Available online at www.sciencedirect.com

SciVerse ScienceDirect

J. Math. Pures Appl. 99 (2013) 544–576

JOURNAL DE MATHÉMATIQUES PURES ET APPLIQUÉES

www.elsevier.com/locate/matpur

Indirect controllability of locally coupled wave-type systems and applications

Fatiha Alabau-Boussouira [a], Matthieu Léautaud [b,c,d,*]

[a] *LMAM, Université de Lorraine and CNRS (UMR 7122), 57045 Metz Cedex 1, France*
[b] *Université Pierre et Marie Curie Paris 6, UMR 7598, Laboratoire Jacques-Louis Lions, Paris, F-75005, France*
[c] *CNRS, UMR 7598 LJLL, Paris, F-75005, France*
[d] *Laboratoire POEMS, Inria Paris-Rocquencourt/ENSTA, CNRS UMR 2706, France*

Received 26 July 2011

Available online 6 September 2012

Abstract

We consider symmetric systems of two wave-type equations only one of them being controlled. The two equations are coupled by zero order terms, localized in part of the domain. We prove an internal and a boundary controllability result in any space dimension, provided that both the coupling and the control regions satisfy the Geometric Control Condition. We deduce similar null-controllability results in any positive time for parabolic systems and Schrödinger-type systems under the same geometric conditions on the coupling and the control regions. This includes several examples in which these two regions have an empty intersection.
© 2012 Elsevier Masson SAS. All rights reserved.

Résumé

On s'intéresse au problème de contrôlabilité exacte pour des systèmes symétriques de deux équations d'ondes, dont une seulement est contrôlée. Les équations sont couplées par des termes d'ordre zéro, localisés dans une partie du domaine. On montre des résultats de contrôlabilité interne et frontière en toute dimension d'espace dès que les zones de couplage et de contrôle satisfont toutes deux la condition de contrôle géométrique. On en déduit des résultats de contrôlabilité à zéro en temps arbitrairement petit pour des systèmes d'équations de la chaleur ou de Schrödinger, sous les mêmes conditions géométriques sur les zones de couplage et de contrôle. Ces résultats incluent de nombreux exemples pour lesquels les deux zones sont disjointes.
© 2012 Elsevier Masson SAS. All rights reserved.

MSC: 35L10; 35L51; 35L90; 35Q93; 93B05; 93B07; 93C05; 93C20

Keywords: Observability; Controllability; Wave equation; Hyperbolic systems; Parabolic systems; Geometric conditions

* Corresponding author at: Université Pierre et Marie Curie Paris 6, UMR 7598, Laboratoire Jacques-Louis Lions, Paris, F-75005, France.
E-mail addresses: fatiha.alabau@univ-lorraine.fr (F. Alabau-Boussouira), leautaud@ann.jussieu.fr (M. Léautaud).

0021-7824/$ – see front matter © 2012 Elsevier Masson SAS. All rights reserved.
http://dx.doi.org/10.1016/j.matpur.2012.09.012

1. Introduction

1.1. Motivation

During the last decade, the controllability properties of coupled parabolic equations like

$$\begin{cases} \partial_t u_1 - \Delta_c u_1 + a u_1 + p u_2 = b f & \text{in } (0,T)\times\Omega,\\ \partial_t u_2 - \Delta_c u_2 + a u_2 + p u_1 = 0 & \text{in } (0,T)\times\Omega,\\ u_1 = u_2 = 0 & \text{on } (0,T)\times\partial\Omega,\\ (u_1,u_2)|_{t=0} = \left(u_1^0,u_2^0\right) & \text{in } \Omega, \end{cases} \tag{1}$$

have been intensively studied. Here, the coefficients p and b are smooth non-negative functions on the bounded domain Ω and $-\Delta_c$ is a second order selfadjoint elliptic operator. The null-controllability problem under view is the following: given a time $T>0$, is it possible to find, for any initial data (u_1^0,u_2^0), a control function f so that the associated solution (u_1,u_2) of (1) is driven to zero in time T? It has been proved in [14,7,16,20] with different methods that System (1) is null-controllable as soon as $\{p>0\}\cap\{b>0\}\neq\emptyset$. In these works, the case $\{p>0\}\cap\{b>0\}=\emptyset$ has been left as an open problem. However, Kavian and de Teresa [19] have proved for a cascade system (i.e. without the term pu_2 in the first equation of (1)) that approximate controllability holds. The natural question is then whether or not null-controllability (which is a stronger property) still holds in the case $\{p>0\}\cap\{b>0\}=\emptyset$:

Question 1. Is System (1) null-controllable in the case $\{p>0\}\cap\{b>0\}=\emptyset$?

The second problem under interest here is the boundary controllability of systems like

$$\begin{cases} \partial_t u_1 - \Delta_c u_1 + a u_1 + p u_2 = 0 & \text{in } (0,T)\times\Omega,\\ \partial_t u_2 - \Delta_c u_2 + a u_2 + p u_1 = 0 & \text{in } (0,T)\times\Omega,\\ u_1 = b_\partial f,\ u_2 = 0 & \text{on } (0,T)\times\partial\Omega,\\ (u_1,u_2)|_{t=0} = \left(u_1^0,u_2^0\right) & \text{in } \Omega, \end{cases} \tag{2}$$

where b_∂ is a smooth function on $\partial\Omega$. The recent work [15] studies such systems in one space dimension and with constant coupling coefficients. The cases of higher space dimensions and varying coupling coefficients (and in particular when the coefficients vanish in a neighborhood of the boundary) are to our knowledge completely open. The second question under interest is then:

Question 2. Is System (2) null-controllable for non-constant coupling coefficients p? Is System (2) null-controllable if the dimension of Ω is larger than one?

Concerning these two open problems, it seems that the parabolic theory and associated tools encounter for the moment some essential difficulties.

On the other hand, it is known from [29] that controllability properties can be transferred from hyperbolic equations to parabolic ones. And it seems (at least for boundary control problems) that the theory for coupled hyperbolic equations of the type

$$\begin{cases} \partial_t^2 u_1 - \Delta_c u_1 + a u_1 + p u_2 = b f & \text{in } (0,T)\times\Omega,\\ \partial_t^2 u_2 - \Delta_c u_2 + a u_2 + p u_1 = 0 & \text{in } (0,T)\times\Omega,\\ u_1 = u_2 = 0 & \text{on } (0,T)\times\partial\Omega,\\ (u_1,u_2,\partial_t u_1,\partial_t u_2)|_{t=0} = \left(u_1^0,u_2^0,u_1^1,u_2^1\right) & \text{in } \Omega, \end{cases} \tag{3}$$

and

$$\begin{cases} \partial_t^2 u_1 - \Delta_c u_1 + a u_1 + p u_2 = 0 & \text{in } (0,T)\times\Omega,\\ \partial_t^2 u_2 - \Delta_c u_2 + a u_2 + p u_1 = 0 & \text{in } (0,T)\times\Omega,\\ u_1 = b_\partial f,\quad u_2 = 0 & \text{on } (0,T)\times\partial\Omega,\\ (u_1,u_2,\partial_t u_1,\partial_t u_2)|_{t=0} = \left(u_1^0,u_2^0,u_1^1,u_2^1\right) & \text{in } \Omega, \end{cases} \tag{4}$$

is better understood (see [1,4]), even less studied. In the case of varying coefficients and several space-dimensions, the associated stabilization problem is addressed in [2,3,6]. In particular in [6], a polynomial stability result is proved for solutions of the system

$$\begin{cases} \partial_t^2 u_1 - \Delta_c u_1 + a u_1 + p u_2 + b \partial_t u_1 = 0 & \text{in } \mathbb{R}^+ \times \Omega, \\ \partial_t^2 u_2 - \Delta_c u_2 + a u_2 + p u_1 = 0 & \text{in } \mathbb{R}^+ \times \Omega, \\ u_1 = u_2 = 0 & \text{on } \mathbb{R}^+ \times \partial\Omega, \end{cases} \tag{5}$$

in some cases where $\{p > 0\} \cap \{b > 0\} = \emptyset$. This gives hope to prove some null-controllability results for (3) in the same situations.

In the present work, we answer Questions 1 and 2 for hyperbolic problems (like (3) and (4)), improving the results of [4,6]. Then, we deduce a (partial) solution to the two open questions raised above for parabolic systems. Indeed, we prove that Systems (1)–(3) are null-controllable (in appropriate spaces) as soon as $\{p > 0\}$ and $\{b > 0\}$ both satisfy the Geometric Control Condition (recalled in Definition 1.1 below) and $\|p\|_{L^\infty(\Omega)}$ satisfies a smallness assumption. This contains several examples with $\{p > 0\} \cap \{b > 0\} = \emptyset$, and partially answer to the first question. We prove as well that similar controllability results hold for the boundary control problems (2) and (4), which partially answers to the second question. Of course, the geometric conditions needed here are essential (and even sharp) for coupled waves, but inappropriate for parabolic equations. However, this is a first step towards a better understanding of these types of systems. In one space dimension in particular, the geometric conditions are reduced to a non-emptiness condition and are hence optimal for parabolic systems as well.

In the end of the present introduction, we state our main results for wave/heat/Schrödinger-type Systems. In Section 2, we introduce an abstract setting adapted to second order (in time) control problems. Then, in Section 3, we present the tools used in the proof of the main theorem, together with a key lemma: an observability inequality for an equation with a right-hand side (for which we give another proof in Appendix A). Section 4 is devoted to the proof of the observability of hyperbolic systems in the abstract setting, and controllability is deduced in Section 5. Finally, in Section 6, we come back to the applications to wave/heat/Schrödinger-type Systems. The results of this paper were announced in [5].

1.2. Main results

Let Ω be a bounded domain in $\mathbb{R}^n$ with smooth (say $\mathscr{C}^\infty$) boundary (or a smooth connected compact Riemannian manifold with or without boundary) and $\Delta_c = \operatorname{div}(c\nabla)$ a (negative) elliptic operator (or the Laplace Beltrami operator with respect to the Riemannian metric) on Ω. Here, c denotes a smooth (say $\mathscr{C}^\infty$) positive symmetric matrix i.e. in particular $C_0^{-1}|\xi|^2 \leqslant c(x)\xi \cdot \xi \leqslant C_0|\xi|^2$ for some $C_0 > 0$, for all $x \in \overline{\Omega}$, $\xi \in \mathbb{R}^n$. We consider the more general first order (in time) control problem

$$\begin{cases} e^{i\theta} \partial_t u_1 - \Delta_c u_1 + a u_1 + p u_2 = b f & \text{in } (0,T) \times \Omega, \\ e^{i\theta} \partial_t u_2 - \Delta_c u_2 + a u_2 + p u_1 = 0 & \text{in } (0,T) \times \Omega, \\ u_1 = u_2 = 0 & \text{on } (0,T) \times \partial\Omega, \\ (u_1, u_2)|_{t=0} = (u_1^0, u_2^0) & \text{in } \Omega, \end{cases} \tag{6}$$

with $\theta \in [-\pi/2, \pi/2]$, including Schrödinger-type systems for $\theta = \pm\pi/2$ and diffusion-type systems for $\theta \in (-\pi/2, \pi/2)$. In particular we recover System (1) when taking (6) for $\theta = 0$.

We also consider System (3), consisting in a wave-type system, with only one control force. In these systems, $a = a(x)$, $p = p(x)$ and $b = b(x)$ are smooth real-valued functions on Ω and f is the control function, that can act on the system.

We shall also consider the same systems controlled from the boundary through the (smooth) real-valued function b_∂:

$$\begin{cases} e^{i\theta} \partial_t u_1 - \Delta_c u_1 + a u_1 + p u_2 = 0 & \text{in } (0,T) \times \Omega, \\ e^{i\theta} \partial_t u_2 - \Delta_c u_2 + a u_2 + p u_1 = 0 & \text{in } (0,T) \times \Omega, \\ u_1 = b_\partial f, \quad u_2 = 0 & \text{on } (0,T) \times \partial\Omega, \\ (u_1, u_2)|_{t=0} = (u_1^0, u_2^0) & \text{in } \Omega, \end{cases} \tag{7}$$

which includes System (2) when $\theta = 0$.

We first notice that, on the space $(L^2(\Omega))^2$ endowed with the natural inner product

$$(U,V)_{L^2(\Omega)\times L^2(\Omega)} = (u_1,v_1)_{L^2(\Omega)} + (u_2,v_2)_{L^2(\Omega)}, \quad U=(u_1,u_2),\ V=(v_1,v_2),$$

the operator

$$A_p = \begin{pmatrix} -\Delta_c + a & p \\ p & -\Delta_c + a \end{pmatrix},$$

with domain $\mathcal{D}(A_p) = (H^2(\Omega)\cap H_0^1(\Omega))^2$, is selfadjoint. As a consequence, for

$$f \in L^2\big((0,T)\times\Omega\big),$$

the Cauchy problem (6), resp. (3), is well-posed in $(L^2(\Omega))^2$, resp. $(H_0^1(\Omega))^2 \times (L^2(\Omega))^2$, in the sense of semigroup theory. Then, taking $f \in L^2((0,T)\times\partial\Omega)$ the initial–boundary value problem (7), resp. (4), is well-posed in $(H^{-1}(\Omega))^2$, resp. $(L^2(\Omega))^2 \times (H^{-1}(\Omega))^2$, in the sense of transposition solution (see [22,31]).

The strategy we adopt here is to prove some controllability results for the hyperbolic systems (3) and (4), extending the two-levels energy method introduced in [4]. Then, using transmutation techniques, we deduce controllability properties of (6) and (7).

An important remark to make before addressing the controllability problem for the hyperbolic systems (3)–(4) is concerned with the regularity of solutions. If one takes for system (3) (resp. (4)) initial data $(u_1^0,u_2^0,u_1^1,u_2^1) \in H_0^1(\Omega)\times(H^2\cap H_0^1(\Omega))\times L^2(\Omega)\times H_0^1(\Omega)$ (resp. $(u_1^0,u_2^0,u_1^1,u_2^1)\in L^2(\Omega)\times H_0^1(\Omega)\times H^{-1}(\Omega)\times L^2(\Omega)$), and a control function $f\in L^2((0,T)\times\Omega)$ (resp. $f\in L^2((0,T)\times\partial\Omega)$), then the state $(u_1,u_2,\partial_t u_1,\partial_t u_2)$ remains in the space $H_0^1(\Omega)\times(H^2\cap H_0^1(\Omega))\times L^2(\Omega)\times H_0^1(\Omega)$ (resp. $L^2(\Omega)\times H_0^1(\Omega)\times H^{-1}(\Omega)\times L^2(\Omega)$) for all time. We recall that for Systems (3) and (4), the null-controllability is equivalent to the exact controllability. As a consequence, it is not possible, taking for instance zero as initial data to reach any target state in $(H_0^1(\Omega))^2\times(L^2(\Omega))^2$ (resp. $(L^2(\Omega))^2\times(H^{-1}(\Omega))^2$). The controllability question for (3)–(4) hence becomes: starting from rest at time $t=0$, is it possible to reach all $H_0^1(\Omega)\times(H^2\cap H_0^1(\Omega))\times L^2(\Omega)\times H_0^1(\Omega)$ (resp. $L^2(\Omega)\times H_0^1(\Omega)\times H^{-1}(\Omega)\times L^2(\Omega)$) in time $t=T$ sufficiently large?

To state our results, we shall need the classical Geometric Control Conditions GCC (resp. GCC_∂). We recall that GCC was introduced in [26] for manifolds without boundaries, in [8] for domains with boundaries and that GCC_∂ was introduced in [9]. From these works, it is known that, making the generic assumption that $\partial\Omega$ has no contact of infinite order with its tangents, GCC (resp. GCC_∂) is a necessary and sufficient condition for the internal (resp. boundary) observability and controllability of one wave equation (see also [11]).

Definition 1.1 *(GCC (resp.* GCC_∂*)).* Let $\omega\subset\Omega$ (resp. $\Gamma\subset\partial\Omega$) and $T>0$. We say that the couple (ω,T) satisfies GCC (resp. (Γ,T) satisfies GCC_∂) if every generalized geodesic (i.e. ray of geometric optics) traveling at speed one in Ω meets ω (resp. meets Γ on a non-diffractive point) in a time $t<T$. We say that ω satisfies GCC (resp. Γ satisfies GCC_∂) if there exists $T>0$ such that (ω,T) satisfies GCC (resp. (Γ,T) satisfies GCC_∂).

We shall make the following key assumptions:

Assumption 1.2.

(i) We have $((-\Delta_c+a)u,u)_{L^2(\Omega)} \geqslant \lambda_0\|u\|^2_{L^2(\Omega)}$, for some $\lambda_0>0$, for all $u\in(H^2(\Omega)\cap H_0^1(\Omega))$.
(ii) We have $p\geqslant 0$ on Ω, $\{p>0\}\supset\overline{\omega}_p$ for some open subset $\omega_p\subset\Omega$ and we set $p^+ := \|p\|_{L^\infty(\Omega)}$.
(iii) We have $b\geqslant 0$ on Ω, $\{b>0\}\supset\overline{\omega}_b$ (resp. $b_\partial\geqslant 0$ on $\partial\Omega$ and $\{b_\partial>0\}\supset\overline{\Gamma}_b$) for some open subset $\omega_b\subset\Omega$ (resp. $\Gamma_b\subset\partial\Omega$).

Note that in the case where $c=\mathrm{Id}$ and $a=0$, the best constant λ_0 is the smallest eigenvalue of the Laplace operator on Ω with Dirichlet boundary conditions. We also have the identity $\lambda_0 = 1/C_{\mathcal{P}}^2$, where $C_{\mathcal{P}}$ is the Poincaré's constant of Ω.

We shall also require that the operator A_p satisfies, for some constant $C>0$,

$$\big(A_p(v_1,v_2),(v_1,v_2)\big)_{L^2(\Omega)\times L^2(\Omega)} \geqslant C\big(\|v_1\|^2_{H_0^1(\Omega)} + \|v_2\|^2_{H_0^1(\Omega)}\big),$$

for all $(v_1,v_2)\in\mathcal{D}(A_p)$. This is the case when assuming $p^+<\lambda_0$.

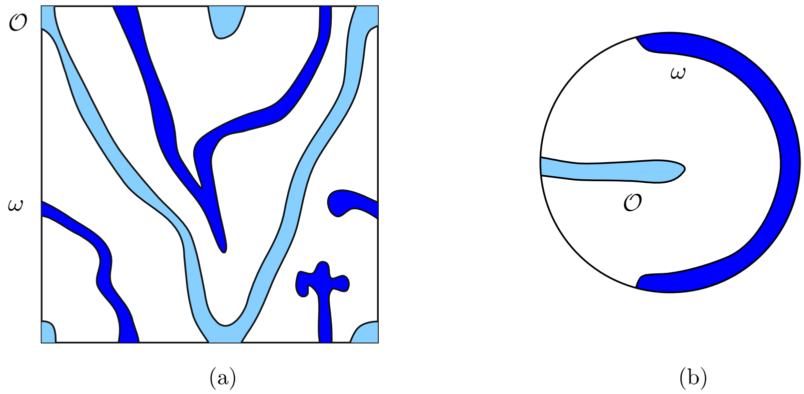

Fig. 1. Examples of open sets $(\Omega, \omega, \mathcal{O})$ such that ω and $\mathcal{O}$ both satisfy GCC in Ω, with $\omega \cap \mathcal{O} = \emptyset$: in the case (a), Ω is the flat torus (or the square), in the case (b), Ω is the disk.

