $c(\alpha_0) \geq 0$ *independent of* t*.*

(ii) *for every* $\alpha_0 \in \pi\mathbb{Q} \cap (-\pi/2, \pi/2)$, *we can build from the sequence of initial conditions* (u_n) *a nonnegative measure* $\sigma_{\alpha_0}(d\omega, ds, dE, dH, dt)$ (*carried by* $\{H = E^2/2\}$) *on* $\mathbb{R}/2\pi\mathbb{Z} \times \mathbb{R}_s \times \mathbb{S}^2_{E,H} \times \mathbb{R}_t$, *taking values in the trace-class operators on* $L^2(0, 2\pi)$, *so that* $\mu^\infty_{\alpha_0}$ *is the measure carried by the set* $\{j = -\sin\alpha_0\} \cap \{H = E^2/2\}$ *such that*

$$\int a_{\mathrm{hom}}(z, \xi, t)\mu^\infty_{\alpha_0}(dz, d\xi, dt, dH)$$
$$= \mathrm{Tr}_{L^2(0,2\pi)}\left(\int m_{a_{\mathrm{hom}}\circ\Phi}(s, \cdot, E, -E\sin\alpha_0, t)\,\sigma_{\alpha_0}(d\omega, ds, dE, dH, dt)\right).$$

If in addition a is symmetric w.r.t. the boundary, we have

$$\int a_{\mathrm{hom}}(z, \xi)\mu^\infty_{\alpha_0}(dz, d\xi, dt, dH)$$
$$= \mathrm{Tr}_{L^2(0,2\pi)}\left(\int U_{\alpha_0,\omega}(t)^* m_{\langle a_{\mathrm{hom}}\rangle_{\alpha_0}\circ\Phi}(\cdot, E, -E\sin\alpha_0)\right.$$
$$\left. U_{\alpha_0,\omega}(t)\,\sigma_{\alpha_0}(d\omega, ds, dE, dH, 0)\, dt\right).$$

The decomposition formula of Theorem 2.7 (i) now holds with

- the distribution $\mu_c \in \mathcal{S}'_0$ described in Sect. 6.1;
- the measure μ_{Leb} given by

$$\mu_{Leb} = \mu^\infty \rceil_{\alpha\notin\pi\mathbb{Q}} + \sum_{\alpha_0\in\pi\mathbb{Q}\cap(-\pi/2,\pi/2)} \mu^{\infty,\alpha_0}\rceil_{\alpha=\alpha_0};$$

- for $\alpha_0 \in \pi\mathbb{Q} \cap (-\pi/2, \pi/2)$ the measure μ_{α_0} given by

$$\mu_{\alpha_0}(t) = \mu^\infty_{\alpha_0}(t);$$

- for $\alpha_0 = \pm\pi/2$ the measure μ_{α_0} given by

$$\mu_{\alpha_0} = \mu^\infty\rceil_{\alpha=\alpha_0}.$$

Theorem 6.1 then implies Theorem 2.7.

7 Proof of Theorems 1.2 and 1.3: Observability inequalities

In this section, we prove Theorems 1.2 and 1.3 using the microlocal version of our results. We could have chosen to do it with semiclassical measures as well. However, since there is no natural frequency-scale, it would have required to perform a dyadic decomposition in frequency (see for instance [6,39]). Note that the idea of proving observability inequalities using microlocal defect measures is due to Lebeau [40].

7.1 Unique continuation for microlocal measures

The goal of this section is to prove a unique continuation result for microlocal measures μ_{ml} associated to solutions of the Schrödinger equation (1.1). According to Theorem 2.7, such a measure decomposes as

$$\mu_{ml} = \mu^\infty + \mu_c,$$

that we shall study independently.

In order to state the result for μ^∞, we introduce the following notation. For $z \in \partial\mathbb{D}$, we define

$$S_z^+ = \{\xi \in \mathbb{R}^2, \xi \cdot z > 0\}, \qquad \overline{S}_z^+ = \{\xi \in \mathbb{R}^2, \xi \cdot z \geq 0\}.$$

The set S^+ defined in Sect. 2.8.1 is $S^+ = \bigcup\{(z,\xi), z \in \partial\mathbb{D}, \xi \in S_z^+\}$ and

$$\bigcup_{z\in\partial\mathbb{D}} \overline{S}_z^+ = \left\{\Phi\left(\left(1-(J/E)^2\right)^{\frac{1}{2}}, \theta, E, J\right), E > 0, |J| \leq E, \theta \in \mathbb{R}/2\pi\mathbb{Z}\right\}.$$

The following two lemmas are respectively useful for the proof of internal and boundary observability.

Lemma 7.1 *Fix $T > 0$. Take $b \in \mathcal{S}^0$ independent of (t, H) and assume that*

$$\text{there exists } z_0 \in \partial\mathbb{D} \text{ such that } b > 0 \text{ in a neighbourhood of } \overline{S}_{z_0}^+. \tag{7.1}$$

Then $\int_0^T \int_{\mathbb{R}^2\times\mathbb{S}^2_{H,\xi}} b^2_{\mathrm{hom}}(z,\xi)\mu^\infty(dz, dH, d\xi, t)dt = 0$ implies $\mu^\infty = 0$ on $\mathbb{R}_t \times \mathbb{R}^2_z \times \mathbb{S}^2_{H,\xi}$.

Lemma 7.2 *Take any nonempty set $\Gamma \subset \partial\mathbb{D}$ and $T > 0$. Then $\mu^\partial_{ml} = 0$ on $T^*((0,T)\times\Gamma)$ implies $\mu^\infty = 0$ on $\mathbb{R}_t \times \mathbb{R}^2_z \times \mathbb{S}^2_{H,\xi}$.*

The proof of these lemmas relies on the properties of μ^∞ together with a unique continuation result for the *one dimensional* Schrödinger flows $U_{\alpha_0,\omega}(t)$ on $L^2(0,2\pi)$ from any nonempty open set $(0,T)\times\Omega$, where $\Omega\subset(0,2\pi)$. Such unique continuation property holds as soon as $\langle V\rangle_{\alpha_0}\in L^\infty((0,T)\times(0,2\pi))$, for instance as a consequence of [37, Appendix B] (see also the references therein).

Concerning μ_c we have the following result.

Lemma 7.3 *Let $\Omega\subset\mathbb{D}$ be a nonempty open set. Assume that the unique continuation property* (UCP$_{V,\Omega,T}$) *holds. Then we have*

$$\langle\mu_c,\mathbb{1}_{(0,T)\times\Omega}\rangle=0\Longrightarrow\mu_c=0.$$

The unique continuation property (UCP$_{V,\Omega,T}$) is for instance known to hold (in any time $T>0$ and for any nonempty open set Ω) if V is analytic in (t,z) as a consequence of the Holmgren theorem (as stated by Hörmander [28, Theorem 5.3.1]). If $V=V(z)$ is smooth and does not depend on t, it is proved in the next section. Note that this last result could be extended to the case where V is continuous outside a set of measure zero, as was done in [6] for the same problem on flat tori.

Remark 7.4 Note that the analogues of the unique continuation results of Lemmas 7.1, 7.2 and 7.3 also hold for semiclassical measures (which do not charge $\{\xi=0\}$). We chose not to state them here for the sake of brevity.

Proof of Lemma 7.1 We decompose μ^∞ as in Theorem 2.7

$$\mu^\infty(t,\cdot)=\mu_{Leb}+\sum_{\alpha_0\in\pi\mathbb{Q}\cap[-\pi/2,\pi/2]}\mu_{\alpha_0}(t,\cdot).$$

As every term in this sum is a non-negative measure, the assumption on μ^∞ implies

$$\int_0^T\int_{\mathbb{R}^2\times\mathbb{S}^2}b_{\rm hom}^2(z,\xi)\mu_{Leb}(dz,dH,d\xi,t)dt=0,\tag{7.2}$$

$$\int_0^T\int_{\mathbb{R}^2\times\mathbb{S}^2}b_{\rm hom}^2(z,\xi)\mu_{\alpha_0}(dz,dH,d\xi,t)dt=0,\tag{7.3}$$

for all $\alpha_0\in\pi\mathbb{Q}\cap[-\pi/2,\pi/2]$.

Still according to Theorem 2.7, μ_{Leb} is of the form $\int_{E>0,|J|\le E}\lambda_{E,J}d\mu'(E,J)$ for some nonnegative measure μ' on he set of pairs (E,J) modulo homotheties. Together with (7.2), this reads

$$0=\int_{E>0,|J|\le E}\int_{T(E,J)}b_{\rm hom}^2\circ\Phi(s,\theta,E,J)\lambda_{E,J}(ds,d\theta)\mu'(dE,dJ).$$

Recall (see Sect. 2.2) that $\lambda_{E,J}(ds, d\theta) = c(E, J) ds d\theta$ where $c(E, J) = (\int_{T(E,J)} ds d\theta)^{-1} > 0$, so that we have

$$0 = \int_{E>0,|J|\leq E} \left(\int_{T(E,J)} b_{\text{hom}}^2 \circ \Phi(s, \theta, E, J) ds d\theta \right) c(E, J) \mu'(dE, dJ).$$

Now, for any (E, J) such that $E > 0$, $|J| \leq E$, there exists $\theta \in \mathbb{S}^1$ (depending only on J/E), such that

$$\Phi\left(\left(1 - (J/E)^2\right)^{\frac{1}{2}}, \theta, E, J \right) \in \overline{S}^+_{z_0}.$$

Assumption (7.1) then implies that $\int_{T(E,J)} b_{\text{hom}}^2 \circ \Phi(s, \theta, E, J) ds d\theta > 0$ for any (E, J). As a consequence, $c(E, J) = 0$ for μ'-almost all (E, J), and μ_{Leb} vanishes identically.

Let us now consider $\alpha_0 = \pm\pi/2$. The rotation invariance given by Theorem 2.7 together with Assumption (7.1) imply that $\mu_{\pm\pi/2}$ vanish.

Let us now consider $\alpha_0 \in \pi\mathbb{Q} \cap (-\pi/2, \pi/2)$. The measure μ_{α_0} is supported by $\mathcal{I}_{\alpha_0}$ and invariant by the billiard flow, so that

$$\begin{aligned}
&\int_0^T \int_{\mathbb{R}^2\times\mathbb{S}^2} b_{\text{hom}}^2 \, \mu_{\alpha_0}(dz, dH, d\xi, t) dt \\
&\quad = \int_0^T \int_{\mathbb{R}^2\times\mathbb{S}^2} \langle b_{\text{hom}}^2 \rangle_{\alpha_0} \, \mu_{\alpha_0}(dz, dH, d\xi, t) dt.
\end{aligned}$$

Using Theorem 2.7 with (7.3), we obtain

$$0 = \int \operatorname{Tr}_{L^2(0,2\pi)} \left(B_{\alpha_0} \sigma_{\alpha_0} \right) d\ell_{\alpha_0} dt, \quad \text{with } B_{\alpha_0} := m^{\alpha_0}_{\langle b_{\text{hom}}^2 \rangle_{\alpha_0}}.$$

According to Corollary 4.18, this yields

$$0 = \int \operatorname{Tr}_{L^2(0,2\pi)} \left(B_{\alpha_0} U_{\alpha_0,\omega}(t) \sigma_{\alpha_0}(\omega, E, H, 0) U^*_{\alpha_0,\omega}(t) \right) \ell_{\alpha_0}(d\omega, dE, dH) dt.$$

Since the integrand is non-negative, we have for ℓ_{α_0}-almost every (ω, E, H),

$$0 = \int_0^T \operatorname{Tr}_{L^2(0,2\pi)} \left(B_{\alpha_0} U_{\alpha_0,\omega}(t) \sigma_{\alpha_0}(\omega, E, H, 0) U^*_{\alpha_0,\omega}(t) \right) dt. \tag{7.4}$$

For ℓ_{α_0}-almost every (ω, E, H), $\sigma_{\alpha_0}(\omega, E, H, 0)$ is a non-negative trace-class operator. We can decompose it as a sum of of orthogonal projectors on its eigenfunctions:

$$\sigma_{\alpha_0}(\cdot,0)=\sum_{k\in\mathbb{N}}\lambda_k|\varphi_k\rangle\langle\varphi_k|,\quad \text{with}\quad \lambda_k\geq 0,$$
$$\sum_{k\in\mathbb{N}}\lambda_k=1,\quad \langle\varphi_k|\varphi_j\rangle_{L^2(0,2\pi)}=\delta_{kj}.$$

Note that λ_k, φ_k depend on (ω, E, H). Now Eq. (7.4) is equivalent to having, for all $k\in\mathbb{N}$, such that $\lambda_k>0$,

$$0=\int_0^T\int_0^{2\pi}\langle b_{\mathrm{hom}}^2\rangle_{\alpha_0}\circ\Phi(s,\theta,E,-E\sin\alpha_0)\left|U_{\alpha_0,\omega}(t)\varphi_k\right|^2(\theta)d\theta dt. \tag{7.5}$$

As above, there exists $\theta\in\mathbb{S}^1$ depending only on α_0, such that

$$\Phi\left(\cos\alpha_0,\theta,E,-E\sin\alpha_0\right)\in\overline{S}_{z_0}^+.$$

Hence, $\langle b_{\mathrm{hom}}^2\rangle_{\alpha_0}>0$ in a neighborhood of this θ. Then (7.5) implies that $U_{\alpha_0,\omega}(t)\varphi_k$ vanishes in a nonempty open subset of $(0,T)\times(0,2\pi)$. One dimensional unique continuation (see e.g. [37, Appendix B] and the references therein) then implies that $\varphi_k=0$. Therefore, $\sigma_{\alpha_0}(\omega,E,H,0)$ vanishes ℓ_{α_0}-almost everywhere, which yields $\mu_{\alpha_0}=0$.

This finally proves that $\mu^\infty=0$ and concludes the proof of the lemma. □

Proof of Lemma 7.2 Let us fix $z_0\in\Gamma$ and prove that μ^∞ vanishes in a neighborhood of $\overline{S}_{z_0}^+$. The result shall then follow from Lemma 7.1. First, according to (2.27), the assumption implies that $\mu^\infty\rceil_{T^*((0,T)\times\Gamma)}$ vanishes. Second, as a consequence of the assumption together with (2.26), we have $\mu_{ml}^S\rceil_{(z,\xi)\in S^+,z\in\Gamma}=0$. Coming back to the definition of the measure μ_{ml}^S in Sect. 2.8.1, this implies that μ^∞ vanishes on all trajectories of the billiard flow touching the boundary on $\{(z,\xi)\in S^+,z\in\Gamma\}$. In particular, this yields $\mu^\infty\rceil_{\overline{S}_{z_0}^+}=0$ and the result follows from Lemma 7.1. □

Proof of Lemma 7.3 Theorem 2.8 together with $\langle\mu_c,\mathbb{1}_{(0,T)\times\Omega}\rangle=0$ imply that

$$0=\int_0^T\|U_V(t)u^0\|_{L^2(\Omega)}^2dt.$$

The unique continuation property ($\mathrm{UCP}_{V,\Omega,T}$) then implies $u^0=0$ and this proves the lemma. □

7.2 Interior observability inequality: proof of Theorem 1.2

7.2.1 Unique continuation implies observability

In this section, we prove the observability inequality (1.5) assuming that ($\mathrm{UCP}_{V,\Omega,T}$) holds. Instead of proving (1.5) for any open set $\Omega \subset \mathbb{D}$ containing a neighbourhood in $\mathbb{D}$ of a point of $\partial\mathbb{D}$, we prove the equivalent statement: for any function $b \in C^0(\mathbb{R}^2)$ (also considered as a function in $C^0(\overline{\mathbb{D}})$) which is positive on a nonempty open subset of $\partial\mathbb{D}$, for any $T > 0$, there exists $C > 0$ such that the following inequality holds:

$$\left\|u^0\right\|^2_{L^2(\mathbb{D})} \leq C \int_0^T \left\|b(z)U_V(t)u^0\right\|^2_{L^2(\mathbb{D})} dt. \tag{7.6}$$

Note that under these conditions on Ω and b, inequalities (7.9) and (7.6) are equivalent.

We proceed by contradiction and suppose that the observability inequality (7.6) is not satisfied. Thus, there exists a sequence $(u_n^0)_{n\in\mathbb{N}}$ in $L^2(\mathbb{D})$ such that

$$\|u_n^0\|_{L^2(\mathbb{D})} = 1, \tag{7.7}$$

$$\int_0^T \left\|b(z)U_V(t)u_n^0\right\|^2_{L^2(\mathbb{D})} dt \to 0, \quad n \to \infty, \tag{7.8}$$

We write $u_n(t) = U_V(t)u_n^0$ the associated solution of (1.1)–(1.2). As previously, we extend u_n to $\mathbb{R}^2$ by zero outside $\mathbb{D}$ (and still use the notation u_n for its extension).

After having extracted a subsequence, we associate to (u_n) a microlocal measure

$$\mu_{ml} = \mu^\infty + \mu_c$$

as in Theorem 2.7. Equation (7.8) implies that

$$\int_0^T \int_{\mathbb{R}^2\times\mathbb{S}^2} b^2(z)\mu^\infty(dz, dH, d\xi, t)dt = 0, \quad \langle \mu_c, \mathbb{1}_{(0,T)} \otimes b^2\rangle = 0.$$

Lemmas 7.1 and 7.3 imply that $\mu^\infty = 0$ and $\mu_c = 0$ respectively. However, Eq. (7.7) implies that

$$\langle \mu_{ml}, \mathbb{1}_{(0,T)} \otimes 1\rangle = T.$$

This yields a contradiction and concludes the proof. Note that ($\mathrm{UCP}_{V,\Omega,T}$) has only been used to apply Lemma 7.3 in order to deal with the term μ_c.

7.2.2 *Observability for time independent potentials*

The structure of the proof in this setting is classical [11,39]. In a first step, we prove the following weakened observability inequality:

$$\left\|u^0\right\|^2_{L^2(\mathbb{D})} \leq C \int_0^T \left\|U_V(t)u^0\right\|^2_{L^2(\Omega)} dt + C \left\|u^0\right\|^2_{H^{-1}(\mathbb{D})}, \tag{7.9}$$

In a second step, we conclude the proof of Theorem 1.2 using a unique continuation property for eigenfunctions of the elliptic operator $-\Delta_D + V$.

The first step is similar to Sect. 7.2.1. We consider a sequence of initial data (u^0_n) contradicting (7.9). It satisfies (7.7), (7.8), together with

$$\|u^0_n\|_{H^{-1}(\mathbb{D})} \to 0, \quad n \to \infty. \tag{7.10}$$

As before, we consider the associated microlocal measure $\mu_{ml} = \mu^\infty + \mu_c$. Note now that (7.10) implies that $\mu_c = 0$. The rest of the proof is completely similar.

We now prove that (7.9) implies the observability inequality (1.5): this step is by now classical [11,39] but we include it for the sake of completeness. We proceed again by contradiction and suppose that the inequality

$$\|u^0\|_{H^{-1}(\mathbb{D})} \leq C \int_0^T \left\|b(z)U_V(t)u^0\right\|^2_{L^2(\Omega)} dt \tag{7.11}$$

is not satisfied. Then, there exists a sequence $(u^0_n)_{n\in\mathbb{N}}$ in $L^2(\mathbb{D})$ such that

$$\|u^0_n\|_{H^{-1}(\mathbb{D})} = 1, \qquad \int_0^T \left\|b(z)U_V(t)u^0_n\right\|^2_{L^2(\mathbb{D})} dt \to 0, \quad n \to \infty. \tag{7.12}$$

Inequality (7.6) implies that u^0_n is bounded in $L^2(\mathbb{D})$, so that, after having extracted a subsequence, we have $u^n_0 \rightharpoonup u^0$ in $L^2(\mathbb{D})$ and $u^0_n \to u^0$ in $H^{-1}(\mathbb{D})$. We deduce from (7.12) that

$$\|u^0\|_{H^{-1}(\mathbb{D})} = 1, \qquad U_V(t)u^0 = 0 \text{ on } \{b^2 > 0\} \text{ for all } t \in (0, T).$$

The weak limit u^0 belongs to the set

$$\mathcal{N} = \{f \in L^2(\mathbb{D}), U_V(t)f = 0 \text{ on } \{b^2 > 0\} \quad \text{for all } t \in (0, T)\}.$$

Then, by linearity, $\mathcal{N}$ is a closed vector subspace of $L^2(\mathbb{D})$. Inequality (7.9) proves that $\mathcal{N}$ is finite dimensional and the time independence of V implies that it is a subspace of $H^2(\mathbb{D}) \cap H_0^1(\mathbb{D})$, stable by the action of the operator $-\Delta_D + V$. If not reduced to $\{0\}$, the space $\mathcal{N}$ hence contains an eigenfunction of $-\Delta_D + V$, vanishing on $\{b^2 > 0\}$. A classical uniqueness result for elliptic operators then implies that this does not occur. This yields $\mathcal{N} = \{0\}$ and thus $u^0 = 0$, which contradicts $\|u^0\|_{H^{-1}(\mathbb{D})} = 1$.

7.3 Boundary observability inequality: proof of Theorem 1.3

We proceed as in the previous section: in a first step, we prove the following weakened observability inequality:

Lemma 7.5 *For all $T > 0$, there exists $C > 0$ such that for all $u^0 \in H_0^1(\mathbb{D})$, we have*

$$\left\|u^0\right\|_{H^1(\mathbb{D})}^2 \leq C \int_0^T \left\|\partial_n (U_V(t)u^0)\right\|_{L^2(\Gamma)}^2 dt + C \left\|u^0\right\|_{L^2(\mathbb{D})}^2, \tag{7.13}$$

With this lemma, we now conclude the proof of the observability inequality (1.6). We proceed by contradiction and suppose that the inequality

$$\|u^0\|_{L^2(\mathbb{D})} \leq \int_0^T \left\|\partial_n (U_V(t)u^0)\right\|_{L^2(\Gamma)}^2 dt \tag{7.14}$$

is not satisfied. Then, there exists a sequence $(u_n^0)_{n\in\mathbb{N}}$ in $L^2(\mathbb{D})$ such that

$$\|u_n^0\|_{L^2(\mathbb{D})} = 1, \qquad \int_0^T \left\|\partial_n (U_V(t)u_n^0)\right\|_{L^2(\Gamma)}^2 dt \to 0, \quad n \to \infty. \tag{7.15}$$

Then, (7.13) implies that u_n^0 is bounded in $H^1(\mathbb{D})$, so that, after having extracted a subsequence, we have $u_n^0 \rightharpoonup u^0$ in $H_0^1(\mathbb{D})$ and $u_n^0 \to u^0$ in $L^2(\mathbb{D})$. We deduce from (7.15) that

$$\|u^0\|_{L^2(\mathbb{D})} = 1, \qquad \partial_n (U_V(t)u^0) = 0 \text{ on } \Gamma \quad \text{for all } \ t \in (0, T).$$

From here, we discuss the two cases with different uniqueness arguments. In the case $V(t, z) = V(z)$, the proof of $u = 0$ follows exactly Sect. 7.2.2 (using unique continuation from the boundary for elliptic operators). The same conclusion holds if we assume ($\mathrm{UCP}_{V,\Gamma,T}$). This contradicts $\|u^0\|_{L^2(\mathbb{D})} = 1$, and proves (7.14). Then, (7.14) and (7.13) imply the sought observability inequality (1.6).

Proof of Lemma 7.5 Assume that (7.13) is not satisfied. Then, there exists a sequence u_n^0 such that

$$\|u_n^0\|^2_{H^1(\mathbb{D})} = 1, \tag{7.16}$$

$$\|u_n^0\|^2_{L^2(\mathbb{D})} \to 0, \tag{7.17}$$

$$\int_0^T \|\partial_n(u_n(t))\|^2_{L^2(\Gamma)} dt \to 0, \tag{7.18}$$

where, as usual, $u_n(t) = U_V(t)u_n^0$. Let us now fix $\chi_T \in C_c^\infty(\mathbb{R})$ such that $\chi_T = 1$ in a neighbourhood of $[0, T]$, $\psi \in C^\infty(\mathbb{R})$, such that $\psi = 0$ on $(-\infty, 1]$ and $\psi = 1$ on $[2, +\infty)$ and set

$$w_n = B(D_t)\chi_T(t)u_n, \quad \text{where} \quad B(D_t) = \mathrm{Op}_1(\psi(H)\sqrt{2H}).$$

The properties of u_n translate into properties of w_n, from which we shall deduce the sought contradiction.

Lemma 7.6 *We set* $A(D_t) = \mathrm{Op}_1(\frac{\psi(H)}{\sqrt{2H}})$. *For any* $R > 0$ *and* $\varepsilon > 0$, *we have*

$$\left\| \left(D_t + \frac{1}{2}\Delta - V \right) w_n \right\|_{L^2((0,T)\times\mathbb{D})} \to 0 \tag{7.19}$$

$$\|A(D_t)w_n\|_{L^2((-R,R)\times\mathbb{D})} \to 0, \tag{7.20}$$

$$T/2 + o_\varepsilon(1) \le \|w_n\|^2_{L^2((-\varepsilon,T+\varepsilon)\times\mathbb{D}),} \tag{7.21}$$

$$\|w_n\|^2_{L^2((-R,R)\times\mathbb{D})} \le R + \varepsilon + o_{R,\varepsilon}(1), \tag{7.22}$$

$$\|\partial_n(A(D_t)w_n)\|_{L^2((-R,R)\times\partial\mathbb{D})} \le C, \tag{7.23}$$

$$\|\partial_n(A(D_t)w_n)\|_{L^2((\varepsilon,T-\varepsilon)\times\Gamma)} \to 0. \tag{7.24}$$

Now, as (w_n) forms a bounded sequence of $L^2_{\mathrm{loc}}(\mathbb{R}\times\mathbb{D})$, we associate to a subsequence a microlocal measure $\mu_{ml} = \mu^\infty + \mu_c$ as in Sect. 2.5. According to (7.19) and Remark 2.6, the measure μ_{ml} satisfies the conclusions of Theorem 2.7. The measure μ_c vanishes as a consequence of (7.20). According to (7.23), the sequence $\partial_n(A(D_t)w_n)$ is bounded in $L^2((0, T) \times \mathbb{D})$, so we may again extract another subsequence and associate a microlocal measure μ^∂_{ml} as in Sect. 2.8.3. According to (7.24), $\mu^\partial_{ml} = 0$ on $T^*((\varepsilon, T - \varepsilon) \times \Gamma)$. As a consequence of Lemma 7.2, μ^∞ vanishes identically on $\mathbb{R}_t \times \mathbb{R}^2_z \times \mathbb{S}^2_{H,\xi}$. Thus, w_n converges to zero in $L^2_{\mathrm{loc}}(\mathbb{R}\times\mathbb{D})$, which is contradiction with (7.21). This concludes the proof of Lemma 7.5. □

Proof of Lemma 7.6 Take $\tilde{\chi} \in C_c^\infty(\mathbb{R})$ such that $\tilde{\chi} = 1$ on $(0, T)$ and $\chi_T = 1$ on $\mathrm{supp}(\tilde{\chi})$. Using that $\left(D_t + \frac{1}{2}\Delta - V\right) u_n = 0$, we have

$$\begin{aligned}
&\left\| \left(D_t + \frac{1}{2}\Delta - V \right) w_n \right\|_{L^2(0,T\times\mathbb{D})} \\
&\quad \le \left\| \tilde{\chi} \left(D_t + \frac{1}{2}\Delta - V \right) w_n \right\|_{L^2(\mathbb{R}\times\mathbb{D})} \\
&\quad \le \left\| \tilde{\chi} B(D_t) \chi_T' u_n \right\|_{L^2(\mathbb{R}\times\mathbb{D})} + \| \tilde{\chi} [V, B(D_t)] \chi_T u_n \|_{L^2(\mathbb{R}\times\mathbb{D})} \\
&\quad \le C \left\| \chi_T' u_n \right\|_{L^2(\mathbb{R}\times\mathbb{D})} + C \| \chi_T u_n \|_{L^2(\mathbb{R}\times\mathbb{D})} \le C \left\| u_n^0 \right\|_{L^2(\mathbb{D})} \to 0,
\end{aligned}$$

as $\tilde{\chi} = 0$ on $\mathrm{supp}(\chi_T')$ and $[V, B(D_t)]$ is bounded on $L^2(\mathbb{R} \times \mathbb{D})$. This proves (7.19).

Let us now take $\tilde{\chi} \in C_c^\infty(\mathbb{R})$ such that $\tilde{\chi} = 1$ on $(-R, R)$. We have

$$\begin{aligned}
\| A(D_t) w_n \|_{L^2((-R,R)\times\mathbb{D})} &\le \left\| \tilde{\chi} \operatorname{Op}_1(\psi^2(H)) \chi_T u_n \right\|_{L^2(\mathbb{R}\times\mathbb{D})} \\
&\le \| \chi_T u_n \|_{L^2(\mathbb{R}\times\mathbb{D})} \to 0,
\end{aligned}$$

which proves (7.20).

Let us fix now $\check{\chi} \in C_c^\infty(-\varepsilon, T + \varepsilon)$ such that $\check{\chi} = 1$ in a neighbourhood of $[0, T]$, and compute

$$\begin{aligned}
&\| w_n \|^2_{L^2((-\varepsilon,T+\varepsilon)\times\mathbb{D})} \\
&\quad \ge \left\| \check{\chi} w_n \right\|^2_{L^2(\mathbb{R}\times\mathbb{D})} = \langle B(D_t) \check{\chi}^2 B(D_t) \chi_T u_n, \chi_T u_n \rangle_{L^2(\mathbb{R}\times\mathbb{D})} \\
&\quad \ge \langle \check{\chi}^2 \operatorname{Op}_1(\psi^2(H)) D_t(\chi_T u_n), \chi_T u_n \rangle_{L^2(\mathbb{R}\times\mathbb{D})} \\
&\qquad + \langle [B(D_t), \check{\chi}^2] B(D_t) \chi_T u_n, \chi_T u_n \rangle_{L^2(\mathbb{R}\times\mathbb{D})},
\end{aligned}$$

where $|\langle [B(D_t), \check{\chi}^2] B(D_t) \chi_T u_n, \chi_T u_n \rangle_{L^2(\mathbb{R}\times\mathbb{D})}| \le \| \chi_T u_n \|_{L^2(\mathbb{R}\times\mathbb{D})} \to 0$. As a consequence, we have

$$\| w_n \|^2_{L^2((-\varepsilon,T+\varepsilon)\times\mathbb{D})} \ge \langle \check{\chi}^2 \operatorname{Op}_1(\psi^2(H)) D_t(\chi_T u_n), \chi_T u_n \rangle_{L^2(\mathbb{R}\times\mathbb{D})} + o(1). \tag{7.25}$$

On the other hand, we have

$$\begin{aligned}
\langle \check{\chi}^2 D_t(\chi_T u_n), \chi_T u_n \rangle_{L^2(\mathbb{R}\times\mathbb{D})} &= \langle \check{\chi}^2 \operatorname{Op}_1(\psi^2(H)) D_t(\chi_T u_n), \chi_T u_n \rangle_{L^2(\mathbb{R}\times\mathbb{D})} \\
&\quad + \langle \check{\chi}^2 \operatorname{Op}_1(\psi^2(-H)) D_t(\chi_T u_n), \chi_T u_n \rangle_{L^2(\mathbb{R}\times\mathbb{D})} + o(1),
\end{aligned} \tag{7.26}$$

where

$$\begin{aligned}&\langle \check{\chi}^2 \operatorname{Op}_1(\psi^2(-H)) D_t(\chi_T u_n), \chi_T u_n\rangle_{L^2(\mathbb{R}\times\mathbb{D})}\\ &\quad = \langle \operatorname{Op}_1(\check{\chi}^2(t)\psi^2(-H)H)(\chi_T u_n), \chi_T u_n\rangle_{L^2(\mathbb{R}\times\mathbb{D})}.\end{aligned}$$

Since the classical symbol $\check{\chi}^2\psi^2(-H)H \in S^1(\mathbb{R}\times\mathbb{R})$ satisfies $\check{\chi}^2\psi^2(-H)H \leq 0$, the sharp Gårding inequality then gives

$$\langle \check{\chi}^2 \operatorname{Op}_1(\psi^2(-H)) D_t(\chi_T u_n), \chi_T u_n\rangle_{L^2(\mathbb{R}\times\mathbb{D})} \leq C\|\chi_T u_n\|^2_{L^2(\mathbb{R}\times\mathbb{D})}.$$

With (7.26), this implies

$$\begin{aligned}&\langle \check{\chi}^2 D_t(\chi_T u_n), \chi_T u_n\rangle_{L^2(\mathbb{R}\times\mathbb{D})}\\ &\quad\leq \langle \check{\chi}^2 \operatorname{Op}_1(\psi^2(H)) D_t(\chi_T u_n), \chi_T u_n\rangle_{L^2(\mathbb{R}\times\mathbb{D})} + C\|\chi_T u_n\|^2_{L^2(\mathbb{R}\times\mathbb{D})}\\ &\quad\leq \langle \check{\chi}^2 \operatorname{Op}_1(\psi^2(H)) D_t(\chi_T u_n), \chi_T u_n\rangle_{L^2(\mathbb{R}\times\mathbb{D})} + o(1).\end{aligned}$$

This, combined with (7.25) now yields

$$\begin{aligned}\|w_n\|^2_{L^2((-\varepsilon,T+\varepsilon)\times\mathbb{D})} &\geq \langle \check{\chi}^2 D_t(\chi_T u_n), \chi_T u_n\rangle_{L^2(\mathbb{R}\times\mathbb{D})} + o(1)\\ &\geq \langle \check{\chi}^2 \chi_T D_t u_n, \chi_T u_n\rangle_{L^2(\mathbb{R}\times\mathbb{D})} + o(1)\\ &\geq \left\langle \check{\chi}^2 \chi_T \left(-\frac{1}{2}\Delta + V\right) u_n, \chi_T u_n\right\rangle_{L^2(\mathbb{R}\times\mathbb{D})} + o(1)\\ &\geq \frac{1}{2}\langle \check{\chi}^2 \chi_T \nabla u_n, \chi_T \nabla u_n\rangle_{L^2(\mathbb{R}\times\mathbb{D})} + o(1)\\ &\geq \frac{T}{2}\|\nabla u_n^0\|^2_{L^2(\mathbb{D})} + o(1) = \frac{T}{2} + o(1).\end{aligned}$$

This concludes the proof of (7.21). The proof of (7.22) follows the same arguments.