In the case of coupled wave equations, our main result can be formulated as follows.

Theorem 1.3 *(Wave-type systems). Suppose that* (i) *holds, that ω_p satisfies GCC and that ω_b* (*resp.* Γ_b) *satisfies GCC* (*resp.* GCC_∂)*. Then, for all b* (*resp.* b_∂) *satisfying* (iii)*, there exists a constant $p_* > 0$* (*depending only on the geometry of Ω and on b, resp. b_∂*) *such that for all $p^+ < p_*$, there exists a time $T_* > 0$ such that for all $T > T_*$, all p satisfying* (ii)*, and all initial data $(u_1^0, u_2^0, u_1^1, u_2^1) \in H_0^1(\Omega) \times (H^2 \cap H_0^1(\Omega)) \times L^2(\Omega) \times H_0^1(\Omega)$* (*resp.* $(u_1^0, u_2^0, u_1^1, u_2^1) \in L^2(\Omega) \times H_0^1(\Omega) \times H^{-1}(\Omega) \times L^2(\Omega)$)*, there exists a control function $f \in L^2((0,T) \times \Omega)$* (*resp.* $f \in L^2((0,T) \times \partial\Omega)$) *such that the solution of* (3) (*resp.* (4)) *satisfies $(u_1, u_2, \partial_t u_1, \partial_t u_2)|_{t=T} = 0$.*

Another way to formulate this result is to say that, under the assumptions of Theorem 1.3, the reachable set at time $T > T_*$ with zero initial data is exactly $H_0^1(\Omega) \times (H^2 \cap H_0^1(\Omega)) \times L^2(\Omega) \times H_0^1(\Omega)$ in the case of L^2 internal control and $L^2(\Omega) \times H_0^1(\Omega) \times H^{-1}(\Omega) \times L^2(\Omega)$ in the case of L^2 boundary control.

Some comments should be made about this result. First this is a generalization of the work [4] where the coupling coefficients considered have to satisfy $p(x) \geqslant C > 0$ for all $x \in \Omega$. The geometric situations covered by Theorem 1.3 are richer, and include in particular several examples of coupling and control regions that do not intersect (see Fig. 1). Second, we do not know if the coercivity assumption (i) for $-\Delta_c + a$ and the smallness assumption on p^+ are only technical and inherent to the method we use here. Moreover, the control time T_* we obtain depends upon all the parameters of the system, and not only the sets ω_p, ω_b and Γ_b (as it is the case for a single wave equation). This feature does not seem to be very natural. Finally, the fact that we consider twice the same elliptic operator Δ_c is a key point in our proof and it is likely that this result does not hold for waves with different speeds (see [4] for results with different speeds and different operators). Similarly, the fact that $p \geqslant 0$ (see Assumption (ii)) is very important here. It is possible that Theorem 1.3 does not work if the sign of p varies.

As a consequence of Theorem 1.3 and using transmutation techniques (due to [29,23] for heat-type equations and to [25,24] for Schrödinger-type equations), we can now state the associated results for Systems (6) and (7).

Corollary 1.4 *(Heat-type systems). Suppose that* (i) *holds, that ω_p satisfies GCC and that ω_b* (*resp.* Γ_b) *satisfies GCC* (*resp.* GCC_∂)*. Then, for all b* (*resp.* b_∂) *satisfying* (iii)*, there exists a constant $p_* > 0$* (*depending only on the geometry of Ω and on b, resp. b_∂*) *such that for all $p^+ < p_*$, for all $T > 0$, $\theta \in (-\pi/2, \pi/2)$, for all p satisfying* (ii)*, and all initial data $(u_1^0, u_2^0) \in (L^2(\Omega))^2$* (*resp.* $(u_1^0, u_2^0) \in (H^{-1}(\Omega))^2$)*, there exists a control function $f \in L^2((0,T) \times \Omega)$* (*resp.* $f \in L^2((0,T) \times \partial\Omega)$) *such that the solution of* (6) (*resp.* (7)) *satisfies $(u_1, u_2)|_{t=T} = 0$.*

To our knowledge, this corollary gives the first controllability result for coupled parabolic symmetric equations when the coupling region ω_p and the control region ω_b do not intersect. Moreover, this seems to be also the first positive result for boundary control of parabolic symmetric systems in several space dimensions or with variable coupling coefficients. A one-dimensional internal controllability result has also been obtained recently in [28] with a different method [13] for cascade systems of two parabolic equations (i.e. without the term pu_2 in the first equation of (1)).

Note that the spaces for which the controllability result of Corollary 1.4 holds are symmetric. This is due to the smoothing effect of parabolic equations. The proof of this result is given in Section 6.2.

The geometric conditions in this theorem are not sharp for parabolic equations. This leads us to think that the same result still holds under the only conditions $\omega_p \neq \emptyset$ (i.e. p is not the null function) and $\omega_b \neq \emptyset$ (i.e. b is not the null function). This remains an open problem. Note that our control result holds for any $T > 0$, which is natural for parabolic equations.

Concerning coupled Schrödinger equations, we have the following result.

Corollary 1.5 *(Schrödinger-type systems). Suppose that* (i) *holds, that ω_p satisfies GCC and that ω_b (resp. Γ_b) satisfies GCC (resp.* GCC$_\partial$*). Then, for all b (resp. b_∂) satisfying* (iii)*, there exists a constant $p_* > 0$ (depending only on the geometry of Ω and on b, resp. b_∂) such that for all $p^+ < p_*$, for all $T > 0$, all p satisfying* (iii) *and all initial data $(u_1^0, u_2^0) \in L^2(\Omega) \times H_0^1(\Omega)$ (resp. $(u_1^0, u_2^0) \in H^{-1}(\Omega) \times L^2(\Omega)$), there exists a control function $f \in L^2((0,T) \times \Omega)$ (resp. $f \in L^2((0,T) \times \partial\Omega)$) such that the solution of* (6) *(resp.* (7)*) with $\theta = \pm\pi/2$ satisfies $(u_1, u_2)|_{t=T} = 0$.*

The proof of this result is given in Section 6.3. Since there is no smoothing effect in this case, we still obtain a controllability result in asymmetric spaces here: the uncontrolled variable u_2 has to be more regular than the other one. This shows that the attainable set from zero for an L^2 internal control (resp. L^2 boundary control) contains $L^2(\Omega) \times H_0^1(\Omega)$ (resp. $H^{-1}(\Omega) \times L^2(\Omega)$). Whether or not a general target in $(L^2(\Omega))^2$ (resp. $(H^{-1}(\Omega))^2$) is reachable for (6) (resp. (7)) with $\theta = \pm\pi/2$ remains open.

Note finally that the geometric conditions GCC and GCC$_\partial$ are not necessary in the case of Schrödinger equations but are not far from being optimal. The most general control result for a single Schrödinger equation [21] is that GCC (resp. GCC$_\partial$) implies null-controllability in any positive time. However, in some cases (see [18,12]), these conditions are not necessary. Here, we do not recover these properties (since our result is deduced from a controllability result for waves). It would be interesting to prove such a result with weaker geometric conditions in particular situations (see [28] in the case of a square). Remark that controllability results have also been obtained in [28] with a different method based on [13] in the periodic case for cascade systems of two Schrödinger equations. Note also that our control result holds for any $T > 0$, which is natural for Schrödinger equations [21] (and which is not the case in the results of [28]).

Remark 1.6. Different boundary conditions (like Neumann or Fourier boundary conditions) can also be addressed with the same techniques since we use the observability inequality for a single wave equation as a black box. In the work [9], the authors prove this observability inequality with all these boundary conditions (all compatible with the Melrose–Sjöstrand theorem of propagation of singularities).

Remark 1.7. Similarly, our results also hold (in the case of Dirichlet boundary conditions) for a boundary $\partial\Omega$ and a metric c having a limited smoothness (i.e. $\mathscr{C}^3$ for $\partial\Omega$ and $\mathscr{C}^2$ for c) according to the article [10]. Note as well that we supposed that the coefficients a, p and b are smooth. It is sufficient that a and p preserve the regularity of u_1 and u_2 and b (resp. b_∂) that of the control function f. For instance, one can take $b \in L^\infty(\Omega)$ (resp. $b_\partial \in L^\infty(\partial\Omega)$) and $a, p \in W^{2,\infty}(\Omega)$.

Remark 1.8. We could also replace the operator A_p in all the systems studied here, by the operator

$$\begin{pmatrix} -\Delta_c + a & \delta p \\ p & -\Delta_c + a \end{pmatrix},$$

for $\delta > 0$. This is what we did in [5]. This operator is selfadjoint on $(L^2(\Omega))^2$ endowed with the inner product $(U, V)_\delta = (u_1, v_1)_{L^2(\Omega)} + \delta(u_2, v_2)_{L^2(\Omega)}$. The controllability results obtained in this case (for all equations, as well as in the abstract setting) hold for all (δ, p^+) such that $\sqrt{\delta} p^+ < p_*$. Such results in this setting seem more general since they allow to consider large p^+ or large δ (provided that the other is small enough). For these choices of δ, the systems obtained are "less symmetric" than the ones for $\delta = 1$.

However, we can pass from the system with δ to the system without δ with a change of variables. Suppose that (z_1, z_2) is the solution of the system

$$\begin{cases} \partial_t^2 z_1 - \Delta_c z_1 + a z_1 + \delta p z_2 = bf & \text{in } (0,T)\times\Omega, \\ \partial_t^2 z_2 - \Delta_c z_2 + a z_2 + p z_1 = 0 & \text{in } (0,T)\times\Omega, \\ z_1 = z_2 = 0 & \text{on } (0,T)\times\partial\Omega, \\ (z_1, z_2, \partial_t z_1, \partial_t z_2)|_{t=0} = (z_1^0, z_2^0, z_1^1, z_2^1) & \text{in } \Omega, \end{cases}$$

then, setting $u_1 = z_1$ and $u_2 = \sqrt{\delta} z_2$, the new variable (u_1, u_2) satisfies the fully symmetric system

$$\begin{cases} \partial_t^2 u_1 - \Delta_c u_1 + a u_1 + \sqrt{\delta} p u_2 = bf & \text{in } (0,T)\times\Omega, \\ \partial_t^2 u_2 - \Delta_c u_2 + a u_2 + \sqrt{\delta} p u_1 = 0 & \text{in } (0,T)\times\Omega, \\ u_1 = u_2 = 0 & \text{on } (0,T)\times\partial\Omega, \\ (u_1, u_2, \partial_t u_1, \partial_t u_2)|_{t=0} = (z_1^0, \sqrt{\delta} z_2^0, z_1^1, \sqrt{\delta} z_2^1) & \text{in } \Omega, \end{cases}$$

to which Theorem 1.3 applies (for $\sqrt{\delta} p^+ < p_*$).

2. Abstract setting

In this section, we describe the abstract setting in which we prove Theorem 1.3 for Systems (3) and (4), and define the appropriate spaces and operators. Let H be a Hilbert space and $(A, \mathcal{D}(A))$ a selfadjoint positive operator on H with compact resolvent. We denote by $(\cdot,\cdot)_H$ the inner product on H and $\|\cdot\|_H$ the associated norm. For $k \in \mathbb{N}$, we set $H_k = \mathcal{D}(A^{\frac{k}{2}})$ endowed with the inner product $(\cdot,\cdot)_{H_k} = (A^{\frac{k}{2}}\cdot, A^{\frac{k}{2}}\cdot)_H$ and associated norm $\|\cdot\|_{H_k} = \|A^{\frac{k}{2}}\cdot\|_H$. We define H_{-k} as the dual space of H_k with respect to the pivot space $H = H_0$. We write $\langle\cdot,\cdot\rangle_{H_k, H_{-k}} = (A^{\frac{k}{2}}\cdot, A^{-\frac{k}{2}}\cdot)_H$ the duality product between H_k and H_{-k}, and $\|\cdot\|_{H_{-k}} = \|A^{-\frac{k}{2}}\cdot\|_H$ is the norm on H_{-k}. The operator A can be extended to an isomorphism from H_k to H_{k-2} for any $k \leqslant 1$, still denoted by A. According to the properties of the operator A, the injection $H_k \hookrightarrow H_{k-1}$ is dense and compact for any $k \in \mathbb{Z}$.

We denote by $\lambda_0 > 0$ the largest constant satisfying

$$\|v\|_{H_1}^2 \geqslant \lambda_0 \|v\|_H^2 \quad \text{for all } v \in H_1, \tag{8}$$

that is, the smallest eigenvalue of the selfadjoint positive operator A. Note that we also have, for all $\alpha \geqslant 0$,

$$\|A^{-\alpha}\|_{\mathcal{L}(H)} = \lambda_0^{-\alpha}.$$

In this abstract setting, we shall denote φ' the derivative with respect to time of a function $\varphi : \mathbb{R} \to H_k$, for some $k \in \mathbb{Z}$. In the following, as in [4], we shall make use of the different energy levels

$$e_k(\varphi(t)) = \frac{1}{2}\left(\|\varphi(t)\|_{H_k}^2 + \|\varphi'(t)\|_{H_{k-1}}^2\right), \quad k \in \mathbb{Z},$$

which are all preserved through time if φ is a solution of $\varphi'' + A\varphi = 0$. Moreover, the coercivity assumption (8) yields, for all $k \in \mathbb{Z}$, $v \in H_k$,

$$\|v\|_{H_k}^2 \geqslant \lambda_0 \|v\|_{H_{k-1}}^2 \quad \text{and} \quad e_k(v) \geqslant \lambda_0 e_{k-1}(v). \tag{9}$$

We consider that the coupling operator P is bounded on H and denote by P^* is its adjoint, $p^+ := \|P\|_{\mathcal{L}(H)} = \|P^*\|_{\mathcal{L}(H)}$.

Before addressing the control problem, let us introduce the adjoint system

$$\begin{cases} v_1'' + A v_1 + P v_2 = 0, \\ v_2'' + A v_2 + P^* v_1 = 0, \\ (v_1, v_2, v_1', v_2')\big|_{t=0} = (v_1^0, v_2^0, v_1^1, v_2^1), \end{cases} \tag{10}$$

which shall stand for our observation system. This system can be recast as a first order differential equation

$$\mathcal{V}' = \mathcal{A}_P \mathcal{V}, \qquad \mathcal{V}(0) = \mathcal{V}^0, \tag{11}$$

where

$$\mathcal{A}_P = \begin{pmatrix} 0 & \mathrm{Id} \\ -A_P & 0 \end{pmatrix}, \qquad A_P = \begin{pmatrix} A & P \\ P^* & A \end{pmatrix}, \qquad V = (v_1, v_2), \qquad \mathcal{V} = (V, V') = (v_1, v_2, v_1', v_2').$$

Note that the operator A_P is selfadjoint on the space $H \times H$ endowed with the inner product $(V, \tilde{V})_{H\times H} = (v_1, \tilde{v}_1)_H + (v_2, \tilde{v}_2)_H$. Using (9) with $k = 1$, we obtain

$$(A_P V, V)_{H\times H} = (Av_1, v_1)_H + (Av_2, v_2)_H + 2(Pv_2, v_1)_H \geqslant \left(1 - \frac{p^+}{\lambda_0}\right)\big(\|v_1\|_{H_1}^2 + \|v_2\|_{H_1}^2\big). \tag{12}$$

As a consequence, we shall suppose that $p^+ < \lambda_0$, so that A_P is coercive. Under this assumption, $(A_P^{\frac{1}{2}} V, A_P^{\frac{1}{2}} \tilde{V})_{H\times H}$ defines an inner product on $(H_1)^2$, equivalent to the natural one. Assuming that $P, P^* \in \mathcal{L}(H_k)$ and writing

$$\mathcal{H}_k = (H_k)^2 \times H_{k-1}^2, \quad k \in \mathbb{Z},$$

the operator $\mathcal{A}_P$ is an isomorphism from $\mathcal{H}_k$ to $\mathcal{H}_{k-1}$ and is skewadjoint on $\mathcal{H}_k$, equipped with the inner product

$$\big((U, V), (\tilde{U}, \tilde{V})\big)_{\mathcal{H}_k} = \big(A_P^{\frac{k}{2}} U, A_P^{\frac{k}{2}} \tilde{U}\big)_{H\times H} + \big(A_P^{\frac{k-1}{2}} V, A_P^{\frac{k-1}{2}} \tilde{V}\big)_{H\times H}.$$

Note that this is an inner product according to the coercivity assumption for A_P, which is equivalent to the natural inner product of $\mathcal{H}_k$. Hence, $\mathcal{A}_P$ generates a group $e^{t\mathcal{A}_P}$ on $\mathcal{H}_k$, and the homogeneous problem (10) is well-posed in these spaces. An important feature of solutions $\mathcal{V}(t)$ of System (10) is that all energies

$$E_k\big(\mathcal{V}(t)\big) = 1/2\big\|\mathcal{V}(t)\big\|_{\mathcal{H}_k}^2, \quad k \in \mathbb{Z},$$

are positive and preserved through time.

2.1. Main results: admissibility, observability and controllability

For System (10), now studied in $\mathcal{H}_1$, we shall observe only the state of the first component, i.e. (u_1, u_1'), and hence define an observation operator $\mathcal{B}^* \in \mathcal{L}(H_2 \times H, Y)$, where Y is a Hilbert space, standing for our observation space. This definition is sufficiently general to take into account both the boundary observation problem (taking $\mathcal{B}^* \in \mathcal{L}(H_2, Y)$) and the internal observation problem (taking $\mathcal{B}^* \in \mathcal{L}(H, Y)$). We assume that $\mathcal{B}^*$ is an admissible observation for one equation:

$$\begin{cases} \text{For all } T > 0 \text{ there exists a constant } C > 0, \\ \text{such that all solutions } \varphi \text{ of } \varphi'' + A\varphi = f \in L^2(0, T; H) \text{ satisfy} \\ \displaystyle\int_0^T \big\|\mathcal{B}^*(\varphi, \varphi')\big\|_Y^2 \, dt \leqslant C\left(e_1\big(\varphi(0)\big) + e_1\big(\varphi(T)\big) + \int_0^T e_1\big(\varphi(t)\big)\, dt + \int_0^T \|f\|_H^2 \, dt\right). \end{cases} \tag{A1}$$

Under this assumption, we have the following lemma:

Lemma 2.1 *(Admissibility). The operator $\mathcal{B}^*$ is an admissible observation for* (10)*. More precisely, for all $p^+ < \frac{\lambda_0}{\sqrt{2}}$ and all $T > 0$, there exists a constant $C > 0$, such that all the solutions of* (10) *satisfy*

$$\int_0^T \big\|\mathcal{B}^*\big(v_1, v_1'\big)(t)\big\|_Y^2 \, dt \leqslant C\big\{e_1\big(v_1(0)\big) + e_0\big(v_2(0)\big)\big\}. \tag{13}$$

Note that only the e_0 energy level of the second component v_2 is necessary in this admissibility estimate. Hence, we cannot hope to observe the whole $\mathcal{H}_1$ energy of $\mathcal{V}$ and the best observability we can expect only involves $e_0(v_2)$. Our aim is now to prove this inverse inequality of (13). For this, we have to suppose some additional assumptions on the operators P and $\mathcal{B}^*$. Let us first precise Assumption (A2), related to the operator P:

$$\begin{cases} \text{We have } \|Pv\|_H^2 \leqslant p^+(Pv, v)_H \text{ and there exists an operator } \Pi_P \in \mathcal{L}(H), \\ \|\Pi_P\|_{\mathcal{L}(H)} = 1, \text{ and a number } p^- > 0 \\ \text{such that } (Pv, v)_H \geqslant p^- \|\Pi_P v\|_H^2 \ \forall v \in H. \end{cases} \tag{A2}$$

Note that $p^- \leqslant p^+ = \|P\|_{\mathcal{L}(H)}$ and that (A2) implies that the operators P and P^* are non-negative.