Finally, according to (7.16) and the hidden regularity result of Proposition 8.1 the sequence $\partial_n(u_n)$ is bounded in $L^2((-R,R)\times\partial\mathbb{D})$. Moreover, we have

$$\begin{aligned}\|\partial_n(A(D_t)w_n)\|_{L^2((-R,R)\times\partial\mathbb{D})} &\leq \left\|\tilde{\chi} \operatorname{Op}_1(\psi^2(H))\chi_T \partial_n(u_n)\right\|_{L^2(\mathbb{R}\times\partial\mathbb{D})}\\ &\leq \|\chi_T \partial_n(u_n)\|_{L^2(\mathbb{R}\times\partial\mathbb{D})} \leq C.\end{aligned}$$

This proves (7.23). The proof of (7.24) comes from a similar computation combined with (7.18). ◻

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Appendix A: Energy estimates, regularization of solutions, and localization on the characteristic set

In this appendix, we present general properties of the Schrödinger equation:

$$\begin{cases} \frac{1}{i}\frac{\partial u}{\partial t} = \left(-\frac{1}{2}\Delta + V\right) u + f, \quad t \in (0, T), \quad z \in \mathbb{D}, \\ u\rceil_{\partial\mathbb{D}} = 0, \\ u\rceil_{t=0} = u^0, \end{cases} \tag{8.1}$$

that are used throughout the article. None of these properties are specific to the disk $\mathbb{D}$, and they all could be stated as well on any smooth manifold with boundary. We first recall basic energy estimates (and hidden regularity of the trace) for the solution u of (8.1). Second, we define an appropriate regularization operator. Third, we prove a localization property on the set $\{2H = |\xi|^2\}$ for solutions of (8.1). Finally, we give a proof of Lemma 2.13.

First recall that for every $V \in L^1(-T, T; L^\infty(\mathbb{D}; \mathbb{C}))$, $u^0 \in L^2(\mathbb{D})$ and $f \in L^1(-T, T; L^2(\mathbb{D}))$, there is a unique solution $u \in C^0([-T, T]; L^2(\mathbb{D}))$ to (8.1). Moreover, there exists $C_{T,V} > 0$ such that for all such u, f and for all $t, s \in [-T, T]$ the following energy estimate holds:

$$\|u(t)\|_{L^2(\mathbb{D})} \leq C_{T,V} \left(\|u(s)\|_{L^2(\mathbb{D})} + \int_{I(s,t)} \|f(\sigma)\|_{L^2(\mathbb{D})} d\sigma \right). \tag{8.2}$$

Above, $I(s, t)$ denotes the interval of $\mathbb{R}$ whose endpoints are t, s. This estimate is obtained by taking the inner product of the equation with u, taking the real part and applying a Gronwall lemma, and using the fact that $u(-t)$ also solves a Schrödinger equation of the form (8.1) (with f, V replaced by their time-reversed counterparts).

Energy estimates at the H^1 level, though classical, are a little subtler. Assume now that $V \in L^1(-T, T; W^{1,\infty}(\mathbb{D}; \mathbb{C}))$ and let u be a smooth solution of (8.1). Using the equation, one obtains:

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\nabla u\|^2_{L^2(\mathbb{D})} &= \frac{d}{dt}\left\langle -\frac{\Delta}{2}u, u\right\rangle \\ &= \langle \partial_t u, (D_t - V)u - f\rangle + \langle (D_t - V)u - f, \partial_t u\rangle \\ &= \langle \partial_t u, -Vu - f\rangle + \langle -Vu - f, \partial_t u\rangle\,,\end{aligned}$$

which simplifies in:

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\nabla u\|^2_{L^2(\mathbb{D})} &= 2\mathrm{Im}\left\langle -\frac{\Delta}{2}u, Vu + f\right\rangle \\ &= \mathrm{Im}\left(\langle \nabla u, V\nabla u\rangle + \langle \nabla u, u\nabla V\rangle + \langle \nabla u, \nabla f\rangle\right).\end{aligned}$$

Gronwall's lemma yields, for any $t, s \in (-T, T)$,

$$\begin{aligned}\|\nabla u(t)\|_{L^2(\mathbb{D})} &\le e^{\int_{I(s,t)} m(\sigma)d\sigma}\|\nabla u(s)\|_{L^2(\mathbb{D})} \\ &+2\int_{I(s,t)} e^{\int_{I(\zeta,t)} m(\sigma)d\sigma}\|\nabla f(\zeta)\|_{L^2(\mathbb{D})}d\zeta,\end{aligned} \tag{8.3}$$

where $m(\sigma) = \|\mathrm{Im}\, V(\sigma)\|_{L^\infty(\mathbb{D})} + C_P\|\nabla V(\sigma)\|_{L^\infty(\mathbb{D})}$ and C_P is the constant in the Poincaré inequality. This estimate implies the well-posedness of (8.1) in $C^0([-T, T]; H^1_0(\mathbb{D}))$ for data $u^0 \in H^1_0(\mathbb{D})$ and $f \in L^1(-T, T; H^1_0(\mathbb{D}))$. Of course, it is possible to relax the $L^1_t W^{1,\infty}_x$ regularity of the potential V; however, in the main part of the paper, much more regularity is required.

We also use the following classical "hidden regularity" estimate for the restriction to the boundary of normal derivatives of solutions of (8.1), whose proof can be found, for instance, in [39, p. 284] or [25, Lemma 2.1].

Proposition 8.1 *For every $T > 0$ there exists a constant $C > 0$ such that, for every $u^0 \in H^1_0(\mathbb{D})$ and every $f \in L^1(-T, T; H^1_0(\mathbb{D}))$, the solution $u \in C^0([-T, T]; H^1_0(\mathbb{D}))$ of (8.1) satisfies*

$$\|\partial_n u\|_{L^2((-T,T)\times\partial\mathbb{D})} \le C\left(\|\nabla u^0\|_{L^2(\mathbb{D})} + \|f\|_{L^1(-T,T;H^1(\mathbb{D}))}\right). \tag{8.4}$$

These estimates will be used to derive properties of the quadratic expression

$$\langle u, \mathrm{Op}_1(a(z, \epsilon\xi, t, \epsilon^2 H))u\rangle,$$

where u is the extension by zero outside $\mathbb{D}$ of a solution to (1.1) and a is smooth and compactly supported in all variables. We prove that, up to a small error in terms of ϵ, we may truncate u in time t and in frequency H, so that the new function w is ϵ-oscillating, and its corresponding quadratic expression is close to the original one. This type of result is rather straightforward in the case of a compact manifold without boundary and a time-independent potential.

We assume $u = U_V(t)u^0$ is the solution to (1.1) with initial datum u^0; take $g \in C_c^\infty(\mathbb{R})$, let $T, \delta > 0$ and take $\chi_T \in C_c^\infty((-\delta - T, T + \delta))$ equal to 1 in a neighborhood of $[-T, T]$. Let us define

$$w(t) = g(\epsilon^2 D_t)\chi_T(t)U_V(t)u^0, \quad \text{and} \quad w^0 = w\rceil_{t=0}. \tag{8.5}$$

We have the following lemma concerning the map $u^0 \mapsto w^0$.

Lemma 8.2 *The time T, the functions χ_T and g being fixed, and the functions w and w^0 being defined by (8.5), we have the following properties*:

(1) *There is $C > 0$ such that for all $u^0 \in L^2(\mathbb{D})$, and all $\epsilon \in (0, 1]$, we have*

$$\|w^0\|_{L^2(\mathbb{D})} \leq C\|u^0\|_{L^2(\mathbb{D})}, \qquad \|\epsilon \nabla w^0\|_{L^2(\mathbb{D})} \leq C\|u^0\|_{L^2(\mathbb{D})}.$$

(2) *For each $\epsilon > 0$, the operator $u^0 \mapsto w^0$ is compact on $L^2(\mathbb{D})$.*
(3) *If $g = 1$ in a neighborhood of zero, then $w^0 \to u^0$ in $L^2(\mathbb{D})$ as $\epsilon \to 0$.*
(4) *For every $a \in C_c^\infty(T^*(\mathbb{R}^2 \times \mathbb{R}))$ such that $g = 1$ in a neighborhood of the H-support of a, for any $\varphi \in C_c^\infty(-T, T)$, we have*

$$\left\|\mathrm{Op}_1\left(a(x, \epsilon\xi, t, \epsilon^2 H)\right)\varphi\left(U_V(t)u^0 - U_V(t)w^0\right)\right\|_{L^2(\mathbb{R}\times\mathbb{R}^2)} \leq C\epsilon\|u^0\|_{L^2(\mathbb{D})},$$

and

$$\Big|\left\langle U_V(t)w^0, \mathrm{Op}_1\left(a(x, \epsilon\xi, t, \epsilon^2 H)\right)\varphi U_V(t)w^0\right\rangle_{L^2(\mathbb{R}^2\times\mathbb{R})} - \left\langle U_V(t)u^0, \mathrm{Op}_1\left(a(x, \epsilon\xi, t, \epsilon^2 H)\right)\varphi U_V(t)u^0\right\rangle_{L^2(\mathbb{R}^2\times\mathbb{R})}\Big| \leq C\epsilon\|u^0\|^2_{L^2(\mathbb{D})}.$$

In the context of this paper, the reader can think of ϵ as being h or R^{-1}. Note that, as a consequence of conclusion (4) in the above lemma, the restriction of semiclassical measures of the sequences $U_V(t)u_n^0$ and $U_V(t)w_n^0$ (w_n^0 being computed from u_n^0 according to (8.5)) to the set $t \in (-T, T)$, $H \in \{g = 1\}$ are the same. When tested with compactly supported symbols, we may thus always assume that the sequence u_n^0 is ϵ-oscillating.

Proof of Lemma 8.2 Using that u solves (1.1), the function w satisfies the equation

$$\begin{cases} \frac{1}{i}\frac{\partial w}{\partial t} = \left(-\frac{1}{2}\Delta + V\right)w - ig(\epsilon^2 D_t)\chi_T' u + [g(\epsilon^2 D_t), V]\chi_T u, \quad t \in \mathbb{R}, \quad z \in \mathbb{D}, \\ w\rceil_{\partial\mathbb{D}} = 0, \\ w\rceil_{t=0} = w^0. \end{cases} \tag{8.6}$$

Using the energy estimate (8.2) for w with $t = 0$, and integrated over $s \in (-T, T)$, we obtain

$$\|w^0\|^2_{L^2(\mathbb{D})} \leq C \int_{-T}^{T} \|w(t)\|^2_{L^2(\mathbb{D})} dt + C \int_{\mathbb{R}} \|\tilde{\chi}_T g(\epsilon^2 D_t) \chi_T' u\|^2_{L^2(\mathbb{D})} dt$$
$$+C \int_{\mathbb{R}} \|\tilde{\chi}_T [g(\epsilon^2 D_t), V] \chi_T u\|^2_{L^2(\mathbb{D})} dt,$$

where $\tilde{\chi}_T \in C_c^\infty(\mathbb{R})$ such that $\tilde{\chi}_T = 1$ in a neighborhood of $[-T, T]$ and $\chi_T = 1$ on a neighborhood on $\mathrm{supp}(\tilde{\chi}_T)$. Note moreover that $\tilde{\chi}_T g(\epsilon^2 D_t) \chi_T' = O_{L^2 \to L^2}(\epsilon^\infty)$ and $\tilde{\chi}_T [g(\epsilon^2 D_t), V] \chi_T = O_{L^2 \to L^2}(\epsilon^2) \|\partial_t V\|_{L^\infty}$. We hence obtain

$$\|w^0\|^2_{L^2(\mathbb{D})} \leq C \int_{-T}^{T} \|w(t)\|^2_{L^2(\mathbb{D})} dt + C\epsilon^2 \int_{-T-\delta}^{T+\delta} \|u(t)\|^2_{L^2(\mathbb{D})} dt.$$

We now notice that, by definition of w, we have

$$\int_{-T}^{T} \|w(t)\|^2_{L^2(\mathbb{D})} dt \leq \int_{\mathbb{R}} \|\chi_T g(\epsilon^2 D_t) \chi_T u\|^2_{L^2(\mathbb{D})} dt \leq C \int_{-T-\delta}^{T+\delta} \|u(t)\|^2_{L^2(\mathbb{D})} dt. \tag{8.7}$$

Since u solves (1.1), the energy estimate (8.2) for u with $s = 0$, integrated over $t \in (-T - \delta, T + \delta)$ then yields $\int_{-T-\delta}^{T+\delta} \|u(t)\|^2_{L^2(\mathbb{D})} dt \leq C \|u^0\|^2_{L^2(\mathbb{D})}$ and thus, combined with the two above estimates, proves the first inequality of Item (1).

Let us now consider the second estimate of Item (1). Using the energy estimate (8.3) for w (satisfying (8.6)) with $t = 0$, and integrated over $s \in (-T, T)$, we obtain (with $\tilde{\chi}_T$ defined above),

$$\begin{aligned} \|\nabla w^0\|^2_{L^2(\mathbb{D})} \leq\ & C \int_{-T}^{T} \|\nabla w(t)\|^2_{L^2(\mathbb{D})} dt \\ & + C \int_{\mathbb{R}} \|\tilde{\chi}_T g(\epsilon^2 D_t) \chi_T' \nabla u\|^2_{L^2(\mathbb{D})} dt \\ & + C \int_{\mathbb{R}} \|\tilde{\chi}_T [g(\epsilon^2 D_t), V] \chi_T \nabla u\|^2_{L^2(\mathbb{D})} dt. \end{aligned}$$

With $B = \tilde{\chi}_T g(\epsilon^2 D_t)\chi_T'$ or $B = \tilde{\chi}_T[g(\epsilon^2 D_t), V]\chi_T$, we have, using that u solves (1.1),

$$\begin{aligned}\frac{1}{2}\|Bu\|^2_{L^2(\mathbb{R}\times\mathbb{D})} &= \langle B\left(-\frac{\Delta}{2}\right)u, u\rangle_{L^2(\mathbb{R}\times\mathbb{D})}\\ &= \langle BD_t u, u\rangle_{L^2(\mathbb{R}\times\mathbb{D})} - \langle BVu, u\rangle_{L^2(\mathbb{R}\times\mathbb{D})}.\end{aligned}$$

For both choices of B, we have $|\langle BVu, u\rangle_{L^2(\mathbb{R}\times\mathbb{D})}| \leq C\epsilon^2 \int_{-T-\delta}^{T+\delta} \|u(t)\|^2_{L^2(\mathbb{D})}dt \leq C\epsilon^2\|u^0\|^2_{L^2(\mathbb{D})}$, and, since g is compactly supported, $BD_t = O_{L^2\to L^2}(\epsilon^{-2})$, so that $|\langle BD_t u, u\rangle_{L^2(\mathbb{R}\times\mathbb{D})}| \leq C\epsilon^{-2}\|u^0\|^2_{L^2(\mathbb{D})}$. As a consequence, we obtain

$$\|\nabla w^0\|^2_{L^2(\mathbb{D})} \leq C\int_{-T}^{T} \|\nabla w(t)\|^2_{L^2(\mathbb{D})}dt + C\epsilon^{-2}\|u^0\|^2_{L^2(\mathbb{D})}. \tag{8.8}$$

Integrating by parts, and using that w solves (8.6), we have

$$\begin{aligned}\frac{1}{2}\int_{-T}^{T} \|\nabla w(t)\|^2_{L^2(\mathbb{D})}dt &= \int_{-T}^{T}\left\langle -\frac{\Delta}{2}w, w\right\rangle_{L^2(\mathbb{D})} dt\\ &= \int_{-T}^{T}\langle (D_t - V)w, w\rangle_{L^2(\mathbb{D})}dt\\ &\quad + \int_{-T}^{T}\left\langle\left(ig(\epsilon^2 D_t)\chi_T' u - [g(\epsilon^2 D_t), V]\chi_T u\right), w\right\rangle_{L^2(\mathbb{D})} dt.\end{aligned}$$

We also have, as above,

$$\begin{aligned}\|D_t w\|^2_{L^2((-T,T)\times\mathbb{D})} &= \|D_t g(\epsilon^2 D_t)\chi_T u\|^2_{L^2((-T,T)\times\mathbb{D})}\\ &\leq C\epsilon^{-4}\int_{-T-\delta}^{T+\delta}\|u(t)\|^2_{L^2(\mathbb{D})}dt \leq C\epsilon^{-4}\|u^0\|^2_{L^2(\mathbb{D})},\end{aligned}$$

together with the L^2 estimate (8.7) for w. This, together with above estimates and (8.8) finally implies $\|\nabla w^0\|^2_{L^2(\mathbb{D})} \leq C\epsilon^{-2}\|u^0\|^2_{L^2(\mathbb{D})}$, which is the second inequality of Item (1).

Item (2) directly follows from the second estimate of Item (1). To prove Item (3), notice first that

$$\|u - w\|_{L^2((-T,T)\times\mathbb{D})} \leq \|\tilde{\chi}_T(1 - g(\epsilon^2 D_t)\chi_T)u\|_{L^2(\mathbb{R}\times\mathbb{D})} \to 0, \quad \text{as } \epsilon \to 0,$$

since $1 - g(\epsilon^2 H) = 0$ on any compact set for ϵ sufficiently small. Then, since w solves (8.6) and u solves (1.1), the function $w - u$ also satisfies (8.6) with the same right hand-side, but with initial data $w^0 - u^0$. Using the same estimates as for the proof of Item (1), we now have

$$\|w^0 - u^0\|^2_{L^2(\mathbb{D})} \leq C \int_{-T}^{T} \|w(t) - u(t)\|^2_{L^2(\mathbb{D})} dt$$
$$+ C \int_{\mathbb{R}} \|\tilde{\chi}_T g(\epsilon^2 D_t) \chi'_T u\|^2_{L^2(\mathbb{D})} dt + C \int_{\mathbb{R}} \|\tilde{\chi}_T [g(\epsilon^2 D_t), V] \chi_T u\|^2_{L^2(\mathbb{D})} dt$$
$$\leq \|u - w\|^2_{L^2((-T,T)\times\mathbb{D})} + C\epsilon^2 \|u^0\|^2_{L^2(\mathbb{D})} \to 0, \quad \text{as } \epsilon \to 0,$$

which proves Item (3).

To prove Item (4), we now write $A = \mathrm{Op}_1(a(x, \epsilon\xi, t, \epsilon^2 H))$, and compute

$$\left\| A\varphi \left(U_V(t) u^0 - U_V(t) w^0 \right) \right\|_{L^2(\mathbb{R}\times\mathbb{R}^2)} \leq \|A\varphi\,(u - w)\|_{L^2((-T,T)\times\mathbb{R}^2)}$$
$$+ \left\| A\varphi \left(w(t) - U_V(t) w^0 \right) \right\|_{L^2((-T,T)\times\mathbb{R}^2)}.$$

Concerning the first term in the right hand-side, we have

$$\|A\varphi\,(u - w)\,\|_{L^2((-T,T)\times\mathbb{R}^2)} = \|A\varphi(1 - g(\epsilon^2 D_t)\chi_T) u\|_{L^2(\mathbb{R}\times\mathbb{R}^2)}$$
$$\leq C_N \epsilon^N \|u\|_{L^2((-\delta, T+\delta)\times\mathbb{D})} \leq C_N \epsilon^N \|u^0\|_{L^2(\mathbb{D})},$$

since the supports of a and $1 - g$ are disjoint. Concerning the second term, we notice that $w(t) - U_V(t) w^0$ satisfies (8.6) with the same right hand-side, but with initial data 0. Thus, using the boundedness of A on L^2, and the same estimates as for the proof of Item (1), we have

$$\left\| A\varphi \left(w(t) - U_V(t) w^0 \right) \right\|_{L^2(\mathbb{R}\times\mathbb{R}^2)} \leq C \|w(t) - U_V(t) w^0\|_{L^2((-T,T)\times\mathbb{D})}$$
$$\leq C\epsilon \|u^0\|_{L^2(\mathbb{D})}.$$

The last three estimates conclude the proof of the first estimate in Item (4). Finally, with $y = U_V(t) w^0$, we have

$$\left| \langle y, A\varphi y \rangle_{L^2(\mathbb{R}^2\times\mathbb{R})} - \langle u, A\varphi u \rangle_{L^2(\mathbb{R}^2\times\mathbb{R})} \right|$$
$$= \left| \langle y, A\varphi(y - u) \rangle_{L^2(\mathbb{R}^2\times\mathbb{R})} - \langle y - u, A\varphi u \rangle_{L^2(\mathbb{R}^2\times\mathbb{R})} \right|$$
$$\leq \|y\|_{L^2(\mathbb{R}^2\times\mathbb{R})} \|A\varphi(y - u)\|_{L^2(\mathbb{R}^2\times\mathbb{R})}$$
$$+ \|u\|_{L^2(\mathbb{R}^2\times\mathbb{R})} \|\varphi A^*(y - u)\|_{L^2(\mathbb{R}^2\times\mathbb{R})}$$
$$\leq C\epsilon \|u^0\|^2_{L^2(\mathbb{D})},$$

according to the first estimate in Item (4) (together with Item (1)) applied both to $A\varphi$ and φA^* (which, as well, is of the form $\varphi A^* \psi$ for some $\psi \in$

$C_c^\infty(-T,T)$, since the t-support of a is contained in $(-T,T)$). This concludes the proof of the second estimate in Item (4), and hence, of the lemma. □

The following proposition states that solutions of (1.1) are localized on the set $\{|\xi|^2 = 2H\}$ at high frequency.

Proposition 8.3 *For all $s \in (1/2, 1)$, for any $a \in C_c^\infty(T^*(\mathbb{R}^2 \times \mathbb{R}))$ supported away from $H = 0$, respectively away from $\xi = 0$, for all $\varphi \in C_c^\infty(\mathbb{R})$, there is $C > 0$ such that for all solutions u to (1.1)–(1.2) with $u^0 \in L^2(\mathbb{D})$, we have for $\epsilon \in (0, 1]$,*

$$\left|\left\langle u, \mathrm{Op}_1\left(a(z,\epsilon\xi,t,\epsilon^2 H)\left(\frac{|\xi|^2}{2H} - 1\right)\right)\varphi u\right\rangle_{L^2(\mathbb{R}^2\times\mathbb{R})}\right| \le C\epsilon^{1-s}\|u^0\|^2_{L^2(\mathbb{D})}, \tag{8.9}$$

respectively,

$$\left|\left\langle u, \mathrm{Op}_1\left(a(z,\epsilon\xi,t,\epsilon^2 H)\left(\frac{2H}{|\xi|^2} - 1\right)\right)\varphi u\right\rangle_{L^2(\mathbb{R}^2\times\mathbb{R})}\right| \le C\epsilon^{1-s}\|u^0\|^2_{L^2(\mathbb{D})}. \tag{8.10}$$

As everywhere in the paper, the notation u stands both for the function on $C^0([0,T]; L^2(\mathbb{D}))$ and its extension by zero to the whole $\mathbb{R}^2$. Note that this holds for all $u^0 \in L^2(\mathbb{D})$, without assuming a priori that u^0 is ϵ-oscillating. This comes from the nice properties of the regularization (8.5) proved in Lemma 8.2.

Proof of Proposition 8.3 Note first that it suffices to prove the estimate

$$\begin{aligned}&\left|\left\langle u, \mathrm{Op}_1\left(a(z,\epsilon\xi,t,\epsilon^2 H)\left(\frac{\epsilon^2|\xi|^2}{2} - \epsilon^2 H\right)\right)\varphi u\right\rangle_{L^2(\mathbb{R}^2\times\mathbb{R})}\right|\\&\quad\le C\epsilon^{1-s}\|u^0\|^2_{L^2(\mathbb{D})}.\end{aligned} \tag{8.11}$$

Estimates (8.9) and (8.10) then follow by changing a into $\frac{a}{\epsilon^2 H}$ and $\frac{a}{\epsilon^2|\xi|^2}$ respectively. According to Lemma 8.2 Item (4), (8.11) is equivalent to the same estimate with u replaced by $U_V(t)w^0$ (extended by zero outside $\mathbb{D}$), where w^0 is defined from u^0 by (8.5). Writing $A = \mathrm{Op}_1(a(z,\epsilon\xi,t,\epsilon^2 H))$, we have $\mathrm{Op}_1(a(z,\epsilon\xi,t,\epsilon^2 H)(\frac{\epsilon^2|\xi|^2}{2} - \epsilon^2 H)) = A(-\frac{\epsilon^2\Delta}{2} - \epsilon^2 D_t)$, where Δ is the Laplace operator on $\mathbb{R}^2$. We thus obtain

$$\begin{aligned}&\mathrm{Op}_1\left(a(z,\epsilon\xi,t,\epsilon^2 H)\left(\frac{\epsilon^2|\xi|^2}{2} - \epsilon^2 H\right)\right)\varphi U_V(t)w^0\\&\quad= A\varphi\left(-\frac{\epsilon^2\Delta}{2} - \epsilon^2 D_t\right)U_V(t)w^0 + i\epsilon^2 A\varphi' U_V(t)w^0.\end{aligned}$$

Recall now that the extended function $U_V(t)w^0$ solves

$$\frac{\epsilon^2}{2}\partial_n\left(U_V(t)w^0\right)\otimes\delta_{\partial\mathbb{D}}=\epsilon^2\left(-\frac{\Delta}{2}+V-D_t\right)U_V(t)w^0,$$

so that

$$\begin{aligned}&\mathrm{Op}_1\left(a(z,\epsilon\xi,t,\epsilon^2 H)\left(\frac{\epsilon^2|\xi|^2}{2}-\epsilon^2 H\right)\right)\varphi U_V(t)w^0\\&=-\epsilon^2 A\varphi V U_V(t)w^0+\frac{\epsilon^2}{2}A\varphi\left(\partial_n\left(U_V(t)w^0\right)\otimes\delta_{\partial\mathbb{D}}\right)+i\epsilon^2 A\varphi' U_V(t)w^0.\end{aligned}\tag{8.12}$$

The operators $A\varphi$ and $A\varphi'$ being bounded on $L^2(-T,T;L^2(\mathbb{D}))$, and according to the energy estimate (8.2) for $U_V(t)w^0$ solution of (1.1), we have

$$\begin{aligned}&\|-\epsilon^2 A\varphi V U_V(t)w^0+i\epsilon^2 A\varphi' U_V(t)w^0\|_{L^2(\mathbb{R}^2\times\mathbb{R})}\leq C\epsilon^2\|w^0\|_{L^2(\mathbb{D})}\\&\leq C\epsilon^2\|u^0\|_{L^2(\mathbb{D})},\end{aligned}\tag{8.13}$$

after having used Lemma 8.2 Item (1), and it only remains to estimate

$$\epsilon^2\left\langle U_V(t)w^0, A\varphi\left(\partial_n\left(U_V(t)w^0\right)\otimes\delta_{\partial\mathbb{D}}\right)\right\rangle_{L^2(\mathbb{R}^2\times\mathbb{R})}.$$

To this aim, we write, for every $s>0$,

$$\begin{aligned}&\left|\left\langle U_V(t)w^0, A\,\varphi\left(\partial_n\left(U_V(t)w^0\right)\otimes\delta_{\partial\mathbb{D}}\right)\right\rangle_{L^2(\mathbb{R}^2\times\mathbb{R})}\right|\\&\leq\|A\|_{L^2((-T,T),H^{-s}(\mathbb{R}^2))\to L^2((-T,T)\times\mathbb{R}^2)}\|U_V(t)w^0\|_{L^2((-T,T)\times\mathbb{D})}\\&\quad\times\|\partial_n U_V(t)w^0\otimes\delta_{\partial\mathbb{D}}\|_{L^2((-T,T),H^{-s}(\mathbb{R}^2))}.\end{aligned}$$

Moreover, for $s>1/2$, the standard trace estimates (see for instance [17, Chapter 2, Section 4]) imply that

$$\|\partial_n U_V(t)w^0\otimes\delta_{\partial\mathbb{D}}\|_{L^2((-T,T),H^{-s}(\mathbb{R}^2))}\leq C\left\|\partial_n\left(U_V(t)w^0\right)\right\|_{L^2((-T,T)\times\partial\mathbb{D})},$$

which, by Proposition 8.1, is bounded by $C\left\|\nabla w^0\right\|_{L^2(\mathbb{D})}$. Using the fact that

$$\|A\|_{L^2((-T,T),H^{-s}(\mathbb{R}^2))\to L^2((-T,T)\times\mathbb{R}^2)}\leq C\epsilon^{-s},$$

we now obtain

$$\left|\epsilon^2 \left\langle U_V(t)w^0, A\varphi\left(\partial_n\left(U_V(t)w^0\right)\otimes\delta_{\partial\mathbb{D}}\right)\right\rangle_{L^2(\mathbb{R}^2\times\mathbb{R})}\right| \\ \leq C\epsilon^{1-s}\|w^0\|_{L^2(\mathbb{D})}\left\|\epsilon\nabla w^0\right\|_{L^2(\mathbb{D})} \leq C\epsilon^{1-s}\|u^0\|^2_{L^2(\mathbb{D})},$$

after having used Lemma 8.2 Item (1). This, together with (8.12) and (8.13), concludes the proof of the proposition. □

Finally, we prove by dyadic decomposition a statement similar to that of (8.9)–(8.10) for homogeneous functions.

Proposition 8.4 *Recall that* $\chi \in C_c^\infty(\mathbb{R})$ *is a nonnegative cut-off function that is identically equal to one near the origin. For all* $s \in (1/2, 1)$, *all* $a \in \mathcal{S}_0$ (*see Definition* 2.1) *vanishing on the set* $\{|\xi|^2 = 2H\}$ *and for all* $\varphi \in C_c^\infty(\mathbb{R})$, *there is* $C > 0$ *such that for all* $u^0 \in L^2(\mathbb{D})$ *and* R *large enough, we have*

$$\left|\left\langle U_V(t)u^0, \mathrm{Op}_1\left(a(z,\xi,t,H)\left(1-\chi\left(\frac{|\xi|^2+|H|}{R^2}\right)\right)\right)\varphi(t)U_V(t)u^0\right\rangle_{L^2(\mathbb{R}^2\times\mathbb{R})}\right| \\ \leq CR^{s-1}\|u^0\|^2_{L^2(\mathbb{D})}. \tag{8.14}$$

Proof To see that, using the homogeneity of a for large R, we write the following decomposition:

$$a(z,\xi,t,H)\left(1-\chi\left(\frac{|\xi|^2+|H|}{R^2}\right)\right) \\ = \sum_{k=0}^{\infty} a(z, 2^{-k}R^{-1}\xi, t, 2^{-2k}R^{-2}H)\left(\chi\left(\frac{|\xi|^2+|H|}{2^{2(k+1)}R^2}\right)-\chi\left(\frac{|\xi|^2+|H|}{2^{2k}R^2}\right)\right).$$

For each k in the sum above, decompose further

$$\chi\left(\frac{|\xi|^2+|H|}{2^{2(k+1)}R^2}\right)-\chi\left(\frac{|\xi|^2+|H|}{2^{2k}R^2}\right) = \left(\chi\left(\frac{|\xi|^2+|H|}{2^{2(k+1)}R^2}\right)-\chi\left(\frac{|\xi|^2+|H|}{2^{2k}R^2}\right)\right) \\ \times\left(\chi\left(\frac{|H|}{2^{2k-1}R^2}\right)+(1-\chi)\left(\frac{|H|}{2^{2k-1}R^2}\right)\right)$$

and note that we must have $|\xi|^2 \geq 2^{2k-1}R^2$ or $|H| \geq 2^{2k-1}R^2$ on the support of this function.

If a vanishes on the set $\{|\xi|^2 = 2H\}$, we can write

$$a(z,\xi,t,H) = b(z,\xi,t,H)\left(\frac{2H}{|\xi|^2}-1\right)$$

where $|\xi|^2 \geq 2^{2k-1}R^2$ and

$$a(z,\xi,t,H) = b(z,\xi,t,H)\left(\frac{|\xi|^2}{2H} - 1\right)$$

where $|H| \geq 2^{2k-1}R^2$. Applying (8.9) and (8.10) for each k (with $\epsilon = 2^{-k}R^{-1}$), we finally obtain

$$\left\langle U_V(t)u^0, \mathrm{Op}_1\left(a(z,\xi,t,H)\left(1 - \chi\left(\frac{|\xi|^2 + |H|}{R^2}\right)\right)\right) U_V(t)u^0\right\rangle_{L^2(\mathbb{R}^2\times\mathbb{R})}$$
$$\leq C \sum_{k=0}^{+\infty} R^{s-1} 2^{k(s-1)} \|u^0\|^2_{L^2(\mathbb{D})}, \tag{8.15}$$

which proves the proposition.

To conclude this section, we give a proof of Lemma 2.13.

Proof of Lemma 2.13 Note that operator $A(D_t)\varphi$ is bounded on $L^2(\mathbb{R}\times\mathbb{D})$. Moreover, we have

$$\begin{aligned} \|\nabla A(D_t)\varphi u\|^2_{L^2(\mathbb{R}\times\mathbb{D})} &= \langle -\Delta A(D_t)\varphi u, A(D_t)\varphi u\rangle_{L^2(\mathbb{R}\times\mathbb{D})} \\ &= \langle A(D_t)\varphi(-\Delta)u, A(D_t)\varphi u\rangle_{L^2(\mathbb{R}\times\mathbb{D})} \\ &\quad + \langle [-\Delta, A(D_t)\varphi]u, A(D_t)\varphi u\rangle_{L^2(\mathbb{R}\times\mathbb{D})}. \end{aligned} \tag{8.16}$$

One the one hand, we have

$$\begin{aligned} &\left|\langle [-\Delta, A(D_t)\varphi]u, A(D_t)\varphi u\rangle_{L^2(\mathbb{R}\times\mathbb{D})}\right| \\ &\quad = \left|-\langle 2\nabla\varphi\cdot\nabla u + u\Delta\varphi, A(D_t)^2\varphi u\rangle_{L^2(\mathbb{R}\times\mathbb{D})}\right| \\ &\quad \leq 2\left|\langle u, \mathrm{div}\left\{\nabla\varphi\left(A(D_t)^2\varphi u\right)\right\}\rangle_{L^2(\mathbb{R}\times\mathbb{D})}\right| + C\|\tilde{\varphi}u\|^2_{L^2(\mathbb{R}\times\mathbb{D})} \\ &\quad \leq 2\left|\langle u, \nabla\varphi\cdot\nabla\left(A(D_t)^2\varphi u\right)\rangle_{L^2(\mathbb{R}\times\mathbb{D})}\right| + C\|\tilde{\varphi}u\|^2_{L^2(\mathbb{R}\times\mathbb{D})} \\ &\quad \leq \varepsilon\|\nabla A(D_t)\varphi u\|^2_{L^2(\mathbb{R}\times\mathbb{D})} + C(1+\varepsilon^{-1})\|\tilde{\varphi}u\|^2_{L^2(\mathbb{R}\times\mathbb{D})} \end{aligned}$$

for some $\tilde{\varphi}$ equal to one on the support of φ, for all $\varepsilon > 0$.