Next, we shall suppose that a single equation is observable both by $\mathcal{B}^*$ and by Π_P in sufficiently large time:

$$\begin{cases} \text{There exist } T_B, T_P > 0 \text{ such that all solutions } \varphi \text{ of } \varphi'' + A\varphi = 0 \text{ satisfy} \\ e_1\big(\varphi(0)\big) \leqslant C_B(T) \displaystyle\int_0^T \big\|\mathcal{B}^*\big(\varphi, \varphi'\big)\big\|_Y^2 \quad \text{for all } T > T_B, \\ e_1\big(\varphi(0)\big) \leqslant C_P(T) \displaystyle\int_0^T \big\|\Pi_P \varphi'\big\|_H^2 \, dt \quad \text{for all } T > T_P. \end{cases} \tag{A3}$$

In the context of Theorem 1.3, these observability assumptions are satisfied as soon as ω_p and ω_b satisfy GCC (resp. Γ_b satisfies GCC_∂). We can now state our main result, i.e. an observability inequality.

Theorem 2.2 *(Observability). Suppose that Assumptions* (A1)–(A3) *hold. Then there exists a constant p_* such that for all $p^+ < p_*$, there exists a time T_* such that for all $T > T_*$ there exists $C > 0$, such that for all $\mathcal{V}^0 \in \mathcal{H}_1$, the solution $\mathcal{V}(t) = e^{t\mathcal{A}_P}\mathcal{V}^0$ of* (10) *satisfies*

$$e_1\big(v_1(0)\big) + e_0\big(v_2(0)\big) \leqslant C \int_0^T \big\|\mathcal{B}^*\big(v_1, v_1'\big)(t)\big\|_Y^2 \, dt. \tag{14}$$

Note that the constants p_* and T_* can be given explicitly in terms of the different parameters of the system. In particular, $T_* \geqslant \max\{T_B, T_P\}$, and p_* depends only on λ_0, on the time T_B and on the observability constant $C_B(T_B^*)$ (given in Assumption (A3)) for some $T_B^* > T_B$. See Proposition 4.7 and Lemma 3.3 below for more precision.

Applying the Hilbert Uniqueness Method (HUM) of [22], we deduce now controllability results for the adjoint system. In this context, we have to define more precisely the observation operator. We shall treat two cases: First, $\mathcal{B}^*(v_1, v_1') = B^* v_1'$ with $B^* \in \mathcal{L}(H, Y)$, corresponding to internal observability (with $Y = L^2(\Omega)$), and second $\mathcal{B}^*(v_1, v_1') = B^* v_1$ with $B^* \in \mathcal{L}(H_2, Y)$, corresponding to boundary observability (with $Y = L^2(\partial\Omega)$). In both cases, we define the control operator B as the adjoint of B^*, and the control problem reads, for a control function f taking its values in Y,

$$\begin{cases} u_1'' + Au_1 + Pu_2 = Bf, \\ u_2'' + Au_2 + P^*u_1 = 0, \\ \big(u_1, u_2, u_1', u_2'\big)\big|_{t=0} = \big(u_1^0, u_2^0, u_1^1, u_2^1\big). \end{cases} \tag{15}$$

This is an abstract version of (3)–(4). Note that under this form, System (15) not only contains (3)–(4), but also locally coupled systems of plate equations, with a distributed or a boundary control.

First case: $\mathcal{B}^*(v_1, v_1') = B^* v_1'$ with $B^* \in \mathcal{L}(H, Y)$. In this case, $B \in \mathcal{L}(Y, H)$ and the control problem (15) is well-posed in $\mathcal{H}_1$ for $f \in L^2(0, T; Y)$. Note that, as in the concrete setting, it also preserves the space $H_1 \times H_2 \times H \times H_1$ through time as soon as $P \in \mathcal{L}(H_1)$. There is thus no hope to control in whole $\mathcal{H}_1$. In this setting, we first deduce from (14) the following other observability estimate for solutions $\mathcal{W}$ of (10) in $\mathcal{H}_0$: $e_0(w_1(0)) + e_{-1}(w_2(0)) \leqslant C \int_0^T \|B^* w_1(t)\|_Y^2 \, dt$. The internal control result of Theorem 1.3 is then a direct consequence of the HUM since Assumptions (A1)–(A3) are satisfied in this application.

Theorem 2.3 *(Controllability). Suppose that Assumptions* (A1)–(A3) *hold. Then, there exists a constant p_* such that for all $p^+ < p_*$, there exists a time T_* such that for all $T > T_*$ and $\mathcal{U}^0 \in H_1 \times H_2 \times H \times H_1$, there exists a control function $f \in L^2(0, T; Y)$ such that the solution $\mathcal{U}(t)$ of* (15) *satisfies $\mathcal{U}(T) = 0$.*

To prove this theorem, we first have to deduce from the observability estimate (14) the following other observability estimate for all $\mathcal{W} = e^{t\mathcal{A}_P}\mathcal{W}^0$ (solution of (10) in $\mathcal{H}_0$): $e_0(w_1(0)) + e_{-1}(w_2(0)) \leqslant C \int_0^T \|B^* w_1(t)\|_Y^2 \, dt$. Then, Theorem 2.3 is a direct consequence of the HUM.

Second case: $\mathcal{B}^*(v_1, v_1') = B^* v_1$ with $B^* \in \mathcal{L}(H_2, Y)$. As a consequence of the admissibility inequality (13), System (15) is well-posed in $\mathcal{H}_0$ in the sense of transposition solutions. Moreover, System (15) also preserves the space $H \times H_1 \times H_{-1} \times H$, and there is no hope to control in whole $\mathcal{H}_0$. In this setting, the boundary control result of Theorem 1.3 is a direct consequence of the HUM and Theorem 2.2 since Assumptions (A1)–(A3) are satisfied in this application.

Theorem 2.4 *(Controllability). Suppose that Assumptions* (A1)–(A3) *hold. Then, there exists a constant p_* such that for all $p^+ < p_*$, there exists a time T_* such that for all $T > T_*$ and $\mathcal{U}^0 \in H \times H_1 \times H_{-1} \times H$, there exists a control function $f \in L^2(0, T; Y)$ such that the solution $\mathcal{U}(t)$ of* (15) *satisfies $\mathcal{U}(T) = 0$.*

Theorem 1.3 is a consequence of Theorem 2.3 in the case of an internal control and of Theorem 2.4 in the case of a boundary control since Assumptions (A1)–(A3) are satisfied in this application. Theorems 2.3 and 2.4 are proved in Section 5.

2.2. Some remarks

Let us make some remarks about these results and their proofs.

First notice that System (15) is reversible in time, so that the concepts of exact controllability, null-controllability and controllability from zero are equivalent.

A consequence of Lemma 2.1 and Theorem 2.2 is that we here describe exactly the attainable set of the control system (15) (which was not considered in [4]). More precisely, starting for zero initial data, we prove that for $T > T_*$ the attainable set is $H_1 \times H_2 \times H \times H_1$ in the first case and $H \times H_1 \times H_{-1} \times H$ in the second case.

Let us briefly describe the method of the proof of Theorem 2.2 which is inspired by the "two energy levels" method of [4]. Everything here is based on energy estimates, considering the H_1 energy of v_1 and the H_0 energy of v_2. There are three main ingredients in our proof.

Our first ingredient is an observability inequality for a single wave-type equation with a right-hand side (see Lemma 3.3). Such observability inequality used to be proved with multiplier techniques [4,6], and thus, under too strong (and not optimal) geometric conditions. Here, we prove such inequalities as a consequence of usual observability inequalities. We also prove for wave equations that such an observability inequality with optimal geometric conditions for an equation with a right-hand side is very natural (see Appendix A). This improvement can in fact be used in several works using multiplier conditions, replacing them with optimal geometric conditions; in particular, the stabilization results of [6] now hold with GCC.

Our second main ingredient is the energy estimate obtained by multiplying the first line of (10) by v_2 and the second one by v_1 and taking the difference of the two equalities. This coupling inequality allows us to estimate the "localized" energy of the unobserved component v_2 by the energy of the observed one v_1 (see Lemma 4.1 below).

Finally to conclude the proof, we use in a crucial way the conservation of the $\mathcal{H}_k$-energy of the solution $\mathcal{V}$. This implies roughly that the integral on $(0, T)$ of the energy is increasing linearly with respect to T, for T sufficiently large.

If the operator B is bounded, we proved in [6] (under similar assumptions) a polynomial decay result for the $E_1(\mathcal{U}(t))$ energy of the system

$$\begin{cases} u_1'' + Au_1 + Pu_2 + BB^*u_1' = 0, \\ u_2'' + Au_2 + P^*u_1 = 0, \\ (u_1, u_2, u_1', u_2')\big|_{t=0} = \mathcal{U}^0 \in H_1 \times H_1 \times H \times H. \end{cases}$$

The observability estimates for a single equation with a right-hand side we prove here (Lemma 3.3) also improve the results of [6] in the case B bounded. Now, these results also hold under optimal geometric conditions for waves. We have the following proposition, where $-\Delta_D$ denotes the Laplace operator with Dirichlet boundary conditions.

Proposition 2.5. *Suppose that Assumption 1.2 holds, and that ω_b and ω_p satisfy GCC. Then there exists $p_* > 0$ such that for all $0 < p^+ < p_*$, the solution $\mathcal{U} = (u_1, u_2, \partial_t u_1, \partial_t u_2)$ of* (5) *satisfies for $n \in \mathbb{N}$, for any initial data*

$$\mathcal{U}^0 = (u_1, u_2, \partial_t u_1, \partial_t u_2)|_{t=0} \in \mathcal{D}\big((-\Delta_D)^{\frac{n+1}{2}}\big)^2 \times \mathcal{D}\big((-\Delta_D)^{\frac{n}{2}}\big)^2,$$

the inequality

$$E_1(\mathcal{U}(t)) \leqslant \frac{C_n}{t^n} \sum_{i=0}^{n} E_1(\mathcal{U}^{(i)}(0)) \quad \forall t > 0.$$

Besides, if $\mathcal{U}^0 \in (H_0^1)^2 \times (L^2)^2$, *then* $E_1(\mathcal{U}(t))$ *converges to zero as* t *goes to infinity.*

The method of energy estimates we use here has several advantages and drawbacks. The main advantage is that it furnishes a systematic method for both internal and boundary controllability problems for a large class of second order in time equations. We only have to check if an observability inequality is known for a single equation, and the results can directly be transferred to systems.

However, we have here to make several assumptions: symmetry of the system, coercivity of the elliptic operator, smallness of the coupling coefficients, large control time… We do not know precisely which assumptions are really needed and which ones are unnecessary. We here provide a general *a priori* analysis of such coupled models. A more precise analysis (for instance for wave systems) remains to be done.

Finally, note that these exact controllability results for abstract second order hyperbolic equations yield null-controllability results for heat or Schrödinger type systems in the abstract setting as well. However, for the sake of clarity, we do not state these results in an abstract setting but only for heat or Schrödinger systems (see Sections 6.2 and 6.3).

3. Two energy levels and two key lemmata

3.1. Two energy levels

In the following sections, when proving Lemma 2.1 and Theorem 2.2, we shall use two different energy levels. Let us consider $\mathcal{V}$ an $\mathcal{H}_1$ solution of (11). We regularize the state $\mathcal{V}$ once by setting

$$\mathcal{W} = \mathcal{A}_P^{-1}\mathcal{V}, \quad \text{i.e.} \begin{cases} w_1' = v_1, \\ w_2' = v_2, \\ Aw_1 + Pw_2 = -v_1', \\ Aw_2 + P^*w_1 = -v_2'. \end{cases} \tag{16}$$

Note that this system has a unique solution (w_1, w_2), since the operator A_P is coercive, that also satisfies

$$\begin{cases} w_1'' + Aw_1 + Pw_2 = 0, \\ w_2'' + Aw_2 + P^*w_1 = 0. \end{cases} \tag{17}$$

Now, the idea, is that, for p^+/λ_0 sufficiently small, the e_0 energy of v_2 is almost equivalent to the e_1 energy of w_2. And we shall see that the e_1 energy level is more practical to handle. This is summarized in the first three identities of the following proposition. The last two identities are technical estimates, used at some points of the proof of the observability inequality.

Proposition 3.1. *We have the following energy estimates for all* $\mathcal{V} = (v_1, v_2, v_1', v_2')$ *in* $\mathcal{H}_1$ *and* $\mathcal{W} = (w_1, w_2, w_1', w_2')$ *defined by* (16):

$$\left(1 - \frac{p^+}{\lambda_0}\right)\left[e_1(v_1) + e_1(v_2)\right] \leqslant E_1(\mathcal{V}) \leqslant \left(1 + \frac{p^+}{\lambda_0}\right)\left[e_1(v_1) + e_1(v_2)\right], \tag{18}$$

$$\left(1 - \frac{p^+}{\lambda_0}\right)\left[e_1(w_1) + e_1(w_2)\right] \leqslant E_1(\mathcal{W}) \leqslant \left(1 + \frac{p^+}{\lambda_0}\right)\left[e_1(w_1) + e_1(w_2)\right], \tag{19}$$

$$\left(\frac{1}{2} - \left(\frac{p^+}{\lambda_0}\right)^2\right)\left(e_1(w_1) + e_1(w_2)\right) \leqslant e_0(v_1) + e_0(v_2) \leqslant \left(2 + 2\left(\frac{p^+}{\lambda_0}\right)^2\right)\left(e_1(w_1) + e_1(w_2)\right), \tag{20}$$

$$e_0(v_2) \leqslant 2\left(1 + \frac{(p^+)^2}{\lambda_0^2}\right)^2 e_1(w_2) + 2\frac{(p^+)^2}{\lambda_0^2} e_0(v_1), \tag{21}$$

$$e_1(v_1) \geqslant \frac{\lambda_0}{2} e_1(w_1) - \frac{(p^+)^2}{\lambda_0} e_1(w_2). \tag{22}$$

As a consequence of this lemma, assuming that $\frac{p^+}{\lambda_0} < \frac{1}{\sqrt{2}}$, we see that the energies $e_1(v_1)+e_1(v_2)$, $e_1(w_1)+e_1(w_2)$ and $e_0(v_1)+e_0(v_2)$ are almost preserved through time for (v_1, v_2) solutions of (10).

Proof of Proposition 3.1. First recall that E_1 is defined by

$$2E_1(\mathcal{V}) = \|v_1'\|_H^2 + \|v_2'\|_H^2 + (Av_1, v_1)_H + (Av_2, v_2)_H + 2(Pv_2, v_1)_H.$$

Then, using the fact that $2|(Pv_2, v_1)_H| \leqslant \frac{p^+}{\lambda_0}(\|A^{\frac{1}{2}}v_1\|_H^2 + \|A^{\frac{1}{2}}v_2\|_H^2)$, we have

$$\begin{aligned} 2E_1(\mathcal{V}) &\geqslant \|v_1'\|_H^2 + \|v_2'\|_H^2 + \left(1 - \frac{p^+}{\lambda_0}\right)\left(\|A^{\frac{1}{2}}v_1\|_H^2 + \|A^{\frac{1}{2}}v_2\|_H^2\right) \\ &\geqslant 2\left(1 - \frac{p^+}{\lambda_0}\right)\left(e_1(v_1) + e_1(v_2)\right), \end{aligned}$$

together with

$$\begin{aligned} 2E_1(\mathcal{V}) &\leqslant \|v_1'\|_H^2 + \|v_2'\|_H^2 + \left(1 + \frac{p^+}{\lambda_0}\right)\left(\|A^{\frac{1}{2}}v_1\|_H^2 + \|A^{\frac{1}{2}}v_2\|_H^2\right) \\ &\leqslant 2\left(1 + \frac{p^+}{\lambda_0}\right)\left(e_1(v_1) + e_1(v_2)\right), \end{aligned}$$

and (18) is proved. Since (18) holds for all $\mathcal{V} \in \mathcal{H}_1$, it also holds for $\mathcal{W}$, which gives (19).