On the other hand, since u solves (1.1), we have

$$\begin{aligned} &\left|\langle A(D_t)\varphi(-\Delta)u, A(D_t)\varphi u\rangle_{L^2(\mathbb{R}\times\mathbb{D})}\right| \\ &\quad = \left|\langle A(D_t)\varphi(2D_t - V)u, A(D_t)\varphi u\rangle_{L^2(\mathbb{R}\times\mathbb{D})}\right| \\ &\quad \leq \left|\langle A(D_t)\varphi(2D_t - V)u, A(D_t)\varphi u\rangle_{L^2(\mathbb{R}\times\mathbb{D})}\right| \\ &\quad \leq 2\left|\langle A(D_t)^2\varphi D_t u, \varphi u\rangle_{L^2(\mathbb{R}\times\mathbb{D})}\right| + C\|\tilde{\varphi}u\|^2_{L^2(\mathbb{R}\times\mathbb{D})} \end{aligned}$$

$$\leq 2 \left| \langle A(D_t)^2 D_t \varphi u, \varphi u \rangle_{L^2(\mathbb{R}\times\mathbb{D})} \right| + C \|\tilde{\varphi} u\|^2_{L^2(\mathbb{R}\times\mathbb{D})}$$
$$\leq C \|\tilde{\varphi} u\|^2_{L^2(\mathbb{R}\times\mathbb{D})},$$

since $A(D_t)^2 D_t = 1/2\,\mathrm{Op}_1(\psi^2(H))$ is bounded. Collecting these estimates in (8.16), recalling (8.2) that $\|\tilde{\varphi} u\|^2_{L^2(\mathbb{R}\times\mathbb{D})} \leq C \|u^0\|^2_{L^2(\mathbb{D})}$, and taking ε sufficiently small concludes the proof of Lemma 2.13. □

Appendix B: Time regularity of Wigner measures

In this section we present a proof of the following (general) result on time regularity of semiclassical measures associated to solutions of the Schrödinger equation (1.1). Even if not stated here, its microlocal counterpart also holds.

Proposition 9.1 *Let μ_{sc} be obtained as a limit (2.5). Then there exists $\mu \in L^\infty(\mathbb{R}_t; \mathcal{M}_+(T^*\mathbb{R}^2))$ such that, for every $a \in C_c^\infty(T^*\mathbb{R}^2 \times T^*\mathbb{R})$ we have:*

$$\int_{T^*\mathbb{R}^2\times T^*\mathbb{R}} a\,(z, \xi, t, H)\,\mu_{sc}\,(dz, d\xi, dt, dH)$$
$$= \int_{\mathbb{R}} \int_{T^*\mathbb{R}^2} a\left(z, \xi, t, \frac{|\xi|^2}{2}\right) \mu\,(t, dz, d\xi)\,dt.$$

Note that if the potential V is complex valued, we only have $\mu \in L^\infty_{\mathrm{loc}}(\mathbb{R}_t; \mathcal{M}_+(T^*\mathbb{R}^2))$.

Proof Let $u_h(\cdot, t) := U_V(t) u_h^0$ and note that the Wigner distributions:

$$\tilde{W}^h_{u_h}(t) : C_c^\infty\left(T^*\mathbb{R}^2\right) \ni l \longmapsto \left\langle U_V\,(t)\,u_h^0, \mathrm{Op}_h\,(l)\,U_V\,(t)\,u_h^0 \right\rangle_{L^2(\mathbb{R}^2)} \in \mathbb{C}$$

are uniformly bounded in $L^\infty(\mathbb{R}_t; \mathcal{D}'(T^*\mathbb{R}^2))$. Hence, possibly after extracting a subsequence (and having used a diagonal extraction argument), we can assume that, for every $b \in C_c^\infty(T^*\mathbb{R}^2 \times \mathbb{R})$:

$$\lim_{h\to 0^+} \int_{\mathbb{R}} \left\langle U_V\,(t)\,u_h^0, \mathrm{Op}_h\,(b\,(\cdot, t))\,U_V\,(t)\,u_h^0 \right\rangle_{L^2(\mathbb{R}^2)} dt$$
$$= \int_{\mathbb{R}} \int_{T^*\mathbb{R}^2} b\,(z, \xi, t)\,\tilde{\mu}_{sc}\,(t, dz, d\xi)\,dt.$$

Moreover, using the sharp Gårding inequality, we see that the limiting Wigner distribution is a nonnegative measure $\tilde{\mu}_{sc} \in L^\infty(\mathbb{R}_t; \mathcal{M}_+(T^*\mathbb{R}^2))$. We next show that for any $b \in C_c^\infty(T^*\mathbb{R}^2 \times \mathbb{R})$ with $b \geq 0$ one has:

$$\int_{T^*\mathbb{R}^2\times T^*\mathbb{R}} b\,(z,\xi,t)\,\mu_{sc}\,(dz,d\xi,dt,dH)$$
$$\leq \int_{\mathbb{R}}\int_{T^*\mathbb{R}^2} b\,(z,\xi,t)\,\tilde{\mu}_{sc}\,(t,dz,d\xi)\,dt. \tag{9.1}$$

To see this, let $\chi \in C_c^\infty(\mathbb{R})$ be a cut-off function satisfying $0 \leq \chi \leq 1$, strictly positive in $(-3/2, 3/2)$, vanishing outside that interval, and such that $\chi\rceil_{(-1,1)} \equiv 1$. Write, for $R > 0$, $\chi_R := \chi(\cdot/R)$ and $\sigma_R := \sqrt{1-\chi_R}$ (which we may also assume smooth). Then we have:

$$\left\langle u_h, \mathrm{Op}_h\,(b)\,\chi_R\left(h^2 D_t\right)u_h\right\rangle_{L^2(\mathbb{R}^2\times\mathbb{R})}$$
$$= \left\langle u_h, \mathrm{Op}_h\,(b)\,u_h\right\rangle_{L^2(\mathbb{R}^2\times\mathbb{R})} + k_{h,R}\,(b) + O\,(h)\,, \tag{9.2}$$

where:

$$k_{h,R}\,(b) := \left\langle \sigma_R\left(h^2 D_t\right)u_h, \mathrm{Op}_h\,(b)\,\sigma_R\left(h^2 D_t\right)u_h\right\rangle_{L^2(\mathbb{R}^2\times\mathbb{R})}.$$

Taking limits in (9.2) as $h \to 0^+$ we find that:

$$\int_{T^*\mathbb{R}^2\times T^*\mathbb{R}} b\,(z,\xi,t)\,\chi_R\,(H)\,\mu_{sc}\,(dz,d\xi,dt,dH)$$
$$= \int_{\mathbb{R}}\int_{T^*\mathbb{R}^2} b\,(z,\xi,t)\,\tilde{\mu}_{sc}\,(t,dz,d\xi)\,dt + \lim_{h\to 0^+} k_{h,R}\,(b)\,. \tag{9.3}$$

But clearly, as $b \geq 0$, we always have

$$\lim_{h\to 0^+} k_{h,R}(b) = \lim_{h\to 0^+}\int_{\mathbb{R}} \tilde{W}^h_{\sigma_R(h^2 D_t)u_h}\,(b\,(t,\cdot))\,dt \geq 0,$$

for every $R > 0$. Taking this into account and letting $R \to \infty$ in (9.3) proves (9.1).

Now, as a consequence of (9.1) we have that the image of μ_{sc} under the projection onto the H-component is of the form $\mu(t,\cdot)dt$ for some $\mu \in L^\infty(\mathbb{R}_t; \mathcal{M}_+(T^*\mathbb{R}^2))$. The disintegration theorem then ensures that μ_{sc} can be written as:

$$\mu_{sc}\,(dz,d\xi,dt,dH) = \mu_{z,\xi,t}\,(dH)\,\mu\,(t,dz,d\xi)\,dt,$$

where, for μ-amost every (z,ξ), $\mu_{z,\xi,t}$ is a probability measure on $\mathbb{R}$. Since μ_{sc} is supported on the characteristic set $|\xi|^2 = 2H$ (see Proposition 8.3), we conclude that $\mu_{z,\xi,t}(dH) = \delta_{|\xi|^2/2}(dH)$ and the result follows. □

Appendix C: From action-angle coordinates to polar coordinates

Here we develop the technical calculations leading to the definitions of the operators $\mathcal{A}_{t,h^2 D_t}(P)$ and $\widetilde{\mathcal{A}}_{t,h^2 D_t}(P)$ used as a black-box in the paper. The point is that our "action-angle" coordinates (s, θ, E, J), well adapted to integrate the dynamics of the billiard flow, are not so convenient to express the Dirichlet boundary condition ($v(z) = 0$ for $|z| = 1$). Actually the best coordinates in which to write the boundary condition are the polar coordinates (which below will be written as $(x = -r \sin u, y = r \cos u)$) since the boundary is simply expressed as the set $\{r = 1\}$.

Let $P(s, \theta, E, J)$ be a function expressed in the new coordinates and let $\mathscr{U}$ be the Fourier integral operator defined in (3.1). The technical calculations done below are aimed at understanding how $\mathscr{U}^* \operatorname{Op}_h(P)\mathscr{U}$ acts in polar coordinates; in particular, under which conditions on the symbol P the boundary condition is preserved by $\mathscr{U}^* \operatorname{Op}_h(P)\mathscr{U}$.

For our purposes we need to understand the operator $\mathscr{U}^* \operatorname{Op}_h(P)\mathscr{U}$ modulo $O(h^2)$. Ideally we would like to separate it into a "tangential part" (involving only angular derivation $\frac{\partial}{\partial u}$) and a "radial part" involving the radial derivative $\frac{\partial}{\partial r}$ in a simple way. Below we calculate the action of the operator $\mathscr{U}^* \operatorname{Op}_h(P)\mathscr{U}$ on a plane wave

$$e_\xi(z) := e^{i\frac{(\xi_x x + \xi_y y)}{h}}$$

(where we use $z = (x, y)$, $\xi = (\xi_x, \xi_y)$ and $|\xi|^2 = \xi_x^2 + \xi_y^2$) and apply the method of stationary phase. The length of the calculation comes from the fact that we explicitly need the term of order h in the expansion.

In this section, we shall assume that $P(s, \theta, E, J)$ satisfies the following properties.

Assumption 10.1 Assume first that $P(s, \theta, E, J)$ is a smooth compactly supported function (possibly depending on h), with support away from $\{E = 0\}$ and inside $\{|J| < E\}$, and being 2π-periodic in the variable θ. We assume further that it satisfies the following estimates

$$\|\partial_s^\alpha \partial_\theta^\beta \partial_E^\gamma \partial_J^\delta P\|_\infty \leq C_{\alpha,\beta,\gamma,\delta} h^{-\delta}, \quad \text{for all } \alpha, \beta, \gamma, \delta \in \mathbb{N}.$$

The function P may also depend on the time variable t and its dual H, but, in this section, we omit them from the notation since they are transparent in the calculation. Typical symbols P for which the calculations below are needed can be found in (4.12) and (4.13).

Using the notation

$$a(E)=\sqrt{E},\quad (s_0,\theta_0,|\xi|,j_0)=\Phi^{-1}(x,y,\xi_x,\xi_y),\\ \text{and}\quad s(x,y,\theta)=-x\sin\theta+y\cos\theta,$$

recalling the expressions of $\mathscr{U}$ and $\mathscr{U}^*$ in (3.2)–(3.3), and unfolding all the integrals, we write

$$\begin{aligned}
&\mathscr{U}^*\,\mathrm{Op}_h(P)\mathscr{U}e_\xi(x,y)\\
&=(2\pi h)^{-5}\int P\left(s,\theta,E',j\right)e^{\frac{ij(\theta-\theta')}{h}}e^{\frac{iE'(s-s')}{h}}e^{-i\frac{S(x',y',\theta',s',E)}{h}}e^{i\frac{(\xi_x x'+\xi_y y')}{h}}e^{i\frac{S(x,y,\theta,s,E'')}{h}}\\
&\quad\times a(E)a(E'')\,d\theta\,ds\,dE''\,dx'\,dy'\,dE\,d\theta'\,ds'\,dE'\,dj\\
&=(2\pi h)^{-3}\int P\left(s,\theta,E',j\right)e^{\frac{ij(\theta-\theta_0)}{h}}e^{\frac{iE'(s-s')}{h}}e^{i\frac{s'|\xi|}{h}}e^{i\frac{S(x,y,\theta,s,E'')}{h}}\frac{a(|\xi|)}{|\xi|}\\
&\quad\times a(E'')\,d\theta\,ds\,dE''\,ds'\,dE'\,dj\\
&=(2\pi h)^{-2}\int P\left(s,\theta,|\xi|,j\right)e^{\frac{ij(\theta-\theta_0)}{h}}e^{i\frac{s|\xi|}{h}}e^{i\frac{S(x,y,\theta,s,E'')}{h}}\frac{a(|\xi|)}{|\xi|}a(E'')\,d\theta\,dE''\,ds\,dj\\
&=(2\pi h)^{-1}\int \big(P\left(s(x,y,\theta),\theta,|\xi|,j\right)a(|\xi|)\\
&\quad-ih\partial_s P\left(s(x,y,\theta),\theta,|\xi|,j\right)a'(|\xi|)\big)\\
&\quad\times e^{\frac{ij(\theta-\theta_0)}{h}}e^{i\frac{-|\xi|x\sin\theta+|\xi|y\cos\theta}{h}}\frac{a(|\xi|)}{|\xi|}d\theta\,dj\quad+\frac{O(h^2)}{\inf_{P(s,\theta,E,J)\neq0}|E|^2}.
\end{aligned}$$

By standard estimates on pseudodifferential operators, the remainder term will correspond to an estimate in the $L^2_{\mathrm{comp}}\to L^2_{\mathrm{loc}}$ topology of operators.

Letting $(x,y)=(-r\sin u,r\cos u)$, we have $r=\sqrt{x^2+y^2}$, $u=\arccos y/r$ and $s(x,y,\theta)=r\cos(\theta-u)$. Modulo $\frac{O(h^2)}{\inf_{P(s,\theta,E,J)\neq0}|E|^2}$, we are thus left with

$$\begin{aligned}
&\mathscr{U}^*\,\mathrm{Op}_h(P)\mathscr{U}e_\xi(x,y)\\
&\quad=(2\pi h)^{-1}\frac{a(|\xi|)}{|\xi|}\int \big(P\left(r\cos(\theta-u),\theta,|\xi|,j\right)a(|\xi|)\\
&\qquad-ih\partial_s P\left(r\cos(\theta-u),\theta,|\xi|,j\right)a'(|\xi|)\big)e^{\frac{ij(\theta-\theta_0)}{h}}e^{i\frac{|\xi|r\cos(\theta-u)}{h}}d\theta dj\\
&\quad=(2\pi h)^{-1}\int \big(P\left(r\cos(\theta-u),\theta,|\xi|,j\right)\\
&\qquad-\frac{ih}{2|\xi|}\partial_s P\left(r\cos(\theta-u),\theta,|\xi|,j\right)e^{\frac{ij(\theta-\theta_0)}{h}}e^{i\frac{|\xi|r\cos(\theta-u)}{h}}d\theta dj.
\end{aligned}$$

Remark 10.2 Note that in the above integral, the functions integrated are all 2π-periodic in the variable θ except for the oscillating factor $e^{i\frac{j\theta}{h}}$. This integral has to be interpreted in one of the following two (equivalent) ways:

- either in the sense of oscillatory integrals [29, Section 7.8]: the integral over $\theta \in \mathbb{R}$ is the Fourier transform of the periodic function of θ, seen as a tempered distribution. The result is a tempered distribution given by a linear combination of Dirac masses carried by $j \in h\mathbb{Z}$. The integration with respect to j has to be interpreted as a duality product;
- or in the sense of Fourier series: assuming $j \in h\mathbb{Z}$, the function of θ is 2π-periodic and the integral over θ takes place on $\mathbb{R}/2\pi\mathbb{Z}$, i.e. on any period. The integral with respect to j has then to be understood as a discrete sum over $h\mathbb{Z}$.

Using for instance the second approach of this remark, we now apply stationary phase w.r.t. θ (while $j \in h\mathbb{Z}$ is kept fixed, since our symbols may be rapidly oscillating in j). We start with the P term (the $ih\partial_s P$-term can be treated exactly the same way). Fixing j and looking at the θ-integral, we let

$$\mathcal{I} = (2\pi h)^{-1/2} \int_{\mathbb{R}/2\pi\mathbb{Z}} P\left(r\cos(\theta-u), \theta, |\xi|, j\right) e^{\frac{ij(\theta-\theta_0)}{h}} e^{i\frac{|\xi| r\cos(\theta-u)}{h}} d\theta. \tag{10.1}$$

The phase in $\mathcal{I}$ has 2 critical points $\theta = u+\theta_1, u+\theta_2$, where θ_k are the solutions of $j - |\xi| r\sin\theta = 0$. Since we are assuming that $P(s,\theta,E,j)$ is supported in $\{|j| < E\}$, these two solutions are distinct for r close to 1, and correspond to non-degenerate stationary points (in all that follows we consider that r is close to 1 since this calculation only serves to understand $\mathscr{U}^* \operatorname{Op}_h(P)\mathscr{U}$ near the boundary of the disk). We will denote by $\theta_1(r,E,j), \theta_2(r,E,j)$ the solutions of $j - Er\sin\theta = 0$. To fix ideas, θ_1 will be the one with $\cos\theta_1 > 0$ and θ_2 the one with $\cos\theta_2 < 0$ (that is, $\theta_1 \in (-\pi/2, \pi/2)$, $\theta_2 \in (\pi/2, 3\pi/2)$).

We let χ_k be smoth cutoff functions such that $\chi_k = 1$ on a neighborhood of θ_k, for $k = 1, 2$ and such that $\operatorname{supp}(\chi_1) \subset (-\pi/2, \pi/2)$ and $\operatorname{supp}(\chi_2) \subset (\pi/2, 3\pi/2)$. Using the non-stationary phase lemma, we have modulo $O(h^\infty)$

$$\begin{aligned}\mathcal{I} &= (2\pi h)^{-1/2} \int \chi_1(\theta-u) P\left(r\cos(\theta-u), \theta, |\xi|, j\right) e^{\frac{ij(\theta-\theta_0)}{h}} e^{i\frac{|\xi| r\cos(\theta-u)}{h}} d\theta \\ &\quad + (2\pi h)^{-1/2} \int \chi_2(\theta-u) P\left(r\cos(\theta-u), \theta, |\xi|, j\right) e^{\frac{ij(\theta-\theta_0)}{h}} e^{i\frac{|\xi| r\cos(\theta-u)}{h}} d\theta.\end{aligned}$$

Below, E will always take the value $E = |\xi|$.

We will work on the integral $\mathcal{I}$ by applying the following lemma (which follows from the method of stationary phase):

Lemma 10.3 *Let*

$$\mathcal{I}_f = \int \chi(\theta) f(\theta) e^{i\frac{S(\theta)}{h}} d\theta,$$

where $S : \mathbb{R} \to \mathbb{R}$ is a smooth function having only one critical point θ_c, which is non degenerate; where χ is a smooth compactly supported function which is constant in a neighborhood of θ_c; and where f is a smooth function. Then, letting $a = f(\theta_c)$, $b = \frac{f'(\theta_c)}{S''(\theta_c)}$ and $c = \frac{1}{2}(\frac{f''(\theta_c)}{S''(\theta_c)^2} - \frac{S^{(3)}(\theta_c) f'(\theta_c)}{S''(\theta_c)^3})$, we have

(i) $\mathcal{I}_f = a \int \chi(\theta) e^{i\frac{S(\theta)}{h}} d\theta + b \int \chi(\theta) S'(\theta) e^{i\frac{S(\theta)}{h}} d\theta + c \int \chi(\theta) S'(\theta)^2 e^{i\frac{S(\theta)}{h}} d\theta + O(h^{2+1/2})$;

(ii) $\mathcal{I}_f = \big(a + ihcS''(\theta_c)\big) \int \chi(\theta) e^{i\frac{S(\theta)}{h}} d\theta + O(h^{2+1/2})$.

Proof (i) The functions f and $g : \theta \mapsto a + bS'(\theta) + cS'(\theta)^2$ coincide up to order 2 at θ_c. The method of stationary phase tells us that $\mathcal{I}_f$ and $\mathcal{I}_g$ coincide modulo $O(h^{2+1/2}) = O(h^{5/2})$.

Item (ii) is obtained from (i) by integration by parts, noting that $S'(\theta) e^{i\frac{S(\theta)}{h}}$ is the derivative of $\frac{h}{i} e^{i\frac{S(\theta)}{h}}$. □

In what follows, this lemma will be applied with $S(\theta) = j(\theta - \theta_0) + Er\cos(\theta - u)$, $f(\theta) = P(r\cos(\theta - u), \theta, E, j)$, $\chi(\theta) = \chi_k(\theta - u)$ ($k = 1, 2$), $\theta_c = u + \theta_k$. Starting with

$$S'(\theta) = j - Er\sin(\theta - u) \sim Er\left[-\cos\theta_k(\theta - u - \theta_k) + \frac{\sin\theta_k}{2}(\theta - u - \theta_k)^2\right] \tag{10.2}$$

(modulo $O(\theta - u - \theta_k)^3$), we have

$$\begin{aligned} P(r\cos(\theta - u), \theta) &\sim P(r\cos\theta_k, u + \theta_k) - \frac{S'(\theta)}{Er\cos\theta_k}\frac{d}{d\theta}P(r\cos\theta_k, u + \theta_k) \\ &\quad + \frac{S'(\theta)^2}{(Er\cos\theta_k)^2}\left[\frac{\sin\theta_k}{2\cos\theta_k}\frac{d}{d\theta}P(r\cos\theta_k, u + \theta_k) + \frac{1}{2}\frac{d^2}{d\theta^2}P(r\cos\theta_k, u + \theta_k)\right]. \end{aligned} \tag{10.3}$$

We have momentarily dropped the j and E variables from the argument of P since they are fixed in the upcoming calculation.

We want to apply the method of Lemma 10.3 with $a = a_k = P(r\cos\theta_k, u + \theta_k)$ $(k = 1, 2)$,

$$c = c_k = \frac{1}{(Er\cos\theta_k)^2}\left[\frac{\sin\theta_k}{2\cos\theta_k}\frac{d}{d\theta}P(r\cos\theta_k, u+\theta_k) + \frac{1}{2}\frac{d^2}{d\theta^2}P(r\cos\theta_k, u+\theta_k)\right].$$

The lemma yields that

$$\begin{aligned}\mathcal{I} &= (2\pi h)^{-1/2}(a_1 + ihc_1 S''(u+\theta_1))\int \chi_1(\theta-u)e^{\frac{ij(\theta-\theta_0)}{h}}e^{i\frac{Er\cos(\theta-u)}{h}}d\theta\\ &\quad +(2\pi h)^{-1/2}(a_2 + ihc_2 S''(u+\theta_2))\int \chi_2(\theta-u)e^{\frac{ij(\theta-\theta_0)}{h}}e^{i\frac{Er\cos(\theta-u)}{h}}d\theta + O(h^2).\end{aligned} \tag{10.4}$$

Remark 10.4 Denoting by $\partial_1 = \partial_s$, $\partial_2 = \partial_\theta$ (to avoid possible confusion), we have

$$\frac{d}{d\theta}P(r\cos\theta, u+\theta) = \partial_2 P(r\cos\theta, u+\theta) - r\sin\theta\,\partial_1 P(r\cos\theta, u+\theta),$$

and

$$\begin{aligned}\frac{d^2}{d\theta^2}P(r\cos\theta, u+\theta) &= \partial_2^2 P(r\cos\theta, u+\theta) - r\cos\theta\,\partial_1 P(r\cos\theta, u+\theta)\\ &\quad -r\sin\theta\,\partial_2\partial_1 P(r\cos\theta, u+\theta)\\ &\quad +r^2\sin^2\theta\,\partial_1^2 P(r\cos\theta, u+\theta).\end{aligned}$$

Remark 10.5 Important remark about symmetry. We keep denoting θ_k for $\theta_k(r, E, j)$. We first note that $\theta_2 = \pi - \theta_1$, $\cos\theta_1 = -\cos\theta_2$, $\sin\theta_1 = \sin\theta_2$.

Moreover, if P satisfies the symmetry condition (B) of Definition 4.1, we have for $r = 1$ (restoring in our notation the dependence of P on the full set of variables)

$$P(\cos\theta_1, u+\theta_1, E, j) = P(\cos\theta_2, u+\theta_2, E, j).$$

And similarly for all partial derivatives of P if we assume the stronger symmetry condition (C) (in Definition 4.1).

Here we don't necessarily want to assume that P is symmetric; but, motivated by the previous remark, we introduce the functions P^σ and P^α, the symmetric and antisymmetric parts of P respectively:

$$
\begin{aligned}
P^{\sigma}(r,\theta,E,j) &:= \frac{P(r\cos\theta_1,\theta+\theta_1,E,j)+P(-r\cos\theta_1,\theta+\pi-\theta_1,E,j)}{2}\\
&= \frac{P(r\cos\theta_1,\theta+\theta_1,E,j)+P(r\cos\theta_2,\theta+\theta_2,E,j)}{2},
\end{aligned}
\tag{10.5}
$$

and

$$
\begin{aligned}
P^{\alpha}(r,\theta,E,j) &:= \frac{P(r\cos\theta_1,\theta+\theta_1,E,j)-P(-r\cos\theta_1,\theta+\pi-\theta_1,E,j)}{2}\\
&= \frac{P(r\cos\theta_1,\theta+\theta_1,E,j)-P(r\cos\theta_2,\theta+\theta_2,E,j)}{2},
\end{aligned}
\tag{10.6}
$$

for $\theta_1=\theta_1(r,E,j),\theta_2=\theta_2(r,E,j)$ defined previously, so that

$$
P(r\cos\theta_1,\theta+\theta_1,E,j)=P^{\sigma}(r,\theta,E,j)+P^{\alpha}(r,\theta,E,j),
$$

$$
P(r\cos\theta_2,\theta+\theta_2,E,j)=P^{\sigma}(r,\theta,E,j)-P^{\alpha}(r,\theta,E,j).
$$

Working from the expression (10.4), the terms

$$
\begin{aligned}
&(2\pi h)^{-1/2}a_1\int\chi_1(\theta-u)e^{\frac{ij(\theta-\theta_0)}{h}}e^{i\frac{Er\cos(\theta-u)}{h}}d\theta\\
&+(2\pi h)^{-1/2}a_2\int\chi_2(\theta-u)e^{\frac{ij(\theta-\theta_0)}{h}}e^{i\frac{Er\cos(\theta-u)}{h}}d\theta
\end{aligned}
$$

maybe grouped as follows:

$$
\begin{aligned}
&(2\pi h)^{-1/2}\int\chi_1(\theta-u)P(r\cos\theta_1,u+\theta_1,E,j)e^{\frac{ij(\theta-\theta_0)}{h}}e^{i\frac{Er\cos(\theta-u)}{h}}d\theta\\
&\quad+(2\pi h)^{-1/2}\int\chi_2(\theta-u)P(r\cos\theta_2,u+\theta_2,E,j)e^{\frac{ij(\theta-\theta_0)}{h}}e^{i\frac{Er\cos(\theta-u)}{h}}d\theta\\
&=(2\pi h)^{-1/2}P^{\sigma}(r,u,E,j)\int_0^{2\pi}e^{\frac{ij(\theta-\theta_0)}{h}}e^{i\frac{Er\cos(\theta-u)}{h}}d\theta\\
&\quad+(2\pi h)^{-1/2}P^{\alpha}(r,u,E,j)\left(\int\chi_1(\theta-u)e^{\frac{ij(\theta-\theta_0)}{h}}e^{i\frac{|\xi|r\cos(\theta-u)}{h}}d\theta\right.\\
&\quad\left.-\int\chi_2(\theta-u)e^{\frac{ij(\theta-\theta_0)}{h}}e^{i\frac{|\xi|r\cos(\theta-u)}{h}}d\theta\right).
\end{aligned}
\tag{10.7}
$$

Applying again Lemma 10.3 (this time with the function $f(\theta)=\cos(\theta-u)$), this expression can be rewritten modulo $O(h^2)$ as

$$
\begin{aligned}
&(2\pi h)^{-1/2}\int_0^{2\pi} P^\sigma(r,u,E,j)e^{\frac{ij(\theta-\theta_0)}{h}}e^{i\frac{Er\cos(\theta-u)}{h}}d\theta\\
&+(2\pi h)^{-1/2}\int_0^{2\pi}\frac{P^\alpha(r,u,E,j)}{E\cos\theta_1(r,E,j)}\\
&\left(E\cos(\theta-u)-ih\frac{1}{2r\cos^2\theta_1(r,E,j)}\right)\\
&\times e^{\frac{ij(\theta-\theta_0)}{h}}e^{i\frac{Er\cos(\theta-u)}{h}}d\theta.
\end{aligned}
$$

With the change of variable $\theta-\theta_0 \rightsquigarrow u-\theta$, this may also be written as

$$
\begin{aligned}
&(2\pi h)^{-1/2}\int_0^{2\pi} P^\sigma(r,u,E,j)e^{\frac{ij(u-\theta)}{h}}e^{i\frac{Er\cos(\theta-\theta_0)}{h}}d\theta\\
&+(2\pi h)^{-1/2}\int_0^{2\pi}\frac{P^\alpha(r,u,E,j)}{E\cos\theta_1(r,E,j)}\\
&\left(E\cos(\theta-\theta_0)-ih\frac{1}{2r\cos^2\theta_1(r,E,j)}\right)\\
&\times e^{\frac{ij(u-\theta)}{h}}e^{i\frac{Er\cos(\theta-\theta_0)}{h}}d\theta.
\end{aligned}
\tag{10.8}
$$

We note that $e^{i\frac{Er\cos(\theta-\theta_0)}{h}} = e^{i\frac{(\xi_x x'+\xi_y y')}{h}} = e_\xi(x',y')$ if $(x',y') = (-r\sin\theta, r\cos\theta)$, and $E\cos(\theta-\theta_0)e^{i\frac{Er\cos(\theta-\theta_0)}{h}} = hD_r e^{i(\xi_x x'+\xi_y y')/h}$ where $D_r=\frac{1}{i}\partial_r$.

Terms of order h. Apart from the term of order h arising in the last line of (10.8), other terms of order h in (10.4) come from evaluation of the integrals

$$
\begin{aligned}
&(2\pi h)^{-1/2} ihc_1 S''(u+\theta_1)\int \chi_1(\theta-u)e^{\frac{ij(\theta-\theta_0)}{h}}e^{i\frac{Er\cos(\theta-u)}{h}}d\theta\\
&\quad+(2\pi h)^{-1/2} ihc_2 S''(u+\theta_2)\int \chi_2(\theta-u)e^{\frac{ij(\theta-\theta_0)}{h}}e^{i\frac{Er\cos(\theta-u)}{h}}d\theta.
\end{aligned}
$$

This is equal to:

$$
\begin{aligned}
&\sum_{k=1,2}\left\{(2\pi h)^{-1/2} ih\int \chi_k(\theta-u)\frac{1}{(Er\cos\theta_k)}\right.\\
&\quad\times\left[\frac{\sin\theta_k}{2\cos\theta_k}\frac{d}{d\theta}P(r\cos\theta_k,u+\theta_k)+\frac{1}{2}\frac{d^2}{d\theta^2}P(r\cos\theta_k,u+\theta_k)\right]\\
&\quad\left.\times e^{\frac{ij(\theta-\theta_0)}{h}}e^{i\frac{Er\cos(\theta-u)}{h}}d\theta\right\}.
\end{aligned}
\tag{10.9}
$$

Here $\frac{d}{d\theta}P(r\cos\theta_k, u+\theta_k)$ and $\frac{d^2}{d\theta^2}P(r\cos\theta_k, u+\theta_k)$ may be replaced by their expressions in terms of partial derivatives of P, as in Remark 10.4.

To summarize our computations, we need to introduce some notation.