Now, applying $A^{-\frac{1}{2}}$ to the last two lines of System (16) gives

$$\begin{cases} w_1' = v_1, \\ w_2' = v_2, \\ A^{\frac{1}{2}}w_1 + A^{-\frac{1}{2}}Pw_2 = -A^{-\frac{1}{2}}v_1', \\ A^{\frac{1}{2}}w_2 + A^{-\frac{1}{2}}P^*w_1 = -A^{-\frac{1}{2}}v_2'. \end{cases} \tag{23}$$

Since we have $\|A^{-\frac{1}{2}}Pw_2\|_H \leqslant \frac{p^+}{\lambda_0}\|A^{\frac{1}{2}}w_2\|_H$ and $\|A^{-\frac{1}{2}}P^*w_1\|_H \leqslant \frac{p^+}{\lambda_0}\|A^{\frac{1}{2}}w_1\|_H$, System (23) yields

$$\begin{aligned} 2\left(e_0(v_1) + e_0(v_2)\right) &= \|v_1\|_H^2 + \|v_2\|_H^2 + \|A^{-\frac{1}{2}}v_1'\|_H^2 + \|A^{-\frac{1}{2}}v_2'\|_H^2 \\ &\leqslant \|w_1'\|_H^2 + \|w_2'\|_H^2 + 2\|A^{\frac{1}{2}}w_1\|_H^2 + 2\left(\frac{p^+}{\lambda_0}\right)^2\|A^{\frac{1}{2}}w_2\|_H^2 \\ &\quad + 2\|A^{\frac{1}{2}}w_2\|_H^2 + 2\left(\frac{p^+}{\lambda_0}\right)^2\|A^{\frac{1}{2}}w_1\|_H^2 \\ &\leqslant \left(2 + 2\left(\frac{p^+}{\lambda_0}\right)^2\right)2\left(e_1(w_1) + e_1(w_2)\right), \end{aligned}$$

together with

$$\begin{aligned} 2\left(e_0(v_1) + e_0(v_2)\right) &\geqslant \|w_1'\|_H^2 + \|w_2'\|_H^2 + \frac{1}{2}\|A^{\frac{1}{2}}w_1\|_H^2 - \left(\frac{p^+}{\lambda_0}\right)^2\|A^{\frac{1}{2}}w_2\|_H^2 \\ &\quad + \frac{1}{2}\|A^{\frac{1}{2}}w_2\|_H^2 - \left(\frac{p^+}{\lambda_0}\right)^2\|A^{\frac{1}{2}}w_1\|_H^2 \\ &\geqslant \left(\frac{1}{2} - \left(\frac{p^+}{\lambda_0}\right)^2\right)2\left(e_1(w_1) + e_1(w_2)\right), \end{aligned}$$

and (20) is proved. To prove (21), we also use the last equation of (23), which gives

$$\left\|A^{-\frac{1}{2}}v_2'\right\|_H \leqslant \frac{p^+}{\lambda_0}\left\|A^{\frac{1}{2}}w_1\right\|_H + \left\|A^{\frac{1}{2}}w_2\right\|_H.$$

Using the third equation of (23) to eliminate $A^{\frac{1}{2}}w_1$ in this estimate, we obtain

$$\begin{aligned}\left\|A^{-\frac{1}{2}}v_2'\right\|_H &\leqslant \frac{p^+}{\lambda_0}\left\|A^{-\frac{1}{2}}Pw_2\right\|_H + \frac{p^+}{\lambda_0}\left\|A^{-\frac{1}{2}}v_1'\right\|_H + \left\|A^{\frac{1}{2}}w_2\right\|_H \\ &\leqslant \left(1+\frac{(p^+)^2}{\lambda_0^2}\right)\left\|A^{\frac{1}{2}}w_2\right\|_H + \frac{p^+}{\lambda_0}\left\|A^{-\frac{1}{2}}v_1'\right\|_H.\end{aligned}$$

Hence, with the second equation of (23), we have

$$\left\|A^{-\frac{1}{2}}v_2'\right\|_H^2 + \|v_2\|_H^2 \leqslant 2\left(1+\frac{(p^+)^2}{\lambda_0^2}\right)^2\left\|A^{\frac{1}{2}}w_2\right\|_H^2 + 2\frac{(p^+)^2}{\lambda_0^2}\left\|A^{-\frac{1}{2}}v_1'\right\|_H^2 + \left\|w_2'\right\|_H^2,$$

which concludes the proof of (21). To prove (22), we first notice that the third line of (23) gives

$$\left\|A^{\frac{1}{2}}w_1\right\|_H^2 \leqslant 2\left\|A^{-\frac{1}{2}}Pw_2\right\|_H^2 + 2\left\|A^{-\frac{1}{2}}v_1'\right\|_H^2 \leqslant 2\lambda_0^{-2}\left(p^+\right)^2\left\|A^{\frac{1}{2}}w_2\right\|_H^2 + 2\lambda_0^{-1}\left\|v_1'\right\|_H^2.$$

Hence, using the first line of (23), we obtain

$$\begin{aligned}2e_1(v_1) = \left\|A^{\frac{1}{2}}w_1'\right\|_H^2 + \left\|v_1'\right\|_H^2 &\geqslant \lambda_0\left\|w_1'\right\|_H^2 + \frac{\lambda_0}{2}\left\|A^{\frac{1}{2}}w_1\right\|_H^2 - \frac{(p^+)^2}{\lambda_0}\left\|A^{\frac{1}{2}}w_2\right\|_H^2 \\ &\geqslant \lambda_0 e_1(w_1) - \frac{2(p^+)^2}{\lambda_0}e_1(w_2),\end{aligned}$$

which yields (22), and concludes the proof of the lemma. □

3.2. Two key lemmata

In this section, we prove Lemma 2.1 together with a key observability inequality for a classical "wave-type" equation with a right-hand side. For both proofs, we shall use the classical well-posedness properties of the equation $\varphi'' + A\varphi = f$ that we recall in the following lemma.

Lemma 3.2. *For any $k \in \mathbb{Z}$, there exists $C > 0$ such that for all $(\varphi^0, \varphi^1) \in H_k \times H_{k-1}$ and $f \in L^1(\mathbb{R}^+; H_{k-1})$ the equation $\varphi'' + A\varphi = f$ has a unique solution, satisfying for all $T > 0$,*

$$e_k\big(\varphi(T)\big) \leqslant C\big(e_k\big(\varphi(0)\big) + \|f\|_{L^1(0,T;H_{k-1})}^2\big). \tag{24}$$

Note that in this energy inequality, the constant C does not depend on the time T.

3.2.1. Proof of Lemma 2.1: admissibility

Here, we prove that Assumption (A1) implies the admissibility inequality (13) for the whole system.

Proof of Lemma 2.1. We suppose that (v_1, v_2) satisfies System (10). In particular, we have,

$$v_1'' + Av_1 = -Pv_2.$$

As a consequence of Assumption (A1), we have for all $T > 0$,

$$\int_0^T \left\|\mathcal{B}^*\big(v_1, v_1'\big)\right\|_Y^2 dt \leqslant C(T)\left(e_1\big(v_1(0)\big) + e_1\big(v_1(T)\big) + \int_0^T e_1\big(v_1(t)\big)\,dt + \int_0^T \left\|Pv_2(t)\right\|_H^2 dt\right). \tag{25}$$

Then, the energy estimate (24) for $k = 1$, the Cauchy–Schwarz inequality and the boundedness of P on H yield

$$e_1\big(v_1(T)\big) \leqslant C\big(e_1\big(v_1(0)\big) + \|Pv_2\|^2_{L^1(0,T;H)}\big)$$
$$\leqslant C\big(e_1\big(v_1(0)\big) + T\big(p^+\big)^2\|v_2\|^2_{L^2(0,T;H)}\big), \tag{26}$$

together with

$$\int_0^T e_1\big(v_1(t)\big)\,dt \leqslant C\big(Te_1\big(v_1(0)\big) + T^2\big(p^+\big)^2\|v_2\|^2_{L^2(0,T;H)}\big). \tag{27}$$

Now, according to (16), (19) and since $p^+ < \lambda_0$, we note that

$$\|v_2\|^2_{L^2(0,T;H)} = \|w_2'\|^2_{L^2(0,T;H)} \leqslant \frac{2}{1-p^+/\lambda_0}\int_0^T E_1\big(\mathcal{W}(t)\big)\,dt.$$

Since $\mathcal{W}$ is a solution of (17), its energy is preserved through time, so that

$$\int_0^T E_1\big(\mathcal{W}(t)\big)\,dt = TE_1\big(\mathcal{W}(0)\big).$$

Using inequalities (19) and (20) (i.e. the equivalence of the different energies), we obtain, for all $p^+ < \frac{\lambda_0}{\sqrt{2}}$,

$$\|v_2\|^2_{L^2(0,T;H)} \leqslant 2T\frac{(1+\frac{p^+}{\lambda_0})}{1-\frac{p^+}{\lambda_0}}\big[e_1\big(w_1(0)\big) + e_1\big(w_2(0)\big)\big]$$
$$\leqslant 2T\frac{(1+\frac{p^+}{\lambda_0})}{1-\frac{p^+}{\lambda_0}}\left(\frac{1}{2} - \left(\frac{p^+}{\lambda_0}\right)^2\right)^{-1}\big[e_0\big(v_1(0)\big) + e_0\big(v_2(0)\big)\big]. \tag{28}$$

We recall that (9) yields $e_0(v_1(0)) \leqslant \lambda_0^{-1}e_1(v_1(0))$. Finally, combining (25)–(28), we obtain

$$\int_0^T \big\|\mathcal{B}^*\big(v_1, v_1'\big)\big\|^2_Y\,dt \leqslant C\big(T, \lambda_0, p^+\big)\big(e_1\big(v_1(0)\big) + e_0\big(v_2(0)\big)\big),$$

with $C(T,\lambda_0,p^+) = C(T)\{1 + C(1+T) + \max(1, 1/\lambda_0)(p^+)^2\,2T(1+CT+CT^2)\frac{(1+\frac{p^+}{\lambda_0})}{1-\frac{p^+}{\lambda_0}}(\frac{1}{2} - (\frac{p^+}{\lambda_0})^2)^{-1}\}$, and the admissibility of $\mathcal{B}^*$ is proved. □

3.2.2. Proof of an observability inequality with a right-hand side

Here, we prove the following lemma:

Lemma 3.3. *Suppose that Assumptions (A1) and (A3) hold. Then, for all $T_B^* > T_B$ and $T_P^* > T_P$, there exist constants $K_B, K_P > 0$ such that for any solution φ of $\varphi'' + A\varphi = f \in L^2(0,T;H)$, we have*

$$\int_0^T e_1\big(\varphi(t)\big)\,dt \leqslant K_B\left(\int_0^T \big\|\mathcal{B}^*\big(\varphi, \varphi'\big)\big\|^2_Y + \int_0^T \big\|f(t)\big\|^2_H\,dt\right) \quad \textit{for all } T \geqslant T_B^* \tag{29}$$

and

$$\int_0^T e_1\big(\varphi(t)\big)\,dt \leqslant K_P\left(\int_0^T \big\|\Pi_P\varphi'\big\|^2_H\,dt + \int_0^T \big\|f(t)\big\|^2_H\,dt\right) \quad \textit{for all } T \geqslant T_P^*. \tag{30}$$

Note that the crucial point in this lemma is that the observability constants K_B, K_P do not depend upon the time T as $T \to +\infty$. It is linked to the invariance properties of the system with respect to translation in time. More precisely, K_B only depends on T_B^* and on $C_B(T_B^*)$ for some $T_B^* > T_B$, where T_B and C_B are defined in Assumption (A3). Such observability inequalities are proved with multiplier techniques for the wave equation or the plate equation in [3–5]. It is then often used to perform energy estimates. Here, proving (29) and (30) as a consequence of the associated observability inequality for the free equation (Assumption (A3)) and an admissibility assumption (A1) has several advantages. In particular, we can use as a black-box the different observability inequalities obtained for the different equations (i.e. for instance [8,9] for waves) and we hence obtain results with the optimal geometric conditions.

However, for the sake of completeness, we also prove Lemma 3.3 in a simple case for the wave equation in a direct way in Appendix A, with optimal geometric conditions. This shows that the norms used here are the natural ones.

Proof of Lemma 3.3. We here only prove (29). The proof of (30) is simpler since the observation operator Π_P is bounded (and we thus do not need an admissibility assumption).

To prove the first inequality of (29), we first split a solution of $\varphi'' + A\varphi = f$ into $\varphi = \phi + \psi$, where ϕ and ψ satisfy

$$\begin{cases} \phi'' + A\phi = f, \\ \big(\phi(0), \phi'(0)\big) = (0, 0), \end{cases}$$

and

$$\begin{cases} \psi'' + A\psi = 0, \\ \big(\psi(0), \psi'(0)\big) = \big(\varphi(0), \varphi(0)\big). \end{cases}$$

We have

$$\int_0^T e_1\big(\varphi(t)\big)\,dt \leqslant 2\int_0^T e_1\big(\phi(t)\big)\,dt + 2\int_0^T e_1\big(\psi(t)\big)\,dt, \tag{31}$$

and we provide upper bounds for both integrals on the right-hand side. The energy estimate (24) applied to ϕ gives, for all $t > 0$,

$$e_1\big(\phi(t)\big) \leqslant C\|f\|^2_{L^1(0,t;H)} \leqslant Ct\|f\|^2_{L^2(0,t;H)}. \tag{32}$$

Then, the observability Assumption (A3) can be applied to ψ, which gives, for all $t > 0$ and $T > T_B$,

$$e_1\big(\psi(t)\big) = e_1\big(\psi(0)\big) \leqslant C(T)\int_0^T \big\|\mathcal{B}^*\big(\psi, \psi'\big)\big\|_Y^2\,dt. \tag{33}$$

Integrating (32) and (33) for $t \in (0, T)$, and using (31), we now have, for all $T > T_B$,

$$\int_0^T e_1\big(\varphi(t)\big)\,dt \leqslant 2CT^2\int_0^T \|f\|_H^2\,dt + 2C(T)T\int_0^T \big\|\mathcal{B}^*\big(\psi, \psi'\big)\big\|_Y^2\,dt. \tag{34}$$

To obtain the observation on φ instead of ψ in the right-hand side, we write

$$\int_0^T \big\|\mathcal{B}^*\big(\psi, \psi'\big)\big\|_Y^2\,dt \leqslant 2\int_0^T \big\|\mathcal{B}^*\big(\varphi, \varphi'\big)\big\|_Y^2\,dt + 2\int_0^T \big\|\mathcal{B}^*\big(\phi, \phi'\big)\big\|_Y^2\,dt. \tag{35}$$

Then, using the admissibility Assumption (A1) for ϕ, we obtain

$$\begin{aligned} \int_0^T \big\|\mathcal{B}^*\big(\phi, \phi'\big)\big\|_Y^2\,dt &\leqslant C(T)\left(e_1\big(\phi(T)\big) + \int_0^T e_1\big(\phi(t)\big)\,dt + \int_0^T \|f\|_H^2\,dt\right) \\ &\leqslant C(T)\big(T + T^2 + 1\big)\int_0^T \|f\|_H^2\,dt, \end{aligned} \tag{36}$$

after having used (32). Combining (34), (35) and (36), we finally obtain for all $T > T_B$ the existence of a constant $D(T)$ such that

$$\int_0^T e_1(\varphi(t))\,dt \leqslant D(T)\left(\int_0^T \|f\|_H^2\,dt + \int_0^T \|\mathcal{B}^*(\varphi,\varphi')\|_Y^2\,dt\right).$$

Now, we explain how this constant can be uniformly bounded for large times. For all $T_B^* > T_B$, we set $D^*(T) = \sup_{T_B^* \leqslant t \leqslant T} D(t)$, so that the application $T \to D^*(T)$ is nondecreasing on $[T_B^*, +\infty)$. Since the equation $\varphi'' + A\varphi = f$ is invariant under time translations, the last inequality yields, for all $T_2 \geqslant T_1 + T_B^*$,

$$\int_{T_1}^{T_2} e_1(\varphi(t))\,dt \leqslant D^*(T_2 - T_1)\left(\int_{T_1}^{T_2} \|f\|_H^2\,dt + \int_{T_1}^{T_2} \|\mathcal{B}^*(\varphi,\varphi')\|_Y^2\,dt\right). \tag{37}$$

For $T \geqslant T_B^*$, there exists an integer $k_0 \geqslant 1$ such that $T \in [k_0 T_B^*, (k_0+1)T_B^*)$. Assume first that $k_0 \geqslant 2$, then we have,

$$\int_0^T e_1(\varphi(t))\,dt = \sum_{k=0}^{k_0-2} \int_{kT_B^*}^{(k+1)T_B^*} e_1(\varphi(t))\,dt + \int_{(k_0-1)T_B^*}^{T} e_1(\varphi(t))\,dt.$$

In each of these integrals the time interval is larger than T_B^* so that we can apply (37). This yields, for all $T \geqslant 2T_B^*$

$$\begin{aligned}\int_0^T e_1(\varphi(t))\,dt &\leqslant D^*(T_B^*) \sum_{k=0}^{k_0-2} \int_{kT_B^*}^{(k+1)T_B^*} \left(\|f\|_H^2 + \|\mathcal{B}^*(\varphi,\varphi')\|_Y^2\right) dt \\ &\quad + D^*\left(T - (k_0-1)T_B^*\right) \int_{(k_0-1)T_B^*}^{T} \left(\|f\|_H^2 + \|\mathcal{B}^*(\varphi,\varphi')\|_Y^2\right) dt \\ &\leqslant D^*(2T_B^*) \int_0^T \left(\|f\|_H^2 + \|\mathcal{B}^*(\varphi,\varphi')\|_Y^2\right) dt,\end{aligned}$$

since $T \in [k_0 T_B^*, (k_0+1)T_B^*)$ and D^* is nondecreasing. This inequality is also true for $T_B^* \leqslant T \leqslant 2T_B^*$, that is in the case $k_0 = 1$. This concludes the proof of (29), taking $K_B = D^*(2T_B^*)$. The proof of (30) is similar. □

4. Proof of Theorem 2.2

In this section, we shall often use the notation $A \lesssim B$, meaning that there exists a universal numerical constant $C > 0$ (depending on none of the parameters of the system) such that $A \leqslant CB$.

4.1. The coupling lemma

In this section, we give the link between v_1 and v_2 that we shall use in the sequel.

Lemma 4.1. *Let $\mathcal{V} = (v_1, v_2, v_1', v_2') \in \mathcal{H}_1$ be solution of* (10) *and $\mathcal{W} = (w_1, w_2, w_1', w_2')$ be defined by* (16)*. Then, for all $T \geqslant 0$, we have*

$$\begin{aligned}\int_0^T (Pv_2, v_2)_H\,dt &\leqslant \int_0^T (Pv_1, v_1)_H\,dt + 2\lambda_0^{\frac{1}{2}}\left(1 + \frac{(p^+)^2}{\lambda_0^2}\right)^2 \left[e_1(w_2(T)) + e_1(w_2(0))\right] \\ &\quad + \frac{1}{\lambda_0^{\frac{1}{2}}}\left(1 + 2\frac{(p^+)^2}{\lambda_0^2}\right)\left[e_1(v_1(T)) + e_1(v_1(0))\right],\end{aligned} \tag{38}$$

and

$$\int_0^T (Pw_2, w_2)_H\,dt \leqslant \int_0^T (Pw_1, w_1)_H\,dt + \frac{1}{\lambda_0^{\frac{1}{2}}}\left(1 + \frac{2(p^+)^2}{\lambda_0^2}\right)\left[e_1\big(w_2(T)\big) + e_1\big(w_2(0)\big)\right] + \frac{2}{\lambda_0^{\frac{3}{2}}}\left[e_1\big(v_1(T)\big) + e_1\big(v_1(0)\big)\right]. \tag{39}$$

Proof. Since $\mathcal{V}$ is a solution of (10), we have

$$\int_0^T \big(v_1'' + Av_1 + Pv_2, v_2\big)_H - \big(v_2'' + Av_2 + P^* v_1, v_1\big)_H\,dt = 0. \tag{40}$$

We first notice that $(Av_1, v_2)_H - (Av_2, v_1)_H = 0$ since A is selfadjoint, and

$$\begin{aligned}
&\left|\int_0^T \big(v_1'', v_2\big)_H - \big(v_2'', v_1\big)_H\,dt\right| \\
&\quad = \left|\left[\big(v_1', v_2\big)_H - \big(v_2', v_1\big)_H\right]_0^T\right| \\
&\quad \leqslant \frac{1}{2}\left[\frac{1}{\varepsilon}\|v_2\|_H^2 + \varepsilon\|v_1'\|_H^2 + \frac{1}{\varepsilon}\|v_2'\|_{H_{-1}}^2 + \varepsilon\|v_1\|_{H_1}^2\right](t=0) \\
&\qquad + \frac{1}{2}\left[\frac{1}{\varepsilon}\|v_2\|_H^2 + \varepsilon\|v_1'\|_H^2 + \frac{1}{\varepsilon}\|v_2'\|_{H_{-1}}^2 + \varepsilon\|v_1\|_{H_1}^2\right](t=T) \\
&\quad \leqslant \frac{1}{\varepsilon}\left[e_0\big(v_2(T)\big) + e_0\big(v_2(0)\big)\right] + \varepsilon\left[e_1\big(v_1(T)\big) + e_1\big(v_1(0)\big)\right],
\end{aligned}$$

for all $\varepsilon > 0$ and $T \geqslant 0$. Once having isolated the term $\int_0^T (Pv_2, v_2)_H\,dt$ in (40), this yields

$$\int_0^T (Pv_2, v_2)_H\,dt \leqslant \int_0^T (Pv_1, v_1)_H\,dt + \frac{1}{\varepsilon}\left[e_0\big(v_2(T)\big) + e_0\big(v_2(0)\big)\right] + \varepsilon\left[e_1\big(v_1(T)\big) + e_1\big(v_1(0)\big)\right].$$

Using (21) in this expression, we now have for all $\varepsilon > 0$ and $T \geqslant 0$,

$$\begin{aligned}
\int_0^T (Pv_2, v_2)_H\,dt &\leqslant \int_0^T (Pv_1, v_1)_H\,dt + \frac{1}{\varepsilon}\left[2\left(1 + \frac{(p^+)^2}{\lambda_0^2}\right)^2 e_1\big(w_2(T)\big) + 2\frac{(p^+)^2}{\lambda_0^2} e_0\big(v_1(T)\big)\right. \\
&\qquad \left. + 2\left(1 + \frac{(p^+)^2}{\lambda_0^2}\right)^2 e_1\big(w_2(0)\big) + 2\frac{(p^+)^2}{\lambda_0^2} e_0\big(v_1(0)\big)\right] + \varepsilon\left[e_1\big(v_1(T)\big) + e_1\big(v_1(0)\big)\right] \\
&\leqslant \int_0^T (Pv_1, v_1)_H\,dt + \frac{2}{\varepsilon}\left(1 + \frac{(p^+)^2}{\lambda_0^2}\right)^2\left[e_1\big(w_2(T)\big) + e_1\big(w_2(0)\big)\right] \\
&\qquad + \left(\varepsilon + 2\frac{(p^+)^2}{\varepsilon\lambda_0^3}\right)\left[e_1\big(v_1(T)\big) + e_1\big(v_1(0)\big)\right],
\end{aligned}$$

since $e_0(v_1) \leqslant \lambda_0^{-1} e_1(v_1)$. We then set $\varepsilon = \lambda_0^{-\frac{1}{2}}$ and estimate (38) is proved.