Definition 10.6 Assume P satisfies Assumption 10.1. Then, we define by $I(P), II(P), III(P), IV(P)$ the operators whose action on e_ξ at the point $(x, y) = (-r\sin u, r\cos u)$ is given as follows: for $\xi = (\xi_x, \xi_y)$, $E = |\xi|$, we have (referring to Remark 10.2 for the meaning of the integrals)

$$I(P)e_\xi(x,y) = \frac{1}{2\pi h}\int A(r,u,E,j)e^{\frac{ij(u-\theta)}{h}} e_\xi(-r\sin\theta, r\cos\theta)d\theta dj$$
$$II(P)e_\xi(x,y) = \frac{1}{2\pi h}\int B(r,u,E,j)e^{\frac{ij(u-\theta)}{h}} hD_r e_\xi(-r\sin\theta, r\cos\theta)d\theta dj$$
$$III(P)e_\xi(x,y) = \frac{1}{2\pi h}\int C(r,u,E,j)e^{\frac{ij(u-\theta)}{h}} e_\xi(-r\sin\theta, r\cos\theta)d\theta dj$$
$$IV(P)e_\xi(x,y) = \frac{1}{2\pi h}\int D(r,u,E,j)e^{\frac{ij(u-\theta)}{h}} hD_r e_\xi(-r\sin\theta, r\cos\theta)d\theta dj,$$

where

$$\begin{aligned}
A(r,u,E,j) &= P^\sigma(r,u,E,j),\\
B(r,u,E,j) &= \frac{P^\alpha(r,u,E,j)}{E\cos\theta_1(r,E,j)},\\
C(r,u,E,j) &= -\frac{1}{2E}\partial_s P^\sigma(r,u,E,j) + c^\sigma(r,u,E,j)\\
&\quad -\frac{1}{2r\cos^2\theta_1(r,E,j)}\frac{P^\alpha(r,u,E,j)}{E\cos\theta_1(r,E,j)},\\
D(r,u,E,j) &= -\frac{1}{2E}\frac{\partial_s P^\alpha(r,u,E,j)}{E\cos\theta_1(r,E,j)} + \frac{c^\alpha(r,u,E,j)}{E\cos\theta_1(r,E,j)},
\end{aligned} \tag{10.10}$$

with the notation P^σ, P^α of (10.5), (10.6), and where, in addition

$$\begin{aligned}
c(s,\theta,E,j) = \frac{1}{(Es)}\Bigg[&\frac{j}{2Es}\left(\partial_2 P(s,\theta,E,j) - \frac{j}{E}\partial_1 P(s,\theta,E,j)\right)\\
&+\frac{1}{2}\Bigg(\partial_2^2 P(s,\theta,E,j) - s\partial_1 P(s,\theta,E,j) - \frac{j}{E}\partial_2\partial_1 P(s,\theta,E,j)\\
&+\frac{j^2}{E^2}\partial_1^2 P(s,\theta,E,j)\Bigg)\Bigg].
\end{aligned} \tag{10.11}$$

Remark first that the expression of $c(s,\theta,E,j)$ is calculated so that

$$\begin{aligned}
&c(r\cos\theta_k, u+\theta_k, E, j)\\
&\quad = \frac{1}{(Er\cos\theta_k)}\left[\frac{\sin\theta_k}{2\cos\theta_k}\frac{d}{d\theta}P(r\cos\theta_k, u+\theta_k)\right.\\
&\quad \left. +\frac{1}{2}\frac{d^2}{d\theta^2}P(r\cos\theta_k, u+\theta_k)\right],
\end{aligned}$$

which is the expression appearing in the last lines of (10.9).

Second, note that A, B, C, D are real-valued functions if P is.

With this notation in hand, we can now summarize our calculations in the following proposition.

Proposition 10.7 *Assume P satisfies Assumption 10.1. Then, modulo a term of order $\frac{O(h^2)}{\inf_{P(s,\theta,E,J)\neq 0}|E|^2}$ in the $L^2_{\mathrm{comp}} \to L^2_{\mathrm{loc}}$ topology of operators, $\mathscr{U}^*\mathrm{Op}_h(P)\mathscr{U}$ satisfies*

$$\mathscr{U}^*\mathrm{Op}_h(P)\mathscr{U} = I(P) + II(P) + ihIII(P) + ihIV(P).$$

Let us now check that operators of the form $I(P), II(P), III(P), IV(P)$ belong to a reasonable class of spatial pseudodifferential operators.

Lemma 10.8 *Let $M(r,u,E,j)$ be a smooth (possibly h-dependent) function, compactly supported in r, E, j, 2π-periodic in u, supported in $\{|j| < E\}$ and away from $\{r=0\}$ and $\{E=0\}$. Assume M satisfies estimates of the form*

$$\sup_{h,r,u,E,j} h^{\gamma+\delta}|\partial_r^\alpha \partial_u^\beta \partial_E^\gamma \partial_j^\delta M| < +\infty, \quad \textit{for all } \alpha,\beta,\gamma,\delta\in\mathbb{N}. \tag{10.12}$$

Then the operators defined by their action on e_ξ at $(x,y) = (-r\sin u, r\cos u)$ by

$$\hat{A}e_\xi(x,y) = \frac{1}{2\pi h}\int M(r,u,|\xi|,j)e^{\frac{ij(u-\theta)}{h}} e_\xi(-r\sin\theta, r\cos\theta)d\theta dj,$$

and

$$\hat{B}e_\xi(x,y) = \frac{1}{2\pi h}\int M(r,u,|\xi|,j)e^{\frac{ij(u-\theta)}{h}} hD_r e_\xi(-r\sin\theta, r\cos\theta)d\theta dj$$

are semiclassical pseudodifferential operators of the form $m_h(z, hD_z)$ where m_h satisfies estimates of the form

$$\sup_{h,z,\xi} h^{|\beta|}|\partial_z^\alpha \partial_\xi^\beta m_h| < +\infty, \quad \textit{for all} \ \ \alpha, \beta \in \mathbb{N}^2. \tag{10.13}$$

In particular, these operators are bounded on $L^2(\mathbb{R}^2)$.

Proof Let us first treat the case of $\hat{A}$. Define $\kappa(r, u) = (-r \sin u, r \cos u)$. The function m_h is given by the formula

$$m_h(\kappa(r, u), h\xi) = \frac{1}{2\pi} \int M(r, u, h|\xi|, hj) e^{ij(u-\theta)} e^{i\xi\cdot(\kappa(r,\theta)-\kappa(r,u))} d\theta dj.$$

The proof of [30, Theorem 18.1.17] applies to prove the desired estimate on m_h.

The operator $\hat{B}$ is an operator of the previous form, composed with hD_r. Since M is assumed to be compactly supported in E, the desired estimate also holds for $\hat{B}$. The bounded follows from the Calderón–Vaillancourt theorem [18].

Coming back to the operators defined in Definition 10.6, we have obtained the following corollary.

Corollary 10.9 *Assume* P *satisfies Assumption* 10.1. *Then, the operators* $I(P)$, $II(P)$, $III(P)$, $IV(P)$ *of Definition* 10.6 *are semiclassical pseudodifferential operators of the form* $m_h(z, hD_z)$ *where* m_h *satisfies estimates of the form* (10.13). *In particular, these operators are bounded on* $L^2(\mathbb{R}^2)$.

Appendix D: The operators $\mathcal{A}_{t,h^2D_t}(P)$ and $\tilde{\mathcal{A}}_{t,h^2D_t}(P)$

We recall that the operators we manipulate are given by $\mathscr{U}^* \operatorname{Op}_h(P(s, \theta, E, J, t, hH))\mathscr{U}$ where the symbol $P(s, \theta, E, J, t, H)$ is typically of the form (4.12) or (4.13), and thus satisfies Assumption 10.1 with respect to the space variable (or more precisely Assumption 11.1 below). The goal of this Appendix is to understand further (and up to order two in powers of h) how $\mathscr{U}^* \operatorname{Op}_h(P)\mathscr{U}$ acts on functions vanishing on the boundary. The euclidean laplacian $\Delta_{\mathbb{R}^2}$ does not preserve the set of functions vanishing on the boundary. That is why we would like to eliminate the dependence of P on the variable E. We use the fact that the semiclassical measures associated with solutions of the Schrödinger equation are supported on $\{E^2 = 2H\}$, to replace E by $\sqrt{2H}$ in the calculations. This induces an additional error term, that will be shown to converge to 0 in Lemma 11.5 below.

This means, in particular, that we need to write explicitly the (t, H) dependence of the symbols (skipped in Appendix A above). As a consequence, the functions A, B, C, D defined from P in Definition 10.6 now also depend on t and H since P does.

The assumptions made on the symbol P in this section are similar to Assumption 10.1 above. We recall that typical symbols P under interest here are given by (4.12) and (4.13).

Assumption 11.1 Assume first that the function P is a smooth compactly supported function in s, E, j, t, H (possibly depending on h), with support in $\{|j| < E\}$ and away from $\{j = 0, s = 0\} \cup \{H = 0\}$, and being 2π-periodic in the variable θ. We assume further that it satisfies the following estimates:

$$\sup_{h,s,u,E,j,t,H} h^\nu h^\delta |\partial_s^\alpha \partial_\theta^\beta \partial_E^\gamma \partial_j^\delta \partial_t^\mu \partial_H^\nu P| < +\infty, \quad \text{for all } \alpha, \beta, \gamma, \delta, \nu, \mu \in \mathbb{N}.$$

In the following formal calculations, it will be convenient to introduce the following notation.

Definition 11.2 If P depends on t and H, we write $I_{t,H}(P) := I(P)$, $II_{t,H}(P) := II(P)$, $III_{t,H}(P) := III(P)$, $IV_{t,H}(P) := IV(P)$: they are t, H-families of operators defined in Definition 10.6. We then denote by $\mathcal{A}_{t,H}(P)$ the family of operators given by

$$\mathcal{A}_{t,H}(P) = I_{t,H}(P) + II_{t,H}(P) + ihIII_{t,H}(P) + ihIV_{t,H}(P). \tag{11.1}$$

We have shown (see Proposition 10.7) that for any given (t, H), $\mathcal{A}_{t,H}(P)$ coincides with $\mathscr{U}^* \operatorname{Op}_h(P(\cdot, t, H))\mathscr{U}$ (where the quantification is only in the variables (s, θ, E, J)) modulo $\frac{O(h^2)}{\inf_{P(s,\theta,E,J)\neq 0} |E|^2}$ in the $L^2_{\text{comp}} \to L^2_{\text{loc}}$ topology of operators.

We now define a modified operator $\widetilde{\mathcal{A}}_{t,H}(P)$ whose action on functions vanishing on $\partial\mathbb{D}$ is easier to understand.

Definition 11.3 We denote by $\widetilde{\mathcal{A}}_{t,H}(P)$ the family of operators

$$\widetilde{\mathcal{A}}_{t,H}(P) = \widetilde{I}_{t,H}(P) + \widetilde{II}_{t,H}(P) + ih\widetilde{III}_{t,H}(P) + ih\widetilde{IV}_{t,H}(P), \tag{11.2}$$

where the four operators involved are defined by their action on e_ξ at the point $(x, y) = (-r \sin u, r \cos u)$ is given by

$$\begin{aligned}
&\widetilde{I}_{t,H}(P)(e_\xi)(x, y) \\
&\quad = \frac{1}{2\pi h} \int A(r, u, \sqrt{2H}, j, t, H) e^{\frac{ij(u-\theta)}{h}} e_\xi(-r \sin\theta, r \cos\theta) d\theta dj, \\
&\widetilde{II}_{t,H}(P)(e_\xi)(x, y) \\
&\quad = \frac{1}{2\pi h} \int B(r, u, \sqrt{2H}, j, t, H) e^{\frac{ij(u-\theta)}{h}} hD_r e_\xi(-r \sin\theta, r \cos\theta) d\theta dj,
\end{aligned}$$

$$\widetilde{III}_{t,H}(P)(e_\xi)(x,y)$$
$$= \frac{1}{2\pi h}\int C(r,u,\sqrt{2H},j,t,H)e^{\frac{ij(u-\theta)}{h}}e_\xi(-r\sin\theta, r\cos\theta)d\theta dj,$$
$$\widetilde{IV}_{t,H}(P)(e_\xi)(x,y)$$
$$= \frac{1}{2\pi h}\int D(r,u,\sqrt{2H},j,t,H)e^{\frac{ij(u-\theta)}{h}}hD_r e_\xi(-r\sin\theta, r\cos\theta)d\theta dj,$$

where A, B, C, D are defined (as functions of P) in Definition 10.6.

In other words, in the definition of $\mathcal{A}_{t,H}(P)$ we have replaced $|\xi|$ by $\sqrt{2H}$ in the symbols. For us, $\widetilde{\mathcal{A}}_{t,H}(P)$ is a very convenient operator to study the Dirichlet boundary problem, since we have

$$\widetilde{I}_{t,H}(P) = A(r,u,\sqrt{2H},hD_u,t,H),$$
$$\widetilde{III}_{t,H}(P) = C(r,u,h\sqrt{2H},hD_u,t,H)$$

(so that they do not involve any derivative w.r.t. r) and

$$\widetilde{II}_{t,H} = B(r,u,\sqrt{2H},hD_u,t,H)\circ hD_r,$$
$$\widetilde{IV}_{t,H}(P) = D(r,u,\sqrt{2H},hD_u,t,H)\circ hD_r$$

which are only of degree 1 w.r.t. the variable r. We define the operators

$$\mathcal{A}_{t,h^2D_t}(P) := \mathrm{Op}_{h^2}\left(\mathcal{A}_{t,H}(P)\right), \quad \text{and} \quad \widetilde{\mathcal{A}}_{t,h^2D_t}(P) := \mathrm{Op}_{h^2}\left(\widetilde{\mathcal{A}}_{t,H}(P)\right),$$

where the quantification only concerns the variables (t, H). We have the analogue of Corollary 10.9 stating that these operators are proper pseudodifferential operators.

Corollary 11.4 *Assume P satisfies Assumption 11.1. Then, the operators $I_{t,h^2D_t}(P)$, $II_{t,h^2D_t}(P)$, $III_{t,h^2D_t}(P)$, $IV_{t,h^2D_t}(P)$, $\mathcal{A}_{t,h^2D_t}(P)$ and the operators $\widetilde{I}_{t,h^2D_t}(P)$, $\widetilde{II}_{t,h^2D_t}(P)$, $\widetilde{III}_{t,h^2D_t}(P)$, $\widetilde{IV}_{t,h^2D_t}(P)$, $\widetilde{\mathcal{A}}_{t,h^2D_t}(P)$ are semiclassical pseudodifferential operators of the form $m_h(z,t,hD_z,h^2D_t)$ where m_h satisfies estimates of the form*:

$$\sup_{h,z,\xi,t,H} h^{|\beta|}h^{\nu}|\partial_z^\alpha\partial_\xi^\beta\partial_t^\mu\partial_H^\nu m_h| < +\infty, \quad \textit{for all } \alpha,\beta\in\mathbb{N}^2, \mu,\nu\in\mathbb{N}.$$

In particular, these operators are bounded on $L^2(\mathbb{R}^2\times\mathbb{R})$.

Now, we want to replace everywhere $\mathcal{A}_{t,h^2D_t}(P)$ by $\widetilde{\mathcal{A}}_{t,h^2D_t}(P)$. This is possible thanks to the fact that our semiclassical measures are supported by the set $\{E^2 = 2H\}$; a precise statement is given in Lemma 11.5 below.

Lemma 11.5 *If u_h is a solution to the Schrödinger equation (1.1) satisfying in addition the assumptions of Remark 2.4, then, for any P satisfying Assumption 11.1, we have*

$$\begin{aligned}&\langle u_h, \mathcal{A}_{t,h^2D_t}(P)u_h\rangle_{L^2(\mathbb{R}^2\times\mathbb{R})} - \langle u_h, \widetilde{\mathcal{A}}_{t,h^2D_t}(P)u_h\rangle_{L^2(\mathbb{R}^2\times\mathbb{R})}\\ &\quad = O_\epsilon\left(h^{1/2-\epsilon}\right)\|u_h^0\|^2_{L^2(\mathbb{D})}.\end{aligned}$$

Proof We write the decompositions (11.1) and (11.2) of the operators $\mathcal{A}_{t,h^2D_t}(P)$ and $\widetilde{\mathcal{A}}_{t,h^2D_t}(P)$. Terms coming from $III_{t,h^2D_t}(P)$, $\widetilde{III}_{t,h^2D_t}(P)$, $IV_{t,h^2D_t}(P)$, $\widetilde{IV}_{t,h^2D_t}(P)$ are of order h, and it suffices to treat the term $I_{t,h^2D_t}(P) - \widetilde{I}_{t,h^2D_t}(P)$ (the term $II_{t,h^2D_t}(P) - \widetilde{II}_{t,h^2D_t}(P)$ is treated similarly).

First, for E restricted to a compact set, we can divide $A(r,u,E,j,t,H) - A(r,u,\sqrt{2H},j,t,H)$ by E^2-2H thanks to the following Taylor formula:

$$\begin{aligned}&A(r,u,E,j,t,H) - A(r,u,\sqrt{2H},j,t,H)\\ &\quad = (E+\sqrt{2H})^{-1}\int_0^1 \frac{\partial A}{\partial E}(r,u,\sqrt{2H}+l(E-\sqrt{2H}),j,t,H)dl\,(E^2-2H).\end{aligned}$$

The function $(E+\sqrt{2H})^{-1}\int_0^1 \frac{\partial A}{\partial E}(r,u,\sqrt{2H}+l(E-\sqrt{2H}),j,t,H)dl$, restricted to a compact set in E, satisfies the estimate of Assumption 11.1. We can apply Corollary 11.4 to see that, for any compactly supported χ, the operator

$$\left(I_{t,h^2D_t}(P) - \widetilde{I}_{t,h^2D_t}(P)\right)\chi(-h^2\Delta_{\mathbb{R}^2})$$

is of the form $\tilde{a}_h(z,hD_z,t,h^2D_t)$ where $\tilde{a}_h$ is of the form

$$\tilde{a}_h(z,\xi,t,H) = a_h(z,\xi,t,H)(E^2-2H)$$

and a_h is compactly supported in ξ, H and satisfies

$$\sup_{h,z,\xi,t,H} h^{|\beta|}h^{2\nu}|\partial_z^\alpha\partial_\xi^\beta\partial_t^\mu\partial_H^\nu a_h| < +\infty, \quad \text{for all } \alpha,\beta\in\mathbb{N}^2, \mu,\nu\in\mathbb{N}.$$

We then apply (8.9) to conclude.