Since $\mathcal{W}$ is a solution of (17), we also have

$$\int_0^T \big(w_1'' + Aw_1 + Pw_2, w_2\big)_H - \big(w_2'' + Aw_2 + P^* w_1, w_1\big)_H\,dt = 0.$$

Following the same procedure, and recalling that $w_1' = v_1$, we obtain,

$$\int_0^T (Pw_2, w_2)_H\,dt \leqslant \int_0^T (Pw_1, w_1)_H\,dt + \left|\int_0^T \left(w_1'', w_2\right)_H - \left(w_2'', w_1\right)_H\,dt\right|$$
$$\leqslant \int_0^T (Pw_1, w_1)_H\,dt + \left|\left[(v_1, w_2)_H - \left(w_2', w_1\right)_H\right]_0^T\right|. \tag{41}$$

Next, we estimate

$$\left|(v_1, w_2)_H\right| + \left|\left(w_2', w_1\right)_H\right| \leqslant \frac{1}{2}\left[\frac{1}{\varepsilon}\|w_2\|_H^2 + \varepsilon\|v_1\|_H^2 + \varepsilon\left\|w_2'\right\|_H^2 + \frac{1}{\varepsilon}\|w_1\|_H^2\right], \tag{42}$$

and notice that System (16) yields $w_1 = -A^{-1}v_1' - A^{-1}Pw_2$, and hence

$$\|w_1\|_H \leqslant \lambda_0^{-1}\left\|v_1'\right\|_H + p^+\lambda_0^{-\frac{3}{2}}\left\|A^{\frac{1}{2}}w_2\right\|_H.$$

This, together with (42) gives

$$\left|(v_1, w_2)_H\right| + \left|\left(w_2', w_1\right)_H\right| \leqslant \frac{1}{2}\left[\frac{1}{\varepsilon\lambda_0}\left\|A^{\frac{1}{2}}w_2\right\|_H^2 + \frac{\varepsilon}{\lambda_0}\left\|A^{\frac{1}{2}}v_1\right\|_H^2 + \varepsilon\left\|w_2'\right\|_H^2 + \frac{2}{\varepsilon\lambda_0^2}\left\|v_1'\right\|_H^2 + \frac{2(p^+)^2}{\varepsilon\lambda_0^3}\left\|A^{\frac{1}{2}}w_2\right\|_H^2\right].$$

Taking $\varepsilon = \lambda_0^{-\frac{1}{2}}$, we obtain

$$\left|(v_1, w_2)_H\right| + \left|\left(w_2', w_1\right)_H\right| \leqslant \frac{1}{\lambda_0^{\frac{1}{2}}}\left[1 + \frac{2(p^+)^2}{\lambda_0^2}\right]e_1(w_2) + \frac{2}{\lambda_0^{\frac{3}{2}}}e_1(v_1),$$

which, together with (41) yields estimate (39), and concludes the proof of the lemma. □

4.2. *A first series of estimates*

Note that until now, we did not assume that p^+/λ_0 is small, except for the coercivity assumption on A_P and the equivalence of the different energies in (19)–(20) (used in the proof of the Admissibility Lemma 2.1). Using the coupling relation (38), we now prove a first series of estimates, that will be made more precise later.

Lemma 4.2. *For all $T \geqslant 0$, all $\frac{(p^+)^2}{\lambda_0^2} \leqslant \frac{1}{2}$, all $\mathcal{V} = (v_1, v_2, v_1', v_2') \in \mathcal{H}_1$ solution of (10) and $\mathcal{W} = (w_1, w_2, w_1', w_2')$ defined by (16), we have the following estimates*

$$e_1\big(w_2(T)\big) + e_1\big(w_2(0)\big) \lesssim \lambda_0^{-1}e_1\big(v_1(0)\big) + \frac{p^+}{\lambda_0^{\frac{3}{2}}}\int_0^T e_1\big(v_1(t)\big)\,dt$$
$$+ e_1\big(w_2(0)\big) + p^+\lambda_0^{\frac{1}{2}}\int_0^T \|w_1\|_H^2\,dt, \tag{43}$$

$$e_1\big(v_1(T)\big) + e_1\big(v_1(0)\big) \lesssim e_1\big(v_1(0)\big) + \frac{p^+}{\lambda_0^{\frac{1}{2}}}\int_0^T e_1\big(v_1(t)\big)\,dt$$
$$+ \lambda_0 e_1\big(w_2(0)\big) + p^+\lambda_0^{\frac{3}{2}}\int_0^T \|w_1\|_H^2\,dt, \tag{44}$$

$$\int_0^T (Pv_2, v_2)_H\,dt \lesssim \frac{1}{\lambda_0^{\frac{1}{2}}} e_1\big(v_1(0)\big) + \frac{p^+}{\lambda_0}\int_0^T e_1\big(v_1(t)\big)\,dt + \lambda_0^{\frac{1}{2}} e_1\big(w_2(0)\big) + p^+\lambda_0 \int_0^T \|w_1\|_H^2\,dt. \tag{45}$$

Proof. Taking the inner product of the first equation of (10) by v_1', we obtain the following H_1-dissipation relation for v_1:

$$\frac{d}{dt} e_1(v_1) = -\big(Pv_2, v_1'\big)_H.$$

Integrated on the time interval $(0, T)$, this yields,

$$\begin{aligned} e_1\big(v_1(T)\big) + e_1\big(v_1(0)\big) &= 2e_1\big(v_1(0)\big) - \int_0^T \big(Pv_2, v_1'\big)_H\,dt \\ &\leqslant 2e_1\big(v_1(0)\big) + \frac{p^+}{2\varepsilon\lambda_0^{\frac{1}{2}}}\int_0^T \big\|v_1'\big\|_H^2\,dt + \frac{\varepsilon\lambda_0^{\frac{1}{2}}}{2}\int_0^T (Pv_2, v_2)_H\,dt, \end{aligned}$$

after having used Assumption (A2) on P and the Young inequality. Using the coupling relation (38) of Lemma 4.1 in this estimate gives

$$\begin{aligned} e_1\big(v_1(T)\big) + e_1\big(v_1(0)\big) \leqslant{}& 2e_1\big(v_1(0)\big) + \frac{p^+}{2\varepsilon\lambda_0^{\frac{1}{2}}}\int_0^T \big\|v_1'\big\|_H^2\,dt + \frac{\varepsilon\lambda_0^{\frac{1}{2}} p^+}{2}\int_0^T \|v_1\|_H^2\,dt \\ &+ \varepsilon\lambda_0\bigg(1 + \frac{(p^+)^2}{\lambda_0^2}\bigg)^2\big[e_1\big(w_2(T)\big) + e_1\big(w_2(0)\big)\big] \\ &+ \frac{\varepsilon}{2}\bigg(1 + 2\frac{(p^+)^2}{\lambda_0^2}\bigg)\big[e_1\big(v_1(T)\big) + e_1\big(v_1(0)\big)\big]. \end{aligned}$$

We obtain for all $\frac{(p^+)^2}{\lambda_0^2} \leqslant \frac{1}{2}$ and ε sufficiently small,

$$\begin{aligned} e_1\big(v_1(T)\big) + e_1\big(v_1(0)\big) \lesssim{}& e_1\big(v_1(0)\big) + \frac{p^+}{\lambda_0^{\frac{1}{2}}}\int_0^T \big\|v_1'\big\|_H^2\,dt + \lambda_0^{\frac{1}{2}} p^+ \int_0^T \|v_1\|_H^2\,dt \\ &+ \lambda_0\big[e_1\big(w_2(T)\big) + e_1\big(w_2(0)\big)\big]. \end{aligned} \tag{46}$$

Putting this back into (38), we also have for all $\frac{(p^+)^2}{\lambda_0^2} \leqslant \frac{1}{2}$,

$$\begin{aligned} \int_0^T (Pv_2, v_2)_H\,dt \lesssim{}& \int_0^T (Pv_1, v_1)_H\,dt + \lambda_0^{\frac{1}{2}}\big[e_1\big(w_2(T)\big) + e_1\big(w_2(0)\big)\big] + \frac{1}{\lambda_0^{\frac{1}{2}}} e_1\big(v_1(0)\big) \\ &+ \frac{p^+}{\lambda_0}\int_0^T \big\|v_1'\big\|_H^2\,dt + p^+\int_0^T \|v_1\|_H^2\,dt \\ \lesssim{}& \frac{1}{\lambda_0^{\frac{1}{2}}} e_1\big(v_1(0)\big) + \frac{p^+}{\lambda_0}\int_0^T e_1\big(v_1(t)\big)\,dt + \lambda_0^{\frac{1}{2}}\big[e_1\big(w_2(T)\big) + e_1\big(w_2(0)\big)\big]. \end{aligned} \tag{47}$$

Now, we take the inner product of the second equation of System (17) on $\mathcal{W}$ by $w_2' = v_2$. We obtain the following H_1-dissipation relation for w_2:

$$\frac{d}{dt}e_1(w_2) = -\big(P^* w_1, w_2'\big)_H = -(w_1, P v_2)_H.$$

Integrated on the time interval $(0, T)$, this yields,

$$\begin{aligned} e_1\big(w_2(T)\big) + e_1\big(w_2(0)\big) &= 2e_1\big(w_2(0)\big) - \int_0^T (w_1, P v_2)_H \, dt \\ &\leqslant 2e_1\big(w_2(0)\big) + \frac{p^+ \lambda_0^{\frac{1}{2}}}{2\varepsilon} \int_0^T \|w_1\|_H^2 \, dt + \frac{\varepsilon}{2\lambda_0^{\frac{1}{2}}} \int_0^T (P v_2, v_2)_H \, dt, \end{aligned}$$

for all $\varepsilon > 0$, after having used Assumption (A2) on P and the Young inequality. Using (47) in this last inequality, we obtain, for all $\varepsilon > 0$,

$$\begin{aligned} e_1\big(w_2(T)\big) + e_1\big(w_2(0)\big) \lesssim e_1\big(w_2(0)\big) + \frac{p^+ \lambda_0^{\frac{1}{2}}}{\varepsilon} \int_0^T \|w_1\|_H^2 \, dt + \frac{\varepsilon}{\lambda_0} e_1\big(v_1(0)\big) \\ + \frac{p^+ \varepsilon}{\lambda_0^{\frac{3}{2}}} \int_0^T e_1\big(v_1(t)\big) \, dt + \varepsilon\big[e_1\big(w_2(T)\big) + e_1\big(w_2(0)\big)\big]. \end{aligned}$$

Taking ε sufficiently small, this yields

$$\begin{aligned} &e_1\big(w_2(T)\big) + e_1\big(w_2(0)\big) \\ &\quad \lesssim e_1\big(w_2(0)\big) + p^+ \lambda_0^{\frac{1}{2}} \int_0^T \|w_1\|_H^2 \, dt + \frac{1}{\lambda_0} e_1\big(v_1(0)\big) + \frac{p^+}{\lambda_0^{\frac{3}{2}}} \int_0^T e_1\big(v_1(t)\big) \, dt. \end{aligned}$$

When using this estimate in (46) and (47), we obtain

$$\begin{aligned} &e_1\big(v_1(T)\big) + e_1\big(v_1(0)\big) \\ &\quad \lesssim e_1\big(v_1(0)\big) + \frac{p^+}{\lambda_0^{\frac{1}{2}}} \int_0^T e_1\big(v_1(t)\big) \, dt + \lambda_0 e_1\big(w_2(0)\big) + p^+ \lambda_0^{\frac{3}{2}} \int_0^T \|w_1\|_H^2 \, dt, \end{aligned}$$

and

$$\begin{aligned} \int_0^T (P v_2, v_2)_H \, dt \lesssim \frac{1}{\lambda_0^{\frac{1}{2}}} e_1\big(v_1(0)\big) + \frac{p^+}{\lambda_0} \int_0^T e_1\big(v_1(t)\big) \, dt \\ + \lambda_0^{\frac{1}{2}} e_1\big(w_2(0)\big) + p^+ \lambda_0 \int_0^T \|w_1\|_H^2 \, dt. \end{aligned}$$

These three inequalities yield the result of the lemma. □

4.3. A second series of estimates

Using the weak coupling relation (39), we now eliminate the terms with $\int_0^T \|w_1\|_H^2 \, dt$ in estimates (43)–(45) of the previous section.

Lemma 4.3. *There exists $\eta > 0$ such that for all $T \geqslant 0$, $\frac{p^+}{\lambda_0} \leqslant \eta$, $\mathcal{V} = (v_1, v_2, v_1', v_2') \in \mathcal{H}_1$ solution of (10) and $\mathcal{W} = (w_1, w_2, w_1', w_2')$ defined by (16), we have the estimates*

$$\int_0^T \|w_1\|_H^2 \, dt \lesssim \frac{1}{\lambda_0^2} \int_0^T e_1\big(v_1(t)\big) \, dt + \frac{p^+}{\lambda_0^{\frac{7}{2}}} e_1\big(v_1(0)\big) + \frac{p^+}{\lambda_0^{\frac{5}{2}}} e_1\big(w_2(0)\big) \tag{48}$$

and

$$\int_0^T (P v_2, v_2)_H \, dt \lesssim \frac{1}{\lambda_0^{\frac{1}{2}}} e_1\big(v_1(0)\big) + \frac{p^+}{\lambda_0} \int_0^T e_1\big(v_1(t)\big) \, dt + \lambda_0^{\frac{1}{2}} e_1\big(w_2(0)\big). \tag{49}$$

Proof. Using estimates (43) and (44) in relation (39), together with $\frac{p^+}{\lambda_0} \leqslant \frac{1}{2}$, we have

$$\begin{aligned}
\int_0^T (P w_2, w_2)_H \, dt &\lesssim p^+ \int_0^T \|w_1\|_H^2 \, dt + \frac{1}{\lambda_0^{\frac{1}{2}}} \left(1 + \frac{2(p^+)^2}{\lambda_0^2}\right) \\
&\quad \times \left[\frac{1}{\lambda_0} e_1\big(v_1(0)\big) + \frac{p^+}{\lambda_0^{\frac{3}{2}}} \int_0^T e_1\big(v_1(t)\big) \, dt + e_1\big(w_2(0)\big) + p^+ \lambda_0^{\frac{1}{2}} \int_0^T \|w_1\|_H^2 \, dt\right] \\
&\quad + \frac{1}{\lambda_0^{\frac{3}{2}}} \left[e_1\big(v_1(0)\big) + \frac{p^+}{\lambda_0^{\frac{1}{2}}} \int_0^T e_1\big(v_1(t)\big) \, dt + \lambda_0 e_1\big(w_2(0)\big) + p^+ \lambda_0^{\frac{3}{2}} \int_0^T \|w_1\|_H^2 \, dt\right] \\
&\lesssim p^+ \int_0^T \|w_1\|_H^2 \, dt + \frac{1}{\lambda_0^{\frac{3}{2}}} e_1\big(v_1(0)\big) + \frac{p^+}{\lambda_0^2} \int_0^T e_1\big(v_1(t)\big) \, dt + \frac{1}{\lambda_0^{\frac{1}{2}}} e_1\big(w_2(0)\big).
\end{aligned} \tag{50}$$

Now, we want to eliminate the term with $\int_0^T \|w_1\|_H^2 \, dt$ in these estimates. According to the third line of (16), we have $w_1 = -A^{-1} v_1' - A^{-1} P w_2$, so that

$$\|w_1\|_H^2 \leqslant 2\big\|A^{-1} v_1'\big\|_H^2 + 2\big\|A^{-1} P w_2\big\|_H^2 \leqslant 2\lambda_0^{-2} \big\|v_1'\big\|_H^2 + 2\lambda_0^{-2} p^+ (P w_2, w_2)_H$$

after having used Assumption (A2) on the operator P. Integrating this estimate on $(0, T)$ and using (50) yields

$$\begin{aligned}
\int_0^T \|w_1\|_H^2 \, dt &\lesssim \frac{1}{\lambda_0^2} \int_0^T \big\|v_1'\big\|_H^2 + \frac{(p^+)^2}{\lambda_0^2} \int_0^T \|w_1\|_H^2 \, dt + \frac{p^+}{\lambda_0^{\frac{7}{2}}} e_1\big(v_1(0)\big) \\
&\quad + \frac{(p^+)^2}{\lambda_0^4} \int_0^T e_1\big(v_1(t)\big) \, dt + \frac{p^+}{\lambda_0^{\frac{5}{2}}} e_1\big(w_2(0)\big).
\end{aligned}$$

Now, we suppose that $\frac{p^+}{\lambda_0} \leqslant \eta$, with η sufficiently small. This yields

$$\begin{aligned}
\int_0^T \|w_1\|_H^2 \, dt &\lesssim \frac{1}{\lambda_0^2} \int_0^T \big\|v_1'\big\|_H^2 + \frac{p^+}{\lambda_0^{\frac{7}{2}}} e_1\big(v_1(0)\big) + \frac{(p^+)^2}{\lambda_0^4} \int_0^T e_1\big(v_1(t)\big) \, dt + \frac{p^+}{\lambda_0^{\frac{5}{2}}} e_1\big(w_2(0)\big) \\
&\lesssim \frac{1}{\lambda_0^2} \int_0^T e_1\big(v_1(t)\big) \, dt + \frac{p^+}{\lambda_0^{\frac{7}{2}}} e_1\big(v_1(0)\big) + \frac{p^+}{\lambda_0^{\frac{5}{2}}} e_1\big(w_2(0)\big),
\end{aligned}$$

which is exactly (48). Finally, using this inequality in (45), we obtain for all $\frac{p^+}{\lambda_0} \leqslant \eta$,

$$\int_0^T (Pv_2, v_2)_H \,dt \lesssim \frac{1}{\lambda_0^{\frac{1}{2}}} e_1\big(v_1(0)\big) + \frac{p^+}{\lambda_0} \int_0^T e_1\big(v_1(t)\big)\,dt + \lambda_0^{\frac{1}{2}} e_1\big(w_2(0)\big)$$
$$+ p^+\lambda_0 \left[\frac{1}{\lambda_0^2}\int_0^T e_1\big(v_1(t)\big)\,dt + \frac{p^+}{\lambda_0^{\frac{7}{2}}} e_1\big(v_1(0)\big) + \frac{p^+}{\lambda_0^{\frac{5}{2}}} e_1\big(w_2(0)\big)\right]$$
$$\lesssim \frac{1}{\lambda_0^{\frac{1}{2}}} e_1\big(v_1(0)\big) + \frac{p^+}{\lambda_0} \int_0^T e_1\big(v_1(t)\big)\,dt + \lambda_0^{\frac{1}{2}} e_1\big(w_2(0)\big).$$

This yields (49), and concludes the proof of the lemma. □

Using the two estimates of this lemma, together with the observability inequality with a right-hand side for the operator Π_P, given in Lemma 3.3, we are now able to prove the following lemma.

Lemma 4.4. *Assume the hypotheses of Lemma 3.3 and that (A2) holds. Then, there exists $\eta > 0$ such that for all $T_P^* > T_P$, there exists a constant K_P such that for all $T > T_P^*$ and $p^+/\lambda_0 \leqslant \eta$, for all $\mathcal{V} = (v_1, v_2, v_1', v_2') \in \mathcal{H}_1$ solution of (10) and $\mathcal{W} = (w_1, w_2, w_1', w_2')$ defined by (16), we have*

$$\int_0^T e_1\big(w_2(t)\big)\,dt \lesssim K_P\left(\frac{1}{p^-} + 1\right)\left(\frac{1}{\lambda_0^{\frac{1}{2}}} e_1\big(v_1(0)\big) + \frac{p^+}{\lambda_0}\int_0^T e_1\big(v_1(t)\big)\,dt + \lambda_0^{\frac{1}{2}} e_1\big(w_2(0)\big)\right). \tag{51}$$

Proof. First, we apply (30) to w_2 for some $T_P^* > T_P$, which, according to (17), satisfies $w_2'' + Aw_2 = -P^* w_1$. We have, for all $T > T_P^*$,

$$\int_0^T e_1\big(w_2(t)\big)\,dt \leqslant K_P\left(\int_0^T \|\Pi_P w_2'\|_H^2\,dt + \int_0^T \|P^* w_1\|_H^2\,dt\right).$$

Using Assumption (A2) on P together with the fact that $w_2' = v_2$, this yields

$$\int_0^T e_1\big(w_2(t)\big)\,dt \leqslant K_P\left(\frac{1}{p^-}\int_0^T (Pv_2, v_2)_H\,dt + \big(p^+\big)^2 \int_0^T \|w_1\|_H^2\,dt\right).$$

Combining this inequality with estimates (48)–(49) of Lemma 4.3, we obtain, for all $T > T_P^*$ and $p^+/\lambda_0 \leqslant \eta$,

$$\int_0^T e_1\big(w_2(t)\big)\,dt \lesssim K_P \frac{1}{p^-}\left(\frac{1}{\lambda_0^{\frac{1}{2}}} e_1\big(v_1(0)\big) + \frac{p^+}{\lambda_0}\int_0^T e_1\big(v_1(t)\big)\,dt + \lambda_0^{\frac{1}{2}} e_1\big(w_2(0)\big)\right)$$
$$+ K_P\big(p^+\big)^2\left(\frac{1}{\lambda_0^2}\int_0^T e_1\big(v_1(t)\big)\,dt + \frac{p^+}{\lambda_0^{\frac{7}{2}}} e_1\big(v_1(0)\big) + \frac{p^+}{\lambda_0^{\frac{5}{2}}} e_1\big(w_2(0)\big)\right)$$
$$\lesssim \frac{K_P}{\lambda_0^{\frac{1}{2}}}\left(\frac{1}{p^-} + 1\right) e_1\big(v_1(0)\big) + \frac{K_P p^+}{\lambda_0}\left(\frac{1}{p^-} + 1\right)\int_0^T e_1\big(v_1(t)\big)\,dt$$
$$+ K_P \lambda_0^{\frac{1}{2}}\left(\frac{1}{p^-} + 1\right) e_1\big(w_2(0)\big),$$

which concludes the proof of the lemma. □

Lemma 4.5 *(Almost conservation of the energy). For all $T \geqslant 0$, all $\frac{p^+}{\lambda_0} \leqslant \frac{1}{\sqrt{2}}$, all $\mathcal{V} = (v_1, v_2, v_1', v_2') \in \mathcal{H}_1$ solution of (10) and $\mathcal{W} = (w_1, w_2, w_1', w_2')$ defined by (16), we have*

$$\int_0^T \left[\lambda_0^{-1} e_1\big(v_1(t)\big) + e_1\big(w_2(t)\big)\right] dt \geqslant \frac{T}{2}\left(1 + \frac{p^+}{\lambda_0}\right)^{-1}\left(1 - \frac{p^+}{\lambda_0}\right)\left[e_1\big(w_1(0)\big) + e_1\big(w_2(0)\big)\right]. \tag{52}$$

Proof. First, as a consequence of (22) of Proposition 3.1, we have

$$\lambda_0^{-1} e_1(v_1) + e_1(w_2) \geqslant \frac{1}{2} e_1(w_1) + \left(1 - \frac{(p^+)^2}{\lambda_0^2}\right) e_1(w_2) \geqslant \frac{1}{2}\left[e_1(w_1) + e_1(w_2)\right],$$

as soon as $\frac{(p^+)^2}{\lambda_0^2} \leqslant \frac{1}{2}$. Integrating this inequality on the interval $(0, T)$, and using Identity (19) together with the conservation of the energy $E_1(\mathcal{W})$, we obtain

$$\begin{aligned}
\int_0^T \left[\lambda_0^{-1} e_1\big(v_1(t)\big) + e_1\big(w_2(t)\big)\right] dt &\geqslant \frac{1}{2}\int_0^T \left[e_1\big(w_1(t)\big) + e_1\big(w_2(t)\big)\right] dt \\
&\geqslant \left(1 + \frac{p^+}{\lambda_0}\right)^{-1} \frac{1}{2}\int_0^T E_1\big(\mathcal{W}(t)\big)\, dt \\
&\geqslant \left(1 + \frac{p^+}{\lambda_0}\right)^{-1} \frac{T}{2} E_1\big(\mathcal{W}(0)\big) \\
&\geqslant \frac{T}{2}\left(1 + \frac{p^+}{\lambda_0}\right)^{-1}\left(1 - \frac{p^+}{\lambda_0}\right)\left[e_1\big(w_1(0)\big) + e_1\big(w_2(0)\big)\right],
\end{aligned}$$

which yields (52), and concludes the proof of the lemma. □

Lemma 4.6 *(Lower bound for $\int_0^T e_1(v_1(t))\,dt$). There exist $C > 0$ and $\eta > 0$ such that for all $T \geqslant 0$, all $\frac{p^+}{\lambda_0} \leqslant \eta$, all $\mathcal{V} = (v_1, v_2, v_1', v_2') \in \mathcal{H}_1$ solution of (10) and $\mathcal{W} = (w_1, w_2, w_1', w_2')$ defined by (16), we have*

$$\left(1 + \frac{Tp^+}{\lambda_0^{\frac{1}{2}}}\right)\int_0^T e_1\big(v_1(t)\big)\, dt \gtrsim T e_1\big(v_1(0)\big) - T\lambda_0 e_1\big(w_2(0)\big). \tag{53}$$

Proof. Taking the inner product of the first equation of (10) with v_1' gives

$$\frac{d}{dt} e_1(v_1) = -\big(Pv_2, v_1'\big)_H.$$

For $0 < t < T$, we integrate this identity on the interval $(0, t)$ and obtain, for all $\varepsilon > 0$,

$$e_1\big(v_1(t)\big) \geqslant e_1\big(v_1(0)\big) - \int_0^t \|Pv_2\|_H \|v_1'\|_H\, ds \geqslant e_1\big(v_1(0)\big) - \frac{\varepsilon}{2}\int_0^T \|Pv_2\|_H^2\, dt - \frac{1}{2\varepsilon}\int_0^T \|v_1'\|_H^2\, dt.$$

Using now Assumption (A2) and integrating the last inequality on the interval $(0, T)$, this gives, for all $\varepsilon > 0$,

$$\int_0^T e_1\big(v_1(t)\big)\, dt \geqslant T e_1\big(v_1(0)\big) - \frac{T\varepsilon p^+}{2}\int_0^T (Pv_2, v_2)_H\, dt - \frac{T}{\varepsilon}\int_0^T e_1\big(v_1(t)\big)\, dt.$$

This, together with (49) yields, for some constant $C > 0$ and for all $\varepsilon > 0$,

$$\int_0^T e_1\big(v_1(t)\big)\,dt \geqslant Te_1\big(v_1(0)\big) - \frac{CT\varepsilon p^+}{\lambda_0^{\frac{1}{2}}} e_1\big(v_1(0)\big) - \left(\frac{CT\varepsilon(p^+)^2}{\lambda_0} + \frac{T}{\varepsilon}\right)\int_0^T e_1\big(v_1(t)\big)\,dt - CT\varepsilon p^+\lambda_0^{\frac{1}{2}} e_1\big(w_2(0)\big).$$

Now we choose $\varepsilon = \frac{\lambda_0^{\frac{1}{2}}}{2Cp^+}$, so that we have

$$\int_0^T e_1\big(v_1(t)\big)\,dt \geqslant \frac{T}{2} e_1\big(v_1(0)\big) - \frac{Tp^+}{\lambda_0^{\frac{1}{2}}}\left(2C + \frac{1}{2}\right)\int_0^T e_1\big(v_1(t)\big)\,dt - \frac{T\lambda_0}{2} e_1\big(w_2(0)\big),$$

and the lemma is proved. □

4.4. End of the proof of Theorem 2.2

In this section, we conclude the proof of Theorem 2.2. Using the key estimates of the preceding sections, we prove in fact the following more precise proposition:

Proposition 4.7. *Suppose that Assumptions (A1)–(A3) hold. Then, there exist $\eta, \gamma > 0$ such that for all $p^+ < p_* := \min\{\eta\lambda_0, \gamma\sqrt{\frac{\lambda_0}{K_B}}\}$, there exists a time $T_* \geqslant \max\{T_B^*, T_P^*\}$ (depending on $p^+, p^-, \lambda_0, K_B, K_P$) such that for all $T > T_*$ there exists $C > 0$, such that for all $\mathcal{V}^0 \in \mathcal{H}_1$, the solution $\mathcal{V}(t) = e^{t\mathcal{A}_P}\mathcal{V}^0$ of (10) satisfies*

$$e_1\big(v_1(0)\big) + e_0\big(v_2(0)\big) \leqslant C\int_0^T \big\|\mathcal{B}^*\big(v_1, v_1'\big)(t)\big\|_Y^2\,dt.$$

A numerical inspection of the proof shows that one can take for instance $\eta = \frac{1}{5}$ and $\gamma = \frac{1}{50}$.

Proof. We proceed as in [4] and use balance of energies. First, we use the observability inequality for a single equation with a right-hand side, given by Lemma 3.3. Since v_1 is a solution of $v_1'' + Av_1 + Pv_2 = 0$ from (10), Assumptions (A1) and (A3) and estimate (29) yield, for all $T \geqslant T_B^*$,

$$\int_0^T e_1\big(v_1(t)\big)\,dt \leqslant K_B\left(\int_0^T \big\|\mathcal{B}^*\big(v_1, v_1'\big)\big\|_Y^2\,dt + \int_0^T \|Pv_2\|_H^2\,dt\right).$$

According to Assumption (A2), this gives

$$\begin{aligned}K_B\int_0^T \big\|\mathcal{B}^*\big(v_1, v_1'\big)\big\|_Y^2 &\geqslant \int_0^T e_1\big(v_1(t)\big)\,dt - K_Bp^+\int_0^T (Pv_2, v_2)_H\,dt\\ &\geqslant \varepsilon\int_0^T \big(e_1\big(v_1(t)\big) + \lambda_0 e_1\big(w_2(t)\big)\big)\,dt + (1-\varepsilon)\int_0^T e_1\big(v_1(t)\big)\,dt\\ &\quad - \varepsilon\lambda_0\int_0^T e_1\big(w_2(t)\big)\,dt - K_Bp^+\int_0^T (Pv_2, v_2)_H\,dt,\end{aligned}$$

for some $\varepsilon \in (0,1)$, to be chosen later on. In this expression, we replace $\int_0^T (Pv_2, v_2)_H\,dt$ by estimate (49) given in Lemma 4.3 and $\int_0^T e_1(w_2(t))\,dt$ by estimate (51) given in Lemma 4.4. We obtain, for some constant $C_0 > 0$ all $T \geqslant \max\{T_B^*, T_P^*\}$,

$$\begin{aligned}
K_B \int_0^T \left\|\mathcal{B}^*(v_1, v_1')\right\|_Y^2
&\geqslant \varepsilon \int_0^T \big(e_1(v_1(t)) + \lambda_0 e_1(w_2(t))\big)\,dt + (1-\varepsilon)\int_0^T e_1(v_1(t))\,dt \\
&\quad - \varepsilon\lambda_0 C_0 K_P \left(\frac{1}{p^-} + 1\right)\left\{\frac{1}{\lambda_0^{\frac12}} e_1(v_1(0)) + \frac{p^+}{\lambda_0}\int_0^T e_1(v_1(t))\,dt + \lambda_0^{\frac12} e_1(w_2(0))\right\} \\
&\quad - C_0\left\{\frac{K_B p^+}{\lambda_0^{\frac12}} e_1(v_1(0)) + \frac{K_B (p^+)^2}{\lambda_0}\int_0^T e_1(v_1(t))\,dt + K_B p^+ \lambda_0^{\frac12} e_1(w_2(0))\right\}.
\end{aligned}$$

This can be rewritten under the form

$$\begin{aligned}
K_B \int_0^T \left\|\mathcal{B}^*(v_1, v_1')\right\|_Y^2 &\geqslant \varepsilon \int_0^T \big(e_1(v_1(t)) + \lambda_0 e_1(w_2(t))\big)\,dt + L_1(\varepsilon)\int_0^T e_1(v_1(t))\,dt \\
&\quad - L_2(\varepsilon) e_1(v_1(0)) - L_3(\varepsilon) e_1(w_2(0)), \qquad (54)
\end{aligned}$$

with

$$\begin{cases}
L_1(\varepsilon) = 1 - \varepsilon - \varepsilon C_0 K_P p^+ \left(\dfrac{1}{p^-} + 1\right) - C_0 \dfrac{K_B (p^+)^2}{\lambda_0}, \\
L_2(\varepsilon) = \varepsilon C_0 K_P \lambda_0^{\frac12}\left(\dfrac{1}{p^-} + 1\right) + C_0 \dfrac{K_B p^+}{\lambda_0^{\frac12}}, \\
L_3(\varepsilon) = \varepsilon C_0 K_P \lambda_0^{\frac32}\left(\dfrac{1}{p^-} + 1\right) + C_0 K_B p^+ \lambda_0^{\frac12}.
\end{cases}$$

Taking p^+ such that $C_0 \frac{K_B (p^+)^2}{\lambda_0} < 1$ and ε sufficiently small, we obtain $L_1(\varepsilon) > 0$.

In inequality (54), we replace the two terms integrated on $(0,T)$ by their estimates (52) given in Lemma 4.5 (almost conservation of the energy) and (53) of Lemma 4.6. We obtain, for some $C_1 > 0$, for $\frac{p^+}{\lambda_0} \leqslant \eta$, $(p^+)^2 < \frac{\lambda_0}{C_0 K_B}$,

$$\begin{aligned}
K_B \int_0^T \left\|\mathcal{B}^*(v_1, v_1')\right\|_Y^2 &\geqslant \varepsilon\lambda_0 \frac{T}{3}\big[e_1(w_1(0)) + e_1(w_2(0))\big] - L_2(\varepsilon) e_1(v_1(0)) - L_3(\varepsilon) e_1(w_2(0)) \\
&\quad + C_1 L_1(\varepsilon)\left(1 + \frac{T p^+}{\lambda_0^{\frac12}}\right)^{-1}\big[T e_1(v_1(0)) - T\lambda_0 e_1(w_2(0))\big].
\end{aligned}$$

This yields

$$K_B \int_0^T \left\|\mathcal{B}^*(v_1, v_1')\right\|_Y^2 \geqslant M_1(\varepsilon) e_1(v_1(0)) + M_2(\varepsilon) e_1(w_2(0)) + \varepsilon\lambda_0 \frac{T}{3} e_1(w_1(0)), \qquad (55)$$

with

$$\begin{cases} M_1(\varepsilon) = C_1 L_1(\varepsilon)\left(1 + \dfrac{Tp^+}{\lambda_0^{\frac{1}{2}}}\right)^{-1} T - L_2(\varepsilon), \\ M_2(\varepsilon) = \varepsilon\lambda_0 \dfrac{T}{3} - C_1 L_1(\varepsilon) T\lambda_0 \left(1 + \dfrac{Tp^+}{\lambda_0^{\frac{1}{2}}}\right)^{-1} - L_3(\varepsilon). \end{cases}$$

Now, it remains to check that these coefficients are positive for a suitable choice of ε (small) and T (large). The coefficients $M_1(\varepsilon)$ and $M_2(\varepsilon)$ are positive if and only if we have

$$\begin{cases} T\left\{C_1 L_1(\varepsilon) - \dfrac{L_2(\varepsilon)p^+}{\lambda_0^{\frac{1}{2}}}\right\} - L_2(\varepsilon) > 0, \\ T^2\left(\dfrac{\varepsilon}{3}\lambda_0^{\frac{1}{2}} p^+\right) + T\left\{\dfrac{\varepsilon\lambda_0}{3} - C_1 L_1(\varepsilon)\lambda_0 - \dfrac{L_3(\varepsilon)p^+}{\lambda_0^{\frac{1}{2}}}\right\} - L_3(\varepsilon) > 0. \end{cases} \tag{56}$$

The first condition of (56) is satisfied for large T if

$$1 - \varepsilon - \varepsilon C_0\left(1 + \frac{1}{C_1}\right) K_P p^+ \left(\frac{1}{p^-} + 1\right) - C_0\left(1 + \frac{1}{C_1}\right)\frac{K_B (p^+)^2}{\lambda_0} > 0,$$

i.e. as soon as

$$\begin{cases} C_0\left(1 + \dfrac{1}{C_1}\right)\dfrac{K_B (p^+)^2}{\lambda_0} < 1, \quad \text{and} \\ \varepsilon\left(1 + C_0\left(1 + \dfrac{1}{C_1}\right) K_P p^+\left(\dfrac{1}{p^-} + 1\right)\right) < 1 - C_0\left(1 + \dfrac{1}{C_1}\right)\dfrac{K_B (p^+)^2}{\lambda_0}. \end{cases}$$

This is the case when taking $p^+ \leqslant \min\{\eta\lambda_0, \gamma\sqrt{\frac{\lambda_0}{K_B}}\}$ for some constant $\gamma > 0$, and ε sufficiently small. Then, the second condition of (56) is always satisfied for large T since $\varepsilon > 0$. Hence, for this choice of p^+, ε sufficiently small, and T sufficiently large, we obtain from (55) the existence of a constant $C > 0$ such that

$$C\int_0^T \left\|\mathcal{B}^*\left(v_1, v_1'\right)\right\|_Y^2 \geqslant e_1\left(v_1(0)\right) + e_1\left(w_2(0)\right).$$

This concludes the proof of Proposition 4.7 (and hence, that of Theorem 2.2), since $e_0(v_2(0))$ can be estimated by $e_1(w_2(0))$ and $e_1(v_1(0))$ according to (21). □

5. From observability to controllability

In this section, we prove that the observability inequality of Theorem 2.2 (or equivalently Proposition 4.7) implies the controllability results of Theorems 2.3 and 2.4. This is done classically with the use of the Hilbert Uniqueness Method (see [22]), that we shall follow here.