Second, since A is compactly supported in the variable H, for sufficiently large E it is clear that we may divide $A(r,u,E,j,t,H) - A(r,u,\sqrt{2H},j,t,H)$ by E^2-2H:

$$\begin{aligned}&A(r,u,E,j,t,H) - A(r,u,\sqrt{2H},j,t,H)\\ &\quad = \left(A(r,u,E,j,t,H) - A(r,u,\sqrt{2H},j,t,H)\right)(E^2-2H)^{-1}(E^2-2H).\end{aligned}$$

The conclusion of the lemma is obtained for large E by an argument similar to those developed in the proof of Proposition 8.4 (we now need a dyadic decomposition only in the variable E, since $A(r, u, E, j, t, H) - A(r, u, \sqrt{2H}, j, t, H)$ is compactly supported in H but not in E). $\square$

Appendix E: Commutators

The goal of this section is to calculate explicitly (in terms of P) the expression of the commutator $[\Delta, \widetilde{\mathcal{A}}_{t,h^2 D_t}(P)]$, where Δ is the laplacian on $\mathbb{R}^2$. This could, in principle, be done by brutal calculation, using the expression of the laplacian in polar coordinates ($\Delta_{r,u} = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial u^2}$). But this is too cumbersome and we try a less frontal approach. We want to use the fact that $[\Delta, \mathcal{A}_{t,h^2 D_t}(P)]$ is known (from the exact Egorov theorem, Eq. (12.1) below) and to see how the calculus is modified when we replace $\mathcal{A}_{t,h^2 D_t}(P)$ by $\widetilde{\mathcal{A}}_{t,h^2 D_t}(P)$.

Recall from Lemma 3.1 and formula (2.10) that we have the exact formula (without remainder term)

$$\left[-\frac{ih\Delta}{2}, \mathscr{U}^* \operatorname{Op}_h(P)\mathscr{U}\right] = \mathscr{U}^* \operatorname{Op}_h\left(E\partial_1 P - \frac{ih}{2}\partial_1^2 P\right)\mathscr{U}, \tag{12.1}$$

where Δ is the Laplacian on $\mathbb{R}^2$.

When doing this commutator analysis, the time variables are completely transparent and (t, H) are frozen parameters. In particular, in the following, Op_h denotes the quantization with respect to space variables only.

E.1 Formal calculation of $[\Delta, \mathcal{A}_{t,H}(P)]$

We use the expression of ∇ in polar coordinates: $\nabla = (\partial_r, r^{-1}\partial_u)$ in the orthonormal frame (e_r, e_u). We also use the formula $\Delta(fg) = f\Delta g + 2\nabla f \cdot \nabla g + g\Delta f$. We obtain the following expression of $[\Delta, I_{t,H}(P)]$ applied to e_ξ at $(x, y) = (-r\sin u, r\cos u)$:

$$\begin{aligned}
[\Delta, I_{t,H}(P)]e_\xi(x, y) &= (2\pi h)^{-1}\int \Delta_{r,u}A(r, u, E, j)e^{\frac{ij(\theta-\theta_0)}{h}}e^{i\frac{Er\cos(\theta-u)}{h}}d\theta dj\\
&+\frac{2i}{h}(2\pi h)^{-1}\int \partial_r A(r, u, E, j)E\cos(\theta-u)e^{\frac{ij(\theta-\theta_0)}{h}}e^{i\frac{Er\cos(\theta-u)}{h}}d\theta dj\\
&+\frac{2i}{h}(2\pi h)^{-1}\int r^{-2}\partial_u A(r, u, E, j)je^{\frac{ij(\theta-\theta_0)}{h}}e^{i\frac{Er\cos(\theta-u)}{h}}d\theta dj
\end{aligned} \tag{12.2}$$

Note that the details of the calculations are actually not important, we only need to know "what the calculations look like" at a formal level (in particular, small errors of calculation are harmless).

Similarly, $[\Delta, II_{t,H}(P)]$ has the expression

$$\begin{aligned}
&[\Delta, II_{t,H}(P)]e_\xi(x,y)\\
&= (2\pi h)^{-1}\int \Delta_{r,u}B(r,u,E,j)E\cos(\theta-u)e^{\frac{ij(\theta-\theta_0)}{h}}e^{i\frac{Er\cos(\theta-u)}{h}}d\theta dj\\
&\quad+\frac{2i}{h}(2\pi h)^{-1}\int \partial_r B(r,u,E,j)(E\cos(\theta-u))^2e^{\frac{ij(\theta-\theta_0)}{h}}e^{i\frac{Er\cos(\theta-u)}{h}}d\theta dj\\
&\quad+\frac{2i}{h}(2\pi h)^{-1}\int r^{-2}\partial_u B(r,u,E,j)jE\cos(\theta-u)e^{\frac{ij(\theta-\theta_0)}{h}}e^{i\frac{Er\cos(\theta-u)}{h}}d\theta dj\\
&= (2\pi h)^{-1}\int \Delta_{r,u}B(r,u,E,j)E\cos(\theta-u)e^{\frac{ij(\theta-\theta_0)}{h}}e^{i\frac{Er\cos(\theta-u)}{h}}d\theta dj\\
&\quad+\frac{2i}{h}(2\pi h)^{-1}\int \partial_r B(r,u,E,j)\\
&\quad\left[(E\cos(\theta_1))^2+ih\frac{\cos\theta_1}{(Er)^2}\right]e^{\frac{ij(\theta-\theta_0)}{h}}e^{i\frac{Er\cos(\theta-u)}{h}}d\theta dj\\
&\quad+\frac{2i}{h}(2\pi h)^{-1}\int r^{-2}\partial_u B(r,u,E,j)j\\
&\quad E\cos(\theta-u)e^{\frac{ij(\theta-\theta_0)}{h}}e^{i\frac{Er\cos(\theta-u)}{h}}d\theta dj+O(h^2)
\end{aligned}\tag{12.3}$$

Similar calculations can be done for $[\Delta, III_{t,H}(P)]$ and $[\Delta, IV_{t,H}(P)]$. We do not need the explicit expressions, but need only to note that it gives a final expression modulo $O(h^2)$ of $[-ih\Delta/2, \mathcal{A}_{t,H}(P)]$ applied to e_ξ at $(x,y)=(-r\sin u, r\cos u)$ in the form:

$$\begin{aligned}
&[-ih\Delta/2, \mathcal{A}_{t,H}(P)]e_\xi(x,y)=\frac{1}{2\pi h}\int K(r,u,E,j)e^{\frac{ij(u-\theta)}{h}}e^{i\frac{Er\cos(\theta-\theta_0)}{h}}d\theta dj\\
&\quad+\frac{1}{2\pi h}\int L(r,u,E,j)E\cos(\theta-\theta_0)e^{\frac{ij(u-\theta)}{h}}e^{i\frac{Er\cos(\theta-\theta_0)}{h}}d\theta dj\\
&\quad+\frac{ih}{2\pi h}\int M(r,u,E,j)e^{\frac{ij(u-\theta)}{h}}e^{i\frac{Er\cos(\theta-\theta_0)}{h}}d\theta dj\\
&\quad+\frac{ih}{2\pi h}\int N(r,u,E,j)E\cos(\theta-\theta_0)e^{\frac{ij(u-\theta)}{h}}e^{i\frac{Er\cos(\theta-\theta_0)}{h}}d\theta dj\\
&\quad+\frac{1}{2\pi h}\int \partial_r B(r,u,E,j)\left[(E\cos(\theta_1))^2+ih\frac{\cos\theta_1}{(Er)^2}\right]e^{\frac{ij(\theta-\theta_0)}{h}}e^{i\frac{Er\cos(\theta-u)}{h}}d\theta dj\\
&\quad+\frac{1}{2\pi h}\int ih\partial_r D(r,u,E,j)(E\cos(\theta_1))^2e^{\frac{ij(\theta-\theta_0)}{h}}e^{i\frac{Er\cos(\theta-u)}{h}}d\theta dj.
\end{aligned}$$

Note that the last two lines may obviously be incorporated into the previous terms; but we shall see later why it is convenient to keep them separate.

The functions K, L, M, N are partial differential operators applied to A, B, C, D, and could in principle be expressed explicitly in terms of P, but we actually do not need these expressions.

E.2 Identification

We know from (12.1) that

$$\left[-\frac{ih\Delta}{2}, \mathscr{U}^* \operatorname{Op}_h(P)\mathscr{U}\right] = \mathscr{U}^* \operatorname{Op}_h\left(E\partial_1 P - \frac{ih}{2}\partial_1^2 P\right)\mathscr{U}$$
$$= \mathcal{A}_{t,H}\left(E\partial_1 P - \frac{ih}{2}\partial_1^2 P\right) + O(h^2).$$

Using the identification Lemma 12.1 below, this leads directly to the identifications:

$$K(r,u,E,j) + \partial_r B(r,u,E,j)(E\cos(\theta_1))^2 = A_{E\partial_1 P}$$
$$L(r,u,E,j) = B_{E\partial_1 P}$$
$$M(r,u,E,j) + \partial_r B(r,u,E,j)\frac{\cos\theta_1}{(Er)^2} + \partial_r D(r,u,E,j)(E\cos(\theta_1))^2$$
$$= C_{E\partial_1 P} - \frac{1}{2}A_{\partial_1^2 P}$$
$$N(r,u,E,j) = D_{E\partial_1 P} - \frac{1}{2}B_{\partial_1^2 P}$$

where $\theta_1 = \theta_1(r,E,j)$ denotes as before the solution in $[-\pi/2, \pi/2)$ of $\sin\theta_1 = j/Er$. On the right-hand sides, notation such as $A_{E\partial_1 P}$, $B_{E\partial_1 P}$ etc. means "the functions A, B etc. associated to $E\partial_1 P$ by the formulas of Definition 10.6".

To justify these identifications we are using the following:

Lemma 12.1 *Let A and B be two* smooth *real-valued functions. Then the values of*

$$\frac{1}{2\pi h}\int A(r,u,E,j)e^{\frac{ij(u-\theta)}{h}}e^{i\frac{Er\cos(\theta-\theta_0)}{h}}d\theta dj$$
$$+\frac{1}{2\pi h}\int B(r,u,E,j)\cos(\theta-\theta_0)e^{\frac{ij(u-\theta)}{h}}e^{i\frac{Er\cos(\theta-\theta_0)}{h}}d\theta dj \quad (12.4)$$

for all r, u, θ_0, E determine A and B uniquely.

Proof Integrating (12.4) along $e^{in\theta_0}d\theta_0$ ($\theta_0 \in [0, 2\pi]$, n an arbitrary integer) yields the value

$$\int A(r, u, E, nh)e^{in(u-\theta)}e^{i\frac{Er\cos(\theta-\theta_0)}{h}}d\theta \\ + \int B(r, u, E, nh)\cos(\theta-\theta_0)e^{in(u-\theta)}e^{i\frac{Er\cos(\theta-\theta_0)}{h}}d\theta. \quad (12.5)$$

If we take $n = n(h)$ a family of even integers growing like $1/h$, application of the method of stationary phase yields that this is (up to $O(h)$)

$$2e^{inu}(2\pi h)^{1/2}[\sin^{1/2}\theta_1\, A(r, u, E, hn(h))\cos(-n\theta_1 + Erh^{-1}\cos\theta_1 + \pi/4) \\ + iB(r, u, E, hn(h))\sin(-n\theta_1 + Erh^{-1}\cos\theta_1 + \pi/4)] \quad (12.6)$$

where θ_1 is the solution in $[-\pi/2, \pi/2)$ of $\sin\theta_1 = \frac{hn(h)}{Er}$. If A and B are continuous and real-valued then (12.6) suffices to determine A and B. □

E.3 Formal calculation of $[\Delta, \tilde{\mathcal{A}}_{t,H}(P)]$

We want to use the previous identities to find the formal expression of $[\Delta, \widetilde{\mathcal{A}}_{t,H}(P)]$. Remember that $\widetilde{\mathcal{A}}_{t,H}(P)$ is the operator we want to use in all our proofs, because it comes naturally into a "tangential" part and a "radial" part of degree 1.

If we compare the formal calculations leading to the expressions of $[\Delta, \mathcal{A}_{t,H}(P)]$ and $[\Delta, \widetilde{\mathcal{A}}_{t,H}(P)]$, we see that they are identical and thus $[-ih\Delta/2, \widetilde{\mathcal{A}}_{t,H}(P)]$ applied to e_ξ at $(-r\sin u, r\cos u)$ has the form

$$\frac{1}{2\pi h}\int K(r, u, \sqrt{2H}, j)e^{\frac{ij(u-\theta)}{h}}e^{i\frac{Er\cos(\theta-\theta_0)}{h}}d\theta dj \\ + \frac{1}{2\pi h}\int L(r, u, \sqrt{2H}, j)E\cos(\theta-\theta_0)e^{\frac{ij(u-\theta)}{h}}e^{i\frac{Er\cos(\theta-\theta_0)}{h}}d\theta dj \\ + \frac{ih}{2\pi h}\int M(r, u, \sqrt{2H}, j)e^{\frac{ij(u-\theta)}{h}}e^{i\frac{Er\cos(\theta-\theta_0)}{h}}d\theta dj \\ + \frac{ih}{2\pi h}\int N(r, u, \sqrt{2H}, j)E\cos(\theta-\theta_0)e^{\frac{ij(u-\theta)}{h}}e^{i\frac{Er\cos(\theta-\theta_0)}{h}}d\theta dj \\ + (2\pi h)^{-1}\int \partial_r B(r, u, \sqrt{2H}, j) \\ \times\left[(E\cos(\theta_1))^2 + ih\frac{\cos\theta_1}{(Er)^2}\right]e^{\frac{ij(\theta-\theta_0)}{h}}e^{i\frac{Er\cos(\theta-u)}{h}}d\theta dj \\ + (2\pi h)^{-1}\int ih\partial_r D(r, u, \sqrt{2H}, j)(E\cos(\theta_1))^2 e^{\frac{ij(\theta-\theta_0)}{h}}e^{i\frac{Er\cos(\theta-u)}{h}}d\theta dj$$

Note that $\theta_1 = \theta_1(r, E, j)$ and that the symbol in the last two lines still depends on E (this is why we treat it separately). Everywhere else in the symbol, E has been replaced by $\sqrt{2H}$. Note also that $(E\cos(\theta_1))^2 = E^2 - \frac{j^2}{r^2}$.

From this and from the identifications of Sect. 5.2, we deduce the following final formula.

Proposition 12.2 *There exists a function $R(r, u, E, \sqrt{2H}, j)$ such that*

$$\begin{aligned}
[-ih\Delta/2, \widetilde{\mathcal{A}}_{t,h^2D_t}(P)] &= \widetilde{\mathcal{A}}_{t,h^2D_t}(E\partial_1 P) - \frac{ih}{2}\widetilde{\mathcal{A}}_{t,h^2D_t}(\partial_1^2 P) + O(h^2)\\
&+\partial_r B(r, u, h\sqrt{2D_t}, hD_u) \circ (-h^2\Delta - 2h^2 D_t)\\
&+ihR(r, u, \sqrt{-h^2\Delta}, h\sqrt{2D_t}, hD_u) \circ (-h^2\Delta - 2h^2 D_t),
\end{aligned} \tag{12.7}$$

where B is the function given from P by Definition 10.6.

Proof Indeed, the identifications of Sect. 5.2 yield

$$\begin{aligned}
[-ih\Delta/2, \widetilde{\mathcal{A}}_{t,h^2D_t}(P)] &= \widetilde{I}_{t,h^2D_t}(E\partial_1 P) + \widetilde{II}_{t,h^2D_t}(E\partial_1 P)\\
&+ih\left(\widetilde{III}_{t,h^2D_t}(E\partial_1 P) - 1/2\widetilde{I}_{t,h^2D_t}(\partial_1^2 P)\right)\\
&+ih\left(\widetilde{IV}_{t,h^2D_t}(E\partial_1 P) - 1/2\widetilde{II}_{t,h^2D_t}(\partial_1^2 P)\right)\\
&+\partial_r B(r, u, h\sqrt{2D_t}, hD_u) \circ (-h^2\Delta - 2h^2 D_t)\\
&+ihR(r, u, \sqrt{-h^2\Delta}, h\sqrt{2D_t}, hD_u) \circ (-h^2\Delta - 2h^2 D_t)
\end{aligned} \tag{12.8}$$

where the function R is defined by the identity

$$\begin{aligned}
&R(r, u, E, \sqrt{2H}, j)(E^2 - 2H)\\
&\quad= \partial_r B(r, u, \sqrt{2H}, j)\left[\frac{\cos\theta_1(r, E, j)}{(Er)^2} - \frac{\cos\theta_1(r, \sqrt{2H}, j)}{2Hr^2}\right].
\end{aligned}$$

Indeed, we can apply a simple division lemma (actually the Taylor integral formula) to write

$$\frac{\cos\theta_1(r, E, j)}{(Er)^2} - \frac{\cos\theta_1(r, \sqrt{2H}, j)}{2Hr^2} = S(r, u, E, j, \sqrt{2H})(E^2 - 2H),$$

and thus

$$R(r, u, E, \sqrt{2H}, j) = \partial_r B(r, u, \sqrt{2H}, j) S(r, u, E, j, \sqrt{2H}).$$

□

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