5.1. First case: $\mathcal{B}^*(v_1, v_1') = B^* v_1'$ with $B^* \in \mathcal{L}(H, Y)$.

In this case, $B \in \mathcal{L}(Y, H)$ and the control problem (15) is well-posed in $\mathcal{H}_1$ for $f \in L^2(0, T; Y)$. Note that, as in the concrete setting, it also preserves the space $H_1 \times H_2 \times H \times H_1$ through time as soon as $P \in \mathcal{L}(H_1)$. There is thus no hope to control in whole $\mathcal{H}_1$.

Note that a direct application of the HUM, at the $H_1 \times H \times H \times H_{-1}$ energy level for the adjoint variable $\mathcal{V}$ would yield a controllability result in $H \times H_{-1} \times H_1 \times H$ with a control $Bf \in L^2(0, T; H_{-1})$ (and we would have to suppose that $B \in \mathcal{L}(Y, H_{-1})$). Since we want the control result to hold at a more regular level, we study an adjoint problem in a less regular space.

Lemma 5.1. *For any $\mathcal{Z}^T = (z_1^T, z_2^T, \tilde{z}_1^T, \tilde{z}_2^T) \in H \times H_{-1} \times H_{-1} \times H_{-2}$ the system*

$$\begin{cases} z_1'' + Az_1 + Pz_2 = 0, \\ z_2'' + Az_2 + P^* z_1 = 0, \\ (z_1, z_2, z_1', z_2')\big|_{t=T} = (z_1^T, z_2^T, \tilde{z}_1^T, \tilde{z}_2^T), \end{cases} \tag{57}$$

is well-posed backward in time and the unique solution satisfies $\mathcal{Z} \in \mathscr{C}^0([0,T]; H \times H_{-1} \times H_{-1} \times H_{-2})$. Moreover suppose that Assumptions (A1)–(A3) hold and that $p^+ < p_* = \min\{\eta\lambda_0, \gamma\sqrt{\frac{\lambda_0}{K_B}}\}$ (where η and γ are given by Proposition 4.7). Then, for all $T > T_*$ (given by Proposition 4.7), the solution $\mathcal{Z}$ satisfies for some constant $C > 1$ the estimates

$$C^{-1} \int_0^T \|B^* z_1(t)\|_Y^2 \, dt \leqslant e_0\big(z_1(0)\big) + e_{-1}\big(z_2(0)\big) \leqslant C \int_0^T \|B^* z_1(t)\|_Y^2 \, dt. \tag{58}$$

Of course, we prove this lemma as a consequence of (13) and (14) for more regular functions. We thus regularize $\mathcal{Z}$, setting $\mathcal{V} := \mathcal{A}_P^{-1}\mathcal{Z}$, end then apply to $\mathcal{V}$ the results of the previous sections.

Proof. First, we know that $\mathcal{A}_P$ generates a group on $\mathcal{H}_{-1}$. Hence, for $\mathcal{Z}^T \in H \times H_{-1} \times H_{-1} \times H_{-2} \subset \mathcal{H}_{-1}$, System (57) has a unique solution $\mathcal{Z}$ in $\mathscr{C}^0([0,T]; \mathcal{H}_{-1})$, and, in particular, $z_2 \in \mathscr{C}^0([0,T]; H_{-1})$. Then, z_1 is solution of the first line of (57) with final data in $H \times H_{-1}$ and a right-hand side $-Pz_2 \in \mathscr{C}^0([0,T]; H_{-1})$. Hence, the solution (z_1, z_1') (which is unique in $H_{-1} \times H_{-2}$) is in $\mathscr{C}^0([0,T]; H \times H_{-1})$. Since $\mathcal{Z}$ in $\mathscr{C}^0([0,T]; \mathcal{H}_{-1})$, this concludes the first part of the lemma.

Second, we set $(v_1^T, v_2^T, \tilde{v}_1^T, \tilde{v}_2^T) = \mathcal{V}^T := \mathcal{A}_P^{-1}\mathcal{Z}^T$. We define

$$\mathcal{V} = e^{(T-t)\mathcal{A}_P}\mathcal{V}^T = \big(v_1, v_2, v_1', v_2'\big),$$

which satisfies

$$\begin{cases} v_1'' + Av_1 + Pv_2 = 0, \\ v_2'' + Av_2 + P^* v_1 = 0, \end{cases}$$

and we have, for all $t \in (0,T)$,

$$\begin{cases} v_1' = z_1, \\ v_2' = z_2, \\ Av_1 + Pv_2 = -z_1', \\ Av_2 + P^* v_1 = -z_2'. \end{cases} \tag{59}$$

Now, let us only consider the smooth solutions, i.e. $\mathcal{Z} \in \mathscr{C}^0([0,T]; \mathcal{H}_0)$, which yields $\mathcal{V} \in \mathscr{C}^0([0,T]; \mathcal{H}_1)$. For these solutions, Theorem 2.2 and Lemma 2.1 yield

$$C^{-1} \int_0^T \|B^* v_1'(t)\|_Y^2 \, dt \leqslant e_1\big(v_1(0)\big) + e_0\big(v_2(0)\big) \leqslant C \int_0^T \|B^* v_1'(t)\|_Y^2 \, dt. \tag{60}$$

We notice that $B^* v_1' = B^* z_1$, so that in order to prove (58), it only remains to show the existence of a constant $C > 1$ such that

$$\frac{1}{C}\big\{e_1\big(v_1(0)\big) + \lambda_0 e_0\big(v_2(0)\big)\big\} \leqslant e_0\big(z_1(0)\big) + \lambda_0 e_{-1}\big(z_2(0)\big) \leqslant C\big\{e_1\big(v_1(0)\big) + \lambda_0 e_0\big(v_2(0)\big)\big\}. \tag{61}$$

According to (59), we have (skipping the time dependence)

$$\begin{aligned} 2\big\{e_0(z_1) + \lambda_0 e_{-1}(z_2)\big\} &= \big\|A^{-\frac{1}{2}} z_1'\big\|_H^2 + \|z_1\|_H^2 + \lambda_0 \big\|A^{-1} z_2'\big\|_H^2 + \lambda_0 \big\|A^{-\frac{1}{2}} z_2\big\|_H^2 \\ &= \big\|A^{\frac{1}{2}} v_1 + A^{-\frac{1}{2}} P v_2\big\|_H^2 + \big\|v_1'\big\|_H^2 + \lambda_0 \big\|v_2 + A^{-1} P^* v_1\big\|_H^2 + \lambda_0 \big\|A^{-\frac{1}{2}} v_2'\big\|_H^2 \\ &\leqslant 2\big\|A^{\frac{1}{2}} v_1\big\|_H^2 + 2\big\|A^{-\frac{1}{2}} P v_2\big\|_H^2 + \big\|v_1'\big\|_H^2 + 2\lambda_0 \|v_2\|_H^2 \end{aligned}$$

$$+ 2\lambda_0 \| A^{-1} P^* v_1 \|_H^2 + \lambda_0 \| A^{-\frac{1}{2}} v_2' \|_H^2$$
$$\leqslant 2\big(1 + (p^+)^2 \lambda_0^{-2}\big) \| A^{\frac{1}{2}} v_1 \|_H^2 + \| v_1' \|_H^2 + 2\big(\lambda_0 + (p^+)^2 \lambda_0^{-1}\big) \| v_2 \|_H^2$$
$$+ \lambda_0 \| A^{-\frac{1}{2}} v_2' \|_H^2 \leqslant 2C\big\{e_1(v_1) + \lambda_0 e_0(v_2)\big\},$$

which proves the right inequality of (61). We also have

$$\begin{aligned}
2\big\{e_1(v_1) + \lambda_0 e_0(v_2)\big\} &= \| v_1' \|_H^2 + \| A^{\frac{1}{2}} v_1 \|_H^2 + \lambda_0 \| A^{-\frac{1}{2}} v_2' \|_H^2 + \lambda_0 \| v_2 \|_H^2 \\
&= \| z_1 \|_H^2 + \| A^{-\frac{1}{2}} z_1' + A^{-\frac{1}{2}} P v_2 \|_H^2 + \lambda_0 \| A^{-\frac{1}{2}} z_2 \|_H^2 + \lambda_0 \| A^{-1} P^* v_1 + A^{-1} z_2' \|_H^2 \\
&\leqslant \| z_1 \|_H^2 + 2 \| A^{-\frac{1}{2}} z_1' \|_H^2 + 2 \| A^{-\frac{1}{2}} P v_2 \|_H^2 + \lambda_0 \| A^{-\frac{1}{2}} z_2 \|_H^2 \\
&\quad + 2\lambda_0 \| A^{-1} P^* v_1 \|_H^2 + 2\lambda_0 \| A^{-1} z_2' \|_H^2 \\
&\leqslant 4\big\{e_0(z_1) + \lambda_0 e_{-1}(z_2)\big\} + 2(p^+)^2 \lambda_0^{-1} \| v_2 \|_H^2 + 2(p^+)^2 \lambda_0^{-2} \| A^{\frac{1}{2}} v_1 \|_H^2.
\end{aligned}$$

For $(p^+)^2 \lambda_0^{-2} < 1/2$, the last two terms in this inequality can be absorbed in the left-hand side, yielding

$$e_1(v_1) + \lambda_0 e_0(v_2) \leqslant C\big\{e_0(z_1) + \lambda_0 e_{-1}(z_2)\big\}.$$

This proves (61) and concludes the proof of the lemma. □

To prove Theorem 2.3 with the HUM, we shall also make use of the following lemma.

Lemma 5.2. *Let (u_1, u_2, u_1', u_2') be the solution of (15) associated with $(u_1^0, u_2^0, u_1^1, u_2^1) \in H_1 \times H_2 \times H \times H_1$, and $f \in L^2(0, T; Y)$ and (z_1, z_2, z_1', z_2') the solution of (57) associated with*

$$\big(z_1^T, z_2^T, \tilde{z}_1^T, \tilde{z}_2^T\big) \in H \times H_{-1} \times H_{-1} \times H_{-2}.$$

Then, we have

$$\big[(u_1', z_1)_H - \langle u_1, z_1' \rangle_{H_1, H_{-1}} + \langle u_2', z_2 \rangle_{H_1, H_{-1}} - \langle u_2, z_2' \rangle_{H_2, H_{-2}}\big]_0^T = \big(f, B^* z_1\big)_{L^2(0,T;Y)}. \tag{62}$$

Proof. It suffices to prove (62) for regular data. The general case can be deduced with a density argument. We take the inner product of the first line of (15) with z_1 and the second line of (15) with z_2 and integrate on $(0, T)$. Summing the two identities, we obtain

$$\int_0^T (Bf, z_1)_H \, dt = \int_0^T \big(\big(u_1'' + Au_1 + Pu_2\big), z_1\big)_H \, dt + \int_0^T \big(\big(u_2'' + Au_2 + P^* u_1\big), z_2\big)_H \, dt.$$

After two integrations by parts, using the selfadjointness of A together with (57), we have

$$\int_0^T \big(f, B^* z_1\big)_Y \, dt = \big[(u_1', z_1)_H - (u_1, z_1')_H + (u_2', z_2)_H - (u_2, z_2')_H\big]_0^T,$$

which directly yields (62) for smooth solutions. We conclude the proof of the lemma with a density argument. □

With Lemmata 5.1 and 5.2, we can now prove Theorem 2.3, following [22].

Proof of Theorem 2.3. Let us fix initial data $(u_1^0, u_2^0, u_1^1, u_2^1) \in H_1 \times H_2 \times H \times H_1$. On the Hilbert space $\mathcal{X} := H \times H_{-1} \times H_{-1} \times H_{-2}$, we consider the bilinear form

$$a\big(\mathcal{Z}^T, \underline{\mathcal{Z}}^T\big) = \int_0^T \big(B^* z_1(t), B^* \underline{z_1}(t)\big)_Y \, dt,$$

and the linear form

$$L(\mathcal{Z}^T) = (u_1^1, z_1(0))_H - \langle u_1^0, z_1'(0)\rangle_{H_1,H_{-1}} + \langle u_2^1, z_2(0)\rangle_{H_1,H_{-1}} - \langle u_2^0, z_2'(0)\rangle_{H_2,H_{-2}},$$

where $\mathcal{Z} = (z_1, z_2, z_1', z_2')$ (resp. $\underline{\mathcal{Z}} = (\underline{z_1}, \underline{z_2}, \underline{z_1}', \underline{z_2}'))$ is the solution of (57) associated with the final data $\mathcal{Z}^T = (z_1^T, z_2^T, \tilde{z}_1^T, \tilde{z}_2^T)$ (resp. $\underline{\mathcal{Z}}^T = (\underline{z_1}^T, \underline{z_2}^T, \underline{\tilde{z}_1}^T, \underline{\tilde{z}_2}^T)$). The linear form L is continuous on $\mathcal{X}$ by definition and, according to (58), the bilinear form a is both continuous and coercive on $\mathcal{X} \times \mathcal{X}$ as soon as $T > T_*$. The Lax–Milgram theorem then yields for $T > T_*$ the existence (and uniqueness) of $\underline{\mathcal{Z}}^T$ such that

$$a(\mathcal{Z}^T, \underline{\mathcal{Z}}^T) = -L(\mathcal{Z}^T), \quad \text{for all } \mathcal{Z}^T \in \mathcal{X}.$$

Now, choosing $f = B^*\underline{z_1} \in L^2(0, T; Y)$ as a control function for the data $(u_1^0, u_2^0, u_1^1, u_2^1)$, we obtain for all test function $\mathcal{Z}^T \in \mathcal{X}$,

$$(u_1^1, z_1(0))_H - \langle u_1^0, z_1'(0)\rangle_{H_1,H_{-1}} + \langle u_2^1, z_2(0)\rangle_{H_1,H_{-1}} - \langle u_2^0, z_2'(0)\rangle_{H_2,H_{-2}} = (f, B^*z_1)_{L^2(0,T;Y)}.$$

According to (62), this yields

$$(u_1(T), u_2(T), u_1'(T), u_2'(T)) = (0, 0, 0, 0),$$

where (u_1, u_2, u_1', u_2') is the solution of (15) associated with $(u_1^0, u_2^0, u_1^1, u_2^1)$ and f. This concludes the proof of Theorem 2.3. □

5.2. *Second case: $\mathcal{B}^*(v_1, v_1') = B^*v_1$ with $B^* \in \mathcal{L}(H_2, Y)$.*

In this setting, we directly apply the HUM in the space $H \times H_1 \times H_{-1} \times H$ for the control problem, and thus in $H_1 \times H \times H \times H_{-1}$ for the observation problem. There is no need of regularizing our observation system and observability inequality.

This means that the adjoint problem of the control problem (15) is directly System (10), for which we proved the admissibility inequality (13) and the observability inequality (14). Recall that in this case, the control operator B is in $\mathcal{L}(Y, H_{-2})$, which is not sufficient for (15) to be well-posed in the classical sense in $\mathcal{H}_0$ for a control function $f \in L^2(0, T; Y)$. Nevertheless, as a consequence of the admissibility inequality (13), System (15) is well-posed in $\mathcal{H}_0$ in the sense of transposition solutions (see [22]). Moreover, these solutions remain in $H \times H_1 \times H_{-1} \times H$ for all time if the initial data are in this space, and there is no hope to control in whole $\mathcal{H}_0$.

In this setting, the boundary control result of Theorem 1.3 is a direct consequence of the HUM and Theorem 2.2 since Assumptions (A1)–(A3) are satisfied in this application.

Lemma 5.3. *Let (u_1, u_2, u_1', u_2') be the transposition solution of (15) associated with $(u_1^0, u_2^0, u_1^1, u_2^1) \in H \times H_1 \times H_{-1} \times H$ and $f \in L^2(0, T; Y)$ and (v_1, v_2, v_1', v_2') the backward solution of (10) associated with $(v_1^T, v_2^T, \tilde{v}_1^T, \tilde{v}_2^T) \in H_1 \times H \times H \times H_{-1}$. Then, we have*

$$\left[\langle u_1', v_1\rangle_{H_{-1},H_1} - (u_1, v_1')_H + (u_2', v_2)_H - \langle u_2, v_2'\rangle_{H_1,H_{-1}}\right]_0^T = (f, B^*v_1)_{L^2(0,T;Y)}. \tag{63}$$

The proof of this lemma is exactly the same as the one of Lemma 5.2. We can now sketch the proof of Theorem 2.4, which follows that of Theorem 2.3.

Proof of Theorem 2.4. We fix initial data $(u_1^0, u_2^0, u_1^1, u_2^1) \in H \times H_1 \times H_{-1} \times H$. On the Hilbert space $\mathcal{X} := H_1 \times H \times H \times H_{-1}$, we consider the bilinear form

$$a(\mathcal{V}^T, \underline{\mathcal{V}}^T) = \int_0^T (B^*v_1(t), B^*\underline{v_1}(t))_Y\, dt,$$

and the linear form

$$L(\mathcal{V}^T) = \langle u_1^1, v_1(0)\rangle_{H_{-1},H_1} - (u_1^0, v_1'(0))_H + (u_2^1, v_2(0))_H - \langle u_2^0, v_2'(0)\rangle_{H_1,H_{-1}},$$

where $\mathcal{V} = (v_1, v_2, v_1', v_2')$ (resp. $\underline{\mathcal{V}} = (\underline{v_1}, \underline{v_2}, \underline{v_1}', \underline{v_2}'))$ is the solution of (10) associated with the final data $\mathcal{V}^T = (v_1^T, v_2^T, \tilde{v}_1^T, \tilde{v}_2^T)$, (resp. $\underline{\mathcal{V}}^T = (\underline{v_1}^T, \underline{v_2}^T, \underline{\tilde{v}_1}^T, \underline{\tilde{v}_2}^T))$. The linear form L is continuous on $\mathcal{X}$ and, according to the admissibility inequality (13) and the observability inequality (14), the bilinear form a is both continuous and coercive on $\mathcal{X} \times \mathcal{X}$ as soon as $T > T_*$. We conclude the proof as the one of Theorem 2.3, using the Lax–Milgram theorem. □

6. Applications

In this section, we explain how Theorem 1.3 and Corollaries 1.4 and 1.5 can be deduced from the abstract results.

6.1. Control of wave systems: Proof of Theorem 1.3

Here, we prove Theorem 1.3. We only have to explain how the situation of this theorem can be put in the abstract setting. Here, $H = L^2(\Omega)$ with usual inner product. For A, we take the operator $-\Delta_c + a$ with domain $H^2(\Omega) \cap H_0^1(\Omega)$, which, according to Assumption 1.2(i) is coercive. Hence $H_1 = H_0^1(\Omega)$ is endowed with the inner product $(u, v)_{H_1} = (c\nabla u, \nabla v)_{L^2(\Omega)} + (au, v)_{L^2(\Omega)}$, $H_{-1} = H^{-1}(\Omega)$ and $H_2 = H^2(\Omega) \cap H_0^1(\Omega)$.

For the operator P we take the multiplication in $L^2(\Omega)$ by the bounded function p, and the operator Π_P needed in Assumption (A2) is the multiplication in $L^2(\Omega)$ by the characteristic function $\mathbb{1}_{\omega_p}$. According to Assumption 1.2(ii), ω_p satisfies GCC, so that the observability inequality of [8,9] directly implies the second part of Assumption (A3).

First case: internal control. The observation space here is $Y = L^2(\Omega)$ and the observation operator B^* is the multiplication in $L^2(\Omega)$ by the bounded (real) function b. In this case, the operator B^* is bounded and we have $B = B^*$. Since B^* is bounded, the admissibility assumption (A1) is directly satisfied. Finally, according to Assumption 1.2(iii), ω_b satisfies GCC, so that the observability inequality of [8,9] directly implies the first part of Assumption (A3).

All the assumptions of Theorem 2.3 are then satisfied, so that it implies Theorem 1.3 in the internal control case.

Second case: boundary control. The observation space here is $Y = L^2(\partial\Omega)$ and the observation operator B^* is defined on $H^2(\Omega) \cap H_0^1(\Omega)$ by

$$B^* v = b_\partial \frac{\partial v}{\partial n},$$

where n denotes the outward normal to $\partial\Omega$. Hence, in this case $B^* \in \mathcal{L}(H^2(\Omega) \cap H_0^1(\Omega); L^2(\partial\Omega))$. The fact that this observation is admissible is a well-known hidden regularity result, see [22] or [31, Section 7.1]. As a consequence, the admissibility assumption (A1) is satisfied. The control operator B is defined in this case as the Dirichlet map for which we refer to [31, Section 10.6]. The duality identity (63) shows in fact that it corresponds to a boundary control, i.e. to Problem (4). Finally, according to Assumption 1.2(iii), Γ_b satisfies GCC$_\partial$, so that the observability inequality of [9] directly implies the first part of Assumption (A3).

All the assumptions of Theorem 2.4 are then satisfied, so that it implies Theorem 1.3 in the boundary control case.

6.2. Control of diffusive systems

Here, we prove Corollary 1.4. Our control strategy consists in first regularizing the initial data (thanks to the natural smoothing effect of the heat equation), and then apply a transmutation argument.

First case: internal control. Let $T > 0$. On the time interval $(0, \frac{T}{2})$, we set $f = 0$. Hence, the initial data $(u_1^0, u_2^0) \in (L^2(\Omega))^2$ are driven to some $(u_1, u_2)|_{t=\frac{T}{2}} \in \mathcal{D}(A_p) = (H^2(\Omega) \cap H_0^1(\Omega))^2 \subset H_0^1(\Omega) \times L^2(\Omega)$. As a consequence of Theorem 1.3, combined with [24, Theorem 3.4] there exists a control function $f \in L^2((\frac{T}{2}, T) \times \Omega)$ such that $(u_1, u_2)|_{t=T} = 0$.

Second case: boundary control. Let $T > 0$. On the time interval $(0, \frac{T}{2})$, we set $f = 0$. Hence, the initial data $(u_1^0, u_2^0) \in (H^{-1}(\Omega))^2$ are driven to some $(u_1, u_2)|_{t=\frac{T}{2}} \in (H_0^1(\Omega))^2 \subset L^2(\Omega) \times H^{-1}(\Omega)$. As a consequence of Theorem 1.3, combined with [24, Theorem 3.4] there exists a control function $f \in L^2((\frac{T}{2}, T) \times \partial\Omega)$ such that $(u_1, u_2)|_{t=T} = 0$.

Note that we could have taken the initial data in less regular spaces, provided that the coefficients a, p are smooth enough.

6.3. Control of Schrödinger systems

The proof of Corollary 1.5 is the same as that of Corollary 1.4, except that the Schrödinger equation does not enjoy smoothing properties. Hence, Corollary 1.5 is a direct consequence of Theorem 1.3, combined with [23, Theorem 3.1].

Acknowledgements

The authors want to thank B. Dehman for discussions on the article [9], S. Ervedoza for having pointed out the papers [23,24], and L. Miller for discussions on these two articles. The first author would like to thank the Fondation des Sciences Mathématiques de Paris, the organizers of the IHP trimester on control of PDE's and the Laboratoire MAPMO for their support. The second author wishes to thank O. Glass and J. Le Rousseau for very fruitful discussions and encouragements. Both authors were partially supported by l'Agence Nationale de la Recherche under grant ANR-07-JCJC-0139-01 and the GDRE CONEDP (CNRS/INDAM/UP).

Appendix A. A direct proof of Lemma 3.3: Observability for a wave equation with a right-hand side

In this section, we provide a direct proof of Lemma 3.3 for a wave equation in a very simple setting. For this, we suppose that (Ω, g) is a compact connected Riemannian manifold without boundary, and we closely follow the proofs of [26,8,9]. This shows in particular that the observability inequality for equations with a right-hand side (29)–(30) are indeed the natural energy estimates in the spaces we consider. In the following, Δ denotes the (negative) Laplace–Beltrami operator on Ω for the metric g, and $P = P(t, x, \partial_t, \partial_x) = \partial_t^2 - \Delta$ denotes the d'Alembert operator on $\mathbb{R} \times \Omega$. Its principal symbol is given by $p(t, x, \tau, \eta) = -\tau^2 + |\eta|_x^2$ for $(t, x, \tau, \eta) \in \mathbb{R} \times \Omega \times \mathbb{R} \times T_x^*\Omega \subset T^*(\mathbb{R} \times \Omega)$, where $|\eta|_x^2 = g_x(\eta, \eta)$ denotes the Riemannian norm in the cotangent space of Ω at x.

Lemma A.1. *Suppose that the couple (ω_b, T_b) satisfies GCC. Then, there exists a constant $C_b > 0$ such that for all $T \geqslant T_b$ and $v \in H^1((0, T) \times \Omega)$ solution of $Pv = f \in L^2((0, T) \times \Omega)$, we have*

$$\int_0^T \|v\|_{H^1(\Omega)}^2 + \|\partial_t v\|_{L^2(\Omega)}^2 \, dt \leqslant C_b \left(\int_0^T \int_\omega |\partial_t v|^2 \, dx \, dt + \int_0^T \|f\|_{L^2(\Omega)}^2 \, dt \right). \tag{64}$$

Proof. We first remark that it is sufficient to prove (64) with a time depending constant $C_b = C(T)$. The time invariance property of the equation $Pv = f$ then yields the desired result.

The proof relies on a compactness-uniqueness method. In a first step, we prove the following weaker energy estimate

$$\int_0^T \|v\|_{H^1(\Omega)}^2 + \|\partial_t v\|_{L^2(\Omega)}^2 \, dt \leqslant C(T) \left(\int_0^T \int_\omega |\partial_t v|^2 \, dx \, dt + \int_0^T \|f\|_{L^2(\Omega)}^2 \, dt + \int_0^T \|v\|_{L^2(\Omega)}^2 \, dt \right), \tag{65}$$

in which a compact term has been added in the right-hand side. In a second step we use a uniqueness argument to get rid of this additional term.

We define the following two vector spaces:

$$E = F = \left\{ v \in H^1\big((0, T) \times \Omega\big);\ Pv \in L^2\big((0, T) \times \Omega\big) \right\},$$

endowed with the norms

$$\|v\|_E^2 = \|v\|_{H^1((0,T)\times\Omega)}^2 + \|Pv\|_{L^2((0,T)\times\Omega)}^2,$$
$$\|v\|_F^2 = \|v\|_{L^2((0,T)\times\Omega)}^2 + \|\partial_t v\|_{L^2((0,T)\times\omega)}^2 + \|Pv\|_{L^2((0,T)\times\Omega)}^2.$$

We first remark that E is a Hilbert space for the norm $\|\cdot\|_E$ and that we have $E \subset F$ with $\|\cdot\|_F \leqslant \|\cdot\|_E$. If we prove that the space F is complete, the Banach isomorphism theorem then yields the inverse inequality: $\|\cdot\|_E \leqslant C\|\cdot\|_F$, which implies (65). Let us consider a Cauchy sequence $(v^k)_{k\in\mathbb{N}}$ of F. Since $L^2((0,T)\times\Omega)$ and $L^2((0,T)\times\omega)$ are complete, there exist $\underline{v} \in L^2((0,T)\times\Omega)$, $\underline{w} \in L^2((0,T)\times\omega)$ and $\underline{f} \in L^2((0,T)\times\Omega)$ such that

$$\begin{aligned} v^k &\to \underline{v} \quad \text{in } L^2\big((0,T)\times\Omega\big),\\ \partial_t v^k\big|_\omega &\to \underline{w} \quad \text{in } L^2\big((0,T)\times\omega\big),\\ Pv^k &\to \underline{f} \quad \text{in } L^2\big((0,T)\times\Omega\big). \end{aligned}$$

Since $\partial_t v^k \to \partial_t \underline{v}$ and $Pv^k \to P\underline{v}$ in $\mathcal{D}'((0,T)\times\Omega)$, we also have $\partial_t \underline{v}|_\omega = \underline{w} \in L^2((0,T)\times\omega)$ and $P\underline{v} = \underline{f} \in L^2((0,T)\times\Omega)$. The first order differential operator ∂_t is microlocally elliptic on $\mathrm{Char}(P) = \{\rho \in T^*(\mathbb{R}\times\Omega)\setminus 0,\ p(\rho)=0\}$, so that $\underline{v} \in H^1((0,T)\times\omega)$. As a consequence, $\underline{v}$ satisfies

$$\begin{cases} P\underline{v} = \underline{f} \in L^2\big((0,T)\times\Omega\big),\\ \underline{v} \in H^1\big((0,T)\times\omega\big). \end{cases} \tag{66}$$

Now, pick any point $\rho = (t,x,\tau,\eta) \in T^*(\mathbb{R}\times\Omega)\setminus 0$ such that $t \in (0,T)$. If $\rho \notin \mathrm{Char}(P)$, then P is elliptic of order two at ρ and the first equation of (66) yields that $\underline{v} \in H^2$ microlocally at ρ. If $\rho \in \mathrm{Char}(P)$, we denote by $\Gamma = \{\gamma(s), s \in (-S_-, S_+)\}$ the maximal bicharacteristic curve of P satisfying $\gamma(0) = \rho$. Since the couple (ω, T) satisfies the geometric control condition, there exists $s^* \in (-S_-, S_+)$ such that $\pi(\gamma(s^*)) \in (0,T)\times\omega$, where $\pi : T^*(\mathbb{R}\times\Omega) \to \mathbb{R}\times\Omega$ denotes the natural projection. The second line of (66) implies that $\underline{v} \in H^1$, microlocally at $\gamma(s^*)$. Hörmander's theorem on propagation of singularities [30, Chapter 6, Theorem 2.1] (see also [17, Theorem 26.1.1]) yields that $\underline{v} \in H^1$ microlocally at ρ since $\underline{v}$ satisfies $P\underline{v} \in L^2$. Note that the L^2 regularity for f is the natural one, required by the propagation theorem.

Finally we obtain $\underline{v} \in H^1((0,T)\times\Omega)$. Hence, the Cauchy sequence $(v^k)_{k\in\mathbb{N}}$ of F converges towards $\underline{v} \in F$, and F is complete. The Banach isomorphism theorem gives the existence of a constant $C > 0$ (depending on T) such that $\|\cdot\|_E \leqslant C\|\cdot\|_F$, which implies (65).

Now, we must get rid of the additional term $\|v\|_{L^2((0,T)\times\Omega)}^2$ in (65). For this, we prove that there exists a constant $C > 0$ such that for all $v \in E$ satisfying $Pv = f \in L^2((0,T)\times\omega)$, we have

$$\|v\|_{L^2((0,T)\times\Omega)}^2 \leqslant C\big(\|\partial_t v\|_{L^2((0,T)\times\omega)}^2 + \|f\|_{L^2((0,T)\times\Omega)}^2\big). \tag{67}$$

We suppose that this inequality is false. Then, there exists a sequence $(v^k)_{k\in\mathbb{N}}$, such that

$$\begin{aligned} \big\|v^k\big\|_{L^2((0,T)\times\Omega)} &= 1,\\ \partial_t v^k\big|_\omega &\to 0 \quad \text{in } L^2\big((0,T)\times\omega\big),\\ Pv^k &\to 0 \quad \text{in } L^2\big((0,T)\times\Omega\big). \end{aligned}$$

Hence, the energy estimate (65) gives $\|v^k\|_{H^1((0,T)\times\Omega)} \leqslant C$ uniformly, so that we can extract a subsequence (also denoted v^k) that converges in $L^2((0,T)\times\Omega)$. Calling $\underline{v} \in L^2((0,T)\times\Omega)$ its limit, we have

$$\begin{cases} \|\underline{v}\|_{L^2((0,T)\times\Omega)} = 1,\\ \partial_t \underline{v}|_\omega = 0 \quad \text{on } (0,T)\times\omega,\\ P\underline{v} = 0 \quad \text{in } (0,T)\times\Omega. \end{cases} \tag{68}$$

Once again, the propagation of regularity [30, Chapter 6, Theorem 2.1] together with GCC gives (for instance) $\underline{v} \in H^2((0,T)\times\Omega)$. According to a uniqueness result [27], the last two lines of (68) yield $\underline{v} = 0$ on $(0,T)\times\Omega$, which contradicts the first line of (68). This gives (67), and concludes the proof of the lemma. □

References

[1] F. Alabau, Observabilité frontière indirecte de systèmes faiblement couplés, C. R. Acad. Sci. Paris, Sér. I Math. 333 (7) (2001) 645–650.
[2] F. Alabau-Boussouira, Stabilisation frontière indirecte de systèmes faiblement couplés, C. R. Acad. Sci. Paris, Sér. I Math. 328 (1999) 1015–1020.
[3] F. Alabau-Boussouira, Indirect boundary stabilization of weakly coupled systems, SIAM J. Control Optim. 41 (2002) 511–541.
[4] F. Alabau-Boussouira, A two-level energy method for indirect boundary observability and controllability of weakly coupled hyperbolic systems, SIAM J. Control Optim. 42 (2003) 871–906.
[5] F. Alabau-Boussouira, M. Léautaud, Indirect controllability of locally coupled systems under geometric conditions, C. R. Acad. Sci. Paris, Sér. I Math. 349 (2011) 395–400.
[6] F. Alabau-Boussouira, M. Léautaud, Indirect stabilization of locally coupled wave-type systems, ESAIM Control Optim. Calc. Var. 18 (2012) 548–582. http://dx.doi.org/10.1051/cocv/2011106.
[7] F. Ammar-Khodja, A. Benabdallah, C. Dupaix, Null controllability of some reaction-diffusion systems with one control force, J. Math. Anal. Appl. 320 (2006) 928–943.
[8] C. Bardos, G. Lebeau, J. Rauch, Un exemple d'utilisation des notions de propagation pour le contrôle et la stabilisation de problèmes hyperboliques, in: Nonlinear Hyperbolic Equations in Applied Sciences, Rend. Sem. Mat. Univ. Politec. Torino Special Issue (1988) 11–31.
[9] C. Bardos, G. Lebeau, J. Rauch, Sharp sufficient conditions for the observation, control, and stabilization of waves from the boundary, SIAM J. Control Optim. 30 (1992) 1024–1065.
[10] N. Burq, Contrôlabilité exacte des ondes dans des ouverts peu réguliers, Asymptot. Anal. 14 (2) (1997) 157–191.
[11] N. Burq, P. Gérard, Condition nécessaire et suffisante pour la contrôlabilité exacte des ondes, C. R. Acad. Sci. Paris, Sér. I Math. 325 (7) (1997) 749–752.
[12] N. Burq, M. Zworski, Geometric control in the presence of a black box, J. Amer. Math. Soc. 17 (2) (2004) 443–471.
[13] R. Dáger, Insensitizing controls for the 1-D wave equation, SIAM J. Control Optim. 45 (5) (2006) 1758–1768.
[14] L. de Teresa, Insensitizing controls for a semilinear heat equation, Comm. Partial Differential Equations 25 (1–2) (2000) 39–72.
[15] E. Fernández-Cara, M. González-Burgos, L. de Teresa, Boundary controllability of parabolic coupled equations, J. Funct. Anal. 259 (7) (2010) 1720–1758.
[16] M. González-Burgos, R. Pérez-García, Controllability results for some nonlinear coupled parabolic systems by one control force, Asymptot. Anal. 46 (2006) 123–162.
[17] L. Hörmander, The analysis of linear partial differential operators. IV, in: Fourier Integral Operators, in: Grundlehren der Mathematischen Wissenschaften, vol. 275, Springer-Verlag, Berlin, 1994, corrected reprint of the 1985 original.
[18] S. Jaffard, Contrôle interne exact des vibrations d'une plaque rectangulaire, Portugal. Math. 47 (4) (1990) 423–429.
[19] O. Kavian, L. de Teresa, Unique continuation principle for systems of parabolic equations, ESAIM Control Optim. Calc. Var. 16 (2) (2010) 247–274.
[20] M. Léautaud, Spectral inequalities for non-selfadjoint elliptic operators and application to the null-controllability of parabolic systems, J. Funct. Anal. 258 (2010) 2739–2778.
[21] G. Lebeau, Contrôle de l'équation de Schrödinger, J. Math. Pures Appl. (9) 71 (3) (1992) 267–291.
[22] J.-L. Lions, Contrôlabilité exacte, perturbations et stabilisation de systèmes distribués. Tome 1, Recherches en Mathématiques Appliquées, vol. 8, Masson, Paris, 1988.
[23] L. Miller, Controllability cost of conservative systems: resolvent condition and transmutation, J. Funct. Anal. 218 (2) (2005) 425–444.
[24] L. Miller, The control transmutation method and the cost of fast controls, SIAM J. Control Optim. 45 (2) (2006) 762–772.
[25] K.-D. Phung, Observability and control of Schrödinger equations, SIAM J. Control Optim. 40 (1) (2001) 211–230.
[26] J. Rauch, M. Taylor, Exponential decay of solutions to hyperbolic equations in bounded domains, Indiana Univ. Math. J. 24 (1974) 79–86.
[27] L. Robbiano, Théorème d'unicité adapté au contrôle des solutions des problèmes hyperboliques, Comm. Partial Differential Equations 16 (4–5) (1991) 789–800.
[28] L. Rosier, L. de Teresa, Exact controllability of a cascade system of conservative equations, C. R. Acad. Sci. Paris, Sér. I Math. 349 (5–6) (2011) 291–296. http://dx.doi.org/10.1016/j.crma.2011.01.014.
[29] D. Russell, A unified boundary controllability theory for hyperbolic and parabolic partial differential equations, Studies in Appl. Math. 52 (1973) 189–221.
[30] M.E. Taylor, Pseudodifferential Operators, Princeton University Press, Princeton, NJ, 1981.
[31] M. Tucsnak, G. Weiss, Observation and Control for Operator Semigroups, Birkhäuser Advanced Texts: Basel Textbooks, Birkhäuser Verlag, Basel, 2009